RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF GENERALIZED ORDER STATISTICS FROM DOUBLY TRUNCATED GOMPERTZ DISTRIBUTION

ZAKI ANWAR and ABDUL NASIR KHAN

Department of Statistics and Operations Research

Aligarh Muslim University

Aligarh-202002

India

e-mail: zakistats@gmail.com

nasirgd4931@gmail.com

Abstract

In this paper, we have established the recurrence relations for single and product moments of generalized order statistics from doubly truncated Gompertz distribution. These relations have also discussed for order statistics, upper record values, sequential order statistics, and order statistics with non-integral sample size as the particular cases of generalized order statistics.

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1. Introduction

Kamps [15] introduced the theory of generalized order statistics (*gos*). The model *gos* contains various sub models such as order statistics, record values, sequential order statistics, order statistics with non-integral sample size, and Pfeifer record values.

Let $\{X_n, n \geq 1\}$ be a sequence of independently and identically distributed random variables (*r. v.*) with *df* $F(x)$ and *pdf* $f(x)$. Then the quantity $X(r, n, m, k)$ is called *r*-th *gos* with parameters $k > 0, n \in \mathbb{N}, m \in \mathbb{R}$ such that $\gamma_r = k + (n - r)(m + 1) > 0 \forall r \in \{1, 2, \dots, n - 1\}$. The joint *pdf* $X(1, n, m, k), X(2, n, m, k), \dots, X(n, n, m, k)$ is given by

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [\bar{F}(x_i)]^m f(x_i) \right) [\bar{F}(x_n)]^{k-1} f(x_n), \quad (1.1)$$

on the cone $F^{-1}(0) < x_1 \leq x_2, \dots, \leq x_n < F^{-1}(1)$ of $\mathfrak{R}^n$.

The density function of *r*-th *gos* $X(r, n, m, k), 1 \leq r < n$ is

$$f_{X(r, n, m, k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)), \quad -\infty < x < \infty. \quad (1.2)$$

The joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k), 1 \leq r < s \leq n$ is,

$$\begin{aligned} f_{X(r, n, m, k), X(s, n, m, k)}(x, y) &= \frac{C_{s-1}}{(r-1)! (s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad -\infty < x < y < \infty, \end{aligned} \quad (1.3)$$

where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1}, & m \neq -1, \\ -\ln(1-x), & m = -1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1).$$

If $m = 0$, $k = 1$, then $X(r, n, m, k)$ reduces to order statistics, $m = -1$, then $X(r, n, m, k)$ tends to upper record values, $m = 0$, $k = \delta - n + 1$, $\delta \in \mathbb{R}^+$, then $X(r, n, m, k)$ tends to order statistics with non-integral sample size and when $m = \delta - 1$, $k = \delta$, then $X(r, n, m, k)$ tends to sequential order statistics.

The literature reveals that the concept of *gos* have been used by various authors such as; Ahmad [1, 2], Ahmad and Fawzy [3], Abdul-Moniem [6], Athar and Islam [5], Balakrishnan and Sandhu [9], Cramer and Kamps [4], Gupta and Anwar [10], Khan and Sharma [8], Khan et al. [14], Khan and Khan [13], Nayabuddin and Athar [11], Pawlas and Szynal [12].

The probability density function (*pdf*) of Gompertz distribution (GD) is given by

$$f_X(x) = \beta \exp \left[\gamma x - \frac{\beta}{\gamma} (e^{\gamma x} - 1) \right], \quad x \geq 0, \beta, \gamma > 0,$$

and the cumulative distribution function (*cdf*) is given by

$$F(x) = 1 - \exp \left[-\frac{\beta}{\gamma} (e^{\gamma x} - 1) \right], \quad x \geq 0, \beta, \gamma > 0.$$

And the *pdf* of doubly truncated GD is given by

$$f_X(x) = \frac{\beta}{P - Q} \exp \left[\gamma x - \frac{\beta}{\gamma} (e^{\gamma x} - 1) \right], \quad Q_1 \leq x \leq P_1, \quad (1.4)$$

where

$$1 - P = \exp \left[-\frac{\beta}{\gamma} (e^{\gamma P_1} - 1) \right] \text{ and } 1 - Q = \exp \left[-\frac{\beta}{\gamma} (e^{\gamma Q_1} - 1) \right].$$

Then

$$1 - F(x) = -P_2 + \frac{e^{-\gamma x}}{\beta} f(x), \quad (1.5)$$

where

$$P_2 = \frac{1 - P}{P - Q}.$$

Recurrence relations are very useful for computation of moments. It is concluded that the obtained results can be used for computational framework of moments of ordered random variables. If the parent population follows GD truncated on both sides.

Hendi et al. [7] presented the same study of the doubly truncated GD for the ordered random variables. However, in this paper, we have considered a more general framework called *gos* for the doubly truncated GD. Also, it is well known that order random variables are special cases of *gos*. Thus, the results proposed in this paper are extension of the work considered by Hendi et al. [7].

The rest of the paper is organized as follows. In Section 2, we have presented recurrence relations for expectation of functions of single *gos* and also discussed the special cases through remarks. In Section 3, recurrence relations are presented for the expectation of functions of two *gos* and special cases through remarks for these recurrence relations for product moments are also discussed.

2. Recurrence Relations for Single Moment

Theorem 2.1. *For doubly truncated Gompertz distribution as given in (1.4) and $n \in \mathbb{N}$, $m \in \mathbb{R}$, $\beta\gamma_1 > 0$, $r > 1$, $n \geq 2$*

$$\begin{aligned} & E[\psi\{X(r, n, m, k)\}] - E[\psi\{X(r-1, n-1, m, k)\}] \\ &= -P_2 K' \{E[\psi\{X(r, n-1, m, k+m)\}] - E[\psi\{X(r-1, n-1, m, k+m)\}]\} \\ &+ \frac{1}{\beta\gamma_1} E[\psi'\{X(r, n, m, k)\}e^{-\gamma X(r, n, m, k)}], \end{aligned} \quad (2.1)$$

where

$$K' = \frac{C_{r-2}^{(n-1)}}{C_{r-2}^{(n-1, k+m)}} = \prod_{i=1}^{r-1} \left(\frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1, k+m)}} \right).$$

Proof. From Khan et al. [14], we have

$$\begin{aligned} E[\psi\{X(r, n, m, k)\}] - E[\psi\{X(r-1, n-1, m, k)\}] \\ = \frac{C_{r-1}}{\gamma_1(r-1)!} \int_{Q_1}^{P_1} \psi'(x) [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx. \end{aligned} \quad (2.2)$$

Substituting (1.5) in (2.2), we get

$$\begin{aligned} E[\psi\{X(r, n, m, k)\}] - E[\psi\{X(r-1, n-1, m, k)\}] \\ = I(x) + \frac{1}{\beta\gamma_1} E[\psi'\{X(r, n, m, k)\} e^{-\gamma X(r, n, m, k)}], \end{aligned} \quad (2.3)$$

where

$$I(x) = -P_2 \frac{C_{r-1}}{\gamma_1(r-1)!} \int_{Q_1}^{P_1} \psi'(x) [\bar{F}(x)]^{\gamma_r^{(n-1, k+m)}} g_m^{r-1}(F(x)) dx.$$

Now after by parts integration of $I(x)$, we get

$$\begin{aligned} I(x) = -P_2 \frac{\gamma_r^{(n-1, k+m)} C_{r-2}^{(n-1)}}{(r-1)!} \int_{Q_1}^{P_1} \psi(x) [\bar{F}(x)]^{\gamma_r^{(n-1, k+m)}-1} f(x) g_m^{r-1}(F(x)) dx \\ + P_2 \frac{C_{r-2}^{(n-1)}}{(r-2)!} \int_{Q_1}^{P_1} \psi(x) [\bar{F}(x)]^{\gamma_r^{(n-1, k+m)}+m} f(x) g_m^{r-2}(F(x)) dx. \end{aligned}$$

After substituting $I(x)$ in (2.3), we get the required result.

Remark 2.1. Putting $\psi(x) = x^p$, $p = 1, 2, \dots$ in (2.1), we get

$$\begin{aligned} E[X^p(r, n, m, k)] - E[X^p(r-1, n-1, m, k)] \\ = -P_2 K' \{E[X^p(r, n-1, m, k+m)] - E[X^p(r-1, n-1, m, k+m)]\} \\ + \frac{p}{\beta\gamma_1} E[X^{p-1}(r, n, m, k) e^{-\gamma X(r, n, m, k)}]. \end{aligned} \quad (2.4)$$

The relation (2.4) represents the recurrence relation for single moments of doubly truncated GD.

Remark 2.2. Putting $k = 1$ and $m = 0$ in (2.4), we get the recurrence relations for single moments of order statistics from doubly truncated GD

$$\mu_{r:n}^p - \mu_{r-1:n-1}^p = -P_2 \left\{ \mu_{r:n-1}^p - \mu_{r-1:n-1}^p \right\} + \frac{P}{\beta n} E[X_{r:n}^{p-1} e^{-\gamma X_{r:n}}],$$

or

$$\mu_{r:n}^p = -P_2 \mu_{r:n-1}^p + Q_2 \mu_{r-1:n-1}^p + \frac{P}{\beta n} E[X_{r:n}^{p-1} e^{-\gamma X_{r:n}}],$$

where

$$\mu_{r:n}^p = E[X_{r:n}^p] \text{ and } P_2 = 1 - Q_2.$$

For $r = 1$ and $n = 1$,

$$\mu_{1:n}^p = -P_2 P_1^p + Q_2 Q_1^p + \frac{P}{\beta} E[X_{1:n}^{p-1} e^{-\gamma X_{1:n}}].$$

These relations have obtained by Hendi et al. [7].

Remark 2.3. Putting $m = -1$ in (2.4), the recurrence relation for single moments of k -th upper record values from doubly truncated GD is given as

$$\begin{aligned} & E[X_{U^{(k)}(r)}^p] - E[X_{U^{(k)}(r-1)}^p] \\ &= -P_2 \left(\frac{k}{k-1} \right)^{r-1} \left\{ E[X_{U^{(k-1)}(r)}^p] - E[X_{U^{(k-1)}(r-1)}^p] \right\} \\ & \quad + \frac{P}{\beta k} E[X_{U^{(k)}(r)}^{p-1} e^{-\gamma X_{U^{(k)}(r)}}], \quad k > 1. \end{aligned}$$

Remark 2.4. For $m = 0$ and $k = \delta - n + 1$, $\delta \in \mathfrak{R}^+$, we obtain the recurrence relation for single moments of order statistics with non-integral sample size from doubly truncated GD

$$\mu_{r:\delta}^p = -P_2 \mu_{r:\delta-1}^p + Q_2 \mu_{r-1:\delta-1}^p + \frac{p}{\beta\delta} E[X_{r:\delta}^{p-1} e^{-\gamma X_{r:\delta}}].$$

Remark 2.5. For $m = \delta - 1$, $k = \delta$, we obtain the recurrence relation for single moments of sequential order statistics from doubly truncated GD

$$\begin{aligned} & E[X^p(r, n, \delta - 1, \delta)] - E[X^p(r - 1, n - 1, \delta - 1, \delta)] \\ &= -P_2 K' \{E[X^p(r, n - 1, \delta - 1, 2\delta - 1)] - E[X^p(r - 1, n - 1, \delta - 1, 2\delta - 1)]\} \\ & \quad + \frac{p}{\delta\beta n} E[X^{p-1}(r, n, \delta - 1, \delta) e^{-\gamma X(r, n, \delta-1, \delta)}], \end{aligned}$$

where

$$K' = \prod_{i=1}^{r-1} \left(\frac{\delta(n-i)}{\delta(n-i+1)-1} \right).$$

Theorem 2.2. For doubly truncated Gompertz distribution given in (1.6), and $n \in N$, $m \in \mathfrak{R}$, $r > 1$, $n \geq 2$, $\beta\gamma_r > 0$

$$\begin{aligned} & E[\psi\{X(r, n, m, k)\}] - E[\psi\{X(r - 1, n, m, k)\}] \\ &= -P_2 K^* \{E[\psi\{X(r, n - 1, m, k + m)\}] - E[\psi\{X(r - 1, n - 1, m, k + m)\}]\} \\ & \quad + \frac{1}{\beta\gamma_r} E[\psi'\{X(r, n, m, k)\} e^{-\gamma X(r, n, m, k)}], \end{aligned}$$

where

$$K^* = \frac{C_{r-2}}{C_{r-2}^{(n-1, k+m)}} = \prod_{i=1}^{r-1} \left(\frac{\gamma_i}{\gamma_i^{(n-1, k+m)}} \right) = \prod_{i=1}^{r-1} \left(\frac{\gamma_i}{\gamma_i - 1} \right).$$

Proof. In view of Khan et al. [14], this theorem can be easily proved on the line proof of Theorem 2.1.

Remark 2.6. Putting $\psi(x) = x^p$, $p = 1, 2, \dots$, in (2.5), we get

$$\begin{aligned} & E[X^p(r, n, m, k)] - E[X^p(r-1, n, m, k)] \\ &= -P_2 K^* \{E[X^p(r, n-1, m, k+m)] - E[X^p(r-1, n-1, m, k+m)]\} \\ &+ \frac{P}{\beta\gamma_r} E[X^{p-1}(r, n, m, k)e^{-\gamma X(r, n, m, k)}]. \end{aligned}$$

This is also a recurrence relation for single moments of *gos* from doubly truncated GD.

Remark 2.7. For non-truncated case $P = 1$ and $Q = 0$, we get the recurrence relation for single moments of *gos* from GD

$$\begin{aligned} E[X^p(r, n, m, k)] &= E[X^p(r-1, n-1, m, k)] \\ &+ \frac{P}{\beta\gamma_r} E[X^{p-1}(r, n, m, k)e^{-\gamma X(r, n, m, k)}], \end{aligned}$$

as obtained by Khan and Sharma [8].

3. Recurrence Relations for Product Moment

Theorem 3.1. For doubly truncated Gompertz distribution as given in (1.4) and $n \in N$, $m \in \mathfrak{R}$, $n \geq 2$, $\beta\gamma_s > 0$, $1 \leq r < s \leq n$

$$\begin{aligned} & E[\psi\{X(r, n, m, k), X(s, n, m, k)\}] - E[\psi\{X(r, n, m, k), X(s-1, n, m, k)\}] \\ &= -P_2 K^* \{E[\psi\{X(r, n-1, m, k+m), X(s, n-1, m, k+m)\}] \\ &- E[\psi\{X(r, n-1, m, k+m), X(s-1, n-1, m, k+m)\}]\} \\ &+ \frac{1}{\beta\gamma_s} E[\psi'\{X(r, n, m, k), X(s, n, m, k)\}e^{\gamma X(s, n, m, k)}], \end{aligned} \quad (3.1)$$

where

$$K^* = \frac{C_{s-2}}{C_{s-2}^{(n-1, k+m)}} = \prod_{j=1}^{s-1} \left(\frac{\gamma_j}{\gamma_j^{(n-1, k+m)}} \right).$$

Proof. In view of Khan et al. [14], we have

$$\begin{aligned} & E[\psi\{X(r, n, m, k), X(s, n, m, k)\}] - E[\psi\{X(r, n, m, k), X(s-1, n, m, k)\}] \\ &= \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_{Q_1}^{P_1} \int_x^{P_1} \psi(x, y) [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{s-1} [\bar{F}(y)] dy dx. \end{aligned} \quad (3.2)$$

Using (1.5) and (3.2), we get

$$\begin{aligned} & E[\psi\{X(r, n, m, k), X(s, n, m, k)\}] - E[\psi\{X(r, n, m, k), X(s-1, n, m, k)\}] \\ &= I(x, y) + \frac{1}{\beta \gamma_s} E[\psi\{X(r, n, m, k), X(s, n, m, k)\} e^{-\gamma X(s, n, m, k)}], \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} I(x, y) &= -P_2 \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_{Q_1}^{P_1} \int_x^{P_1} \psi'(x, y) [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s^{(n-1, k+m)}} dy dx. \end{aligned}$$

Now integrating $I(x, y)$ by parts with respect to y , we get

$$\begin{aligned} I(x, y) &= P_2 \frac{C_{s-2}}{(r-1)!(s-r-2)!} \int_{Q_1}^{P_1} \int_x^{P_1} \psi(x, y) [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-2} [\bar{F}(y)]^{\gamma_{s-1}^{(n-1, k+m)}-1} f(y) dy dx \\ & \quad - P_2 \frac{\gamma_{s-1}^{(n-1, k+m)} C_{s-2}}{(r-1)!(s-r-2)!} \int_{Q_1}^{P_1} \int_x^{P_1} \psi(x, y) [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-2} [\bar{F}(y)]^{\gamma_s^{(n-1, k+m)}-1} f(y) dy dx, \end{aligned}$$

$$\begin{aligned} \Rightarrow I(x, y) = & -P_2 K^* \{E[\psi\{X(r, n-1, m, k+m), X(s, n-1, m, k+m)\}] \\ & - E[\psi\{X(r, n-1, m, k+m), X(s-1, n-1, m, k+m)\}]\}. \end{aligned} \quad (3.4)$$

Substituting (3.3) in (3.4), we the required result.

Remark 3.1. Let $\psi(x, y) = x^p y^q$, $p, q = 1, 2, \dots$ in (3.4), we get

$$\begin{aligned} & E[X^p(r, n, m, k)X^q(s, n, m, k)] - E[X^p(r, n, m, k)X^q(s-1, n, m, k)] \\ & = -P_2 K^* \{E[X^p(r, n-1, m, k+m)X^q(s, n-1, m, k+m)] \\ & \quad - E[X^p(r, n-1, m, k+m)X^q(s-1, n-1, m, k+m)]\} \\ & \quad + \frac{q}{\beta\gamma_s} E[X^p(r, n, m, k)X^{q-1}(s, n, m, k)e^{-\gamma X(s, n, m, k)}]. \end{aligned} \quad (3.5)$$

The relation (3.5) represents the recurrence relation for product moments of doubly truncated GD.

Remark 3.2. For non-truncated case $P = 1$ and $Q = 0$ in (2.5), we get the recurrence relation for product moments of *gos* from GD

$$\begin{aligned} E[X^p(r, n, m, k)X^q(s, n, m, k)] & = E[X^p(r, n, m, k)X^q(s-1, n, m, k)] \\ & \quad + \frac{q}{\beta\gamma_s} E[X^p(r, n, m, k)X^{q-1}(s, n, m, k)e^{-\gamma X(s, n, m, k)}], \end{aligned}$$

as obtained by Khan and Sharma [8].

Remark 3.3. Putting $m = 0$ and $k = 1$ in (3.5), we get the recurrence relation for product moments of order statistics from doubly truncated GD

$$\begin{aligned} E[X_{r:n}^p X_{s:n}^q] - E[X_{r:n}^p X_{s-1:n}^q] & = -\frac{nP_2}{n-s+1} \{E[X_{r:n-1}^p X_{s:n-1}^q] \\ & \quad - E[X_{r:n-1}^p X_{s-1:n-1}^q]\} + \frac{q}{\beta\gamma_s} E[X_{r:n}^p X_{s:n}^{q-1} e^{-\gamma X_{s:n}}]. \end{aligned}$$

as obtained by Hendi et al. [7].

Remark 3.4. Putting $m = -1$ in (3.5), we obtain the recurrence relation for product moments of k -th upper record values from doubly truncated GD

$$\begin{aligned} & E[X_{U^{(k)}(r)}^p X_{U^{(k)}(s)}^q] - E[X_{U^{(k)}(r)}^p X_{U^{(k)}(s-1)}^q] \\ &= -P_2 \left(\frac{k}{k-1} \right)^{s-1} \{ E[X_{U^{(k-1)}(r)}^p X_{U^{(k-1)}(s)}^q] - E[X_{U^{(k-1)}(r)}^p X_{U^{(k-1)}(s-1)}^q] \} \\ &+ \frac{q}{\beta k} E[X_{U^{(k)}(r)}^p X_{U^{(k)}(s-1)}^{q-1} e^{-\gamma X_{U^{(k)}(s)}}], \quad k > 1. \end{aligned}$$

Remark 3.5. For $m = 0$ and $k = \delta - n + 1$, $\delta \in \mathfrak{R}_+$, we obtain the recurrence relation for single moments of order statistics with non-integral sample size from doubly truncated GD

$$\begin{aligned} & E[X_{r:\delta}^p X_{s:\delta}^q] - E[X_{r:\delta}^p X_{s-1:\delta}^q] = -P_2 \frac{\delta}{(\delta - s + 1)} \{ E[X_{r:\delta-1}^p X_{s:\delta-1}^q] \\ & - E[X_{r:\delta-1}^p X_{s-1:\delta-1}^q] \} + \frac{q}{\beta(\delta - s + 1)} E[X_{r:\delta}^p X_{s:\delta}^{q-1} e^{-\gamma X_{s:\delta}}]. \end{aligned}$$

Remark 3.6. For $m = \delta - 1$, $k = \delta$, we obtain the recurrence relation for single moments of sequential order statistics from doubly truncated GD

$$\begin{aligned} & E[X^p(r, n, \delta - 1, \delta) X^q(s, n, \delta - 1, \delta)] - E[X^p(r, n, \delta - 1, \delta) X^q(s - 1, n, \delta - 1, \delta)] \\ &= -P_2 K^* \{ E[X^p(r, n - 1, \delta - 1, 2\delta - 1) X^q(s, n - 1, \delta - 1, 2\delta - 1)] \\ & - E[X^p(r, n - 1, \delta - 1, 2\delta - 1) X^q(s - 1, n - 1, \delta - 1, 2\delta - 1)] \} \\ &+ \frac{q}{\beta \delta (n - s + 1)} E[X^p(r, n, \delta - 1, \delta) X^{q-1}(s, n, \delta - 1, \delta) e^{-\gamma X(s, n, \delta - 1, \delta)}], \end{aligned}$$

where

$$K^* = \prod_{j=1}^{s-1} \left(\frac{\delta(n - j + 1)}{\delta(n - j + 1) - 1} \right).$$

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