PROGRESSIVELY CENSORED DATA FROM THE GENERALIZED LINEAR EXPONENTIAL DISTRIBUTION MOMENTS AND ESTIMATION

N. M. YHIEA

Department of Mathematics

Faculty of Science

Suez Canal University

Ismailia

Egypt

e-mail: n_yhiea@hotmail.com

Abstract

In this paper, we derive approximate moments of progressively type-II right censored order statistics from the generalized linear exponential distribution. Depending on these moments, the best linear unbiased estimators and maximum likelihoods estimators of the location and scale parameters are found. In addition, we use Monte-Carlo simulation method to obtain the mean square error of the best linear unbiased estimates, maximum likelihood estimates and make comparison between them. Finally, we determine the optimal progressive censoring scheme for some practical choices of n and m when progressively type-II right censored samples are from the considered distribution and present numerical example to illustrate the developed inference procedures.

2010 Mathematics Subject Classification: 62Fxx.

Keywords and phrases: progressively type-II censored sample, generalized linear exponential distribution, approximate moments, best linear unbiased estimates (BLUEs), maximum likelihood estimates (MLEs), Monte-Carlo method, optimal scheme.

Received May 24, 2018; Revised June 23, 2018

1. Introduction

The exponential distribution is the most frequently used distribution in reliability theory and applications. The linear exponential distribution, having exponential as special case, has many applications in applied statistics and reliability analysis. It possesses several important statistical properties, and yet exhibits great mathematical tractability. Also, it has the ability to model failure rate which are quite common in reliability and biological studies.

In this paper, we derive approximate moments of progressively type-II right censored order statistics from the generalized linear exponential distribution. The best linear unbiased estimates and the maximum likelihood methods are used to drive the point estimators of the scale and location parameters from considered distribution. Several interesting mathematical results for inference procedures have been developed by the authors, see, for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], and [12]. [13] have derived approximate moments of progressively type-II right censored order statistic from the Weibull Gamma distribution and using these moments to derive the best linear unbiased estimates and maximum likelihood estimates.

In addition, we use Monte-Carlo simulation method to make comparison between the MSE of BLUEs and MLEs. Finally, we determine the optimal progressive censoring scheme and present numerical example to illustrate the developed inference procedures.

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \dots, R_m)$ drawn from the generalized exponential distribution whose probability function is given by:

$$f(x) = \alpha \left(\lambda x + \frac{\theta}{2} x^2 \right)^{\alpha-1} (\lambda + \theta x) e^{-(\lambda x + \frac{\theta}{2} x^2)^\alpha}, \quad \theta, \alpha > 0 \text{ and } \lambda \geq 0, \quad (1)$$

and distribution function is given by

$$F(x) = 1 - e^{-(\lambda x + \frac{\theta}{2} x^2)^\alpha}. \quad (2)$$

So, we can write the joint density function of $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ of a progressively type-II right censored sample from the generalized linear exponential distribution, with censoring scheme $(R_1, R_2, \dots, R_m)$, in the form:

$$\begin{aligned} f_{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}}(x_1, x_2, \dots, x_m) \\ = A(n, m-1) * \prod_{i=1}^m \alpha(\lambda + \theta x) \left(\lambda x + \frac{\theta}{2} x^2 \right)^{\alpha-1} \\ \times e^{-(\lambda x + \frac{\theta}{2} x^2)^{\alpha(R_i+1)}}, \end{aligned} \quad (3)$$

$$0 < x_1 < x_2 < \dots < x_m < \infty, \dots, \theta, \alpha > 0 \text{ and } \lambda \geq 0,$$

where

$$A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - R_1 - R_2 - \dots R_{m-1} - m + 1). \quad (4)$$

See [2].

It can be noted that the moments can not be derived in closed forms, so the relation between the generalized linear exponential distribution and uniform distribution is used to derive approximate moments.

Let $U_{1:m:n}, U_{2:m:n}, \dots, U_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \dots, R_m)$ taken from the uniform $(0, 1)$ distribution.

In [2], Gibbons and Chakraborti have derived exact moments of progressively type-II right censored order statistics from the uniform $(0, 1)$ distribution. These expressions enable us to derive the approximate means, variances, and covariances for the generalized linear exponential distribution using resulting of [1].

2. The Approximate Moments

Since, the joint density function of the form (3) is more difficult to use it to find the moments, so we get relationship between the generalized linear exponential distribution and uniform distribution. Let

$$U = 1 - e^{-(\lambda X + \frac{\theta}{2} X^2)^\alpha} \sim U(0, 1). \quad (5)$$

We use this relation to find the mean, variance and covariance of the generalized linear exponential distribution. In [2], Gibbons and Chakraborti have derived exact and explicit expressions for the single and product moments of progressively type-II right censored order statistics from $U(0, 1)$ distribution. These expressions, will enable us to derive approximate means, variances and covariances of progressively type-II right censored order statistics from the generalized linear exponential distribution using the two theorems in [1], pages 134-135.

For this purpose, let us use these notations:

$$(1) \quad E(U) = \mu, D^2(U) = \sigma^2, X = \phi(U) = (1 - U^{\frac{1}{\theta}})^{-\frac{1}{\alpha}} - 1,$$

and U 's are progressively type-II right censored order statistics from $U(0, 1)$.

(2) $X_{i:m:n}$, $i = 1, 2, \dots, m$ are progressively type-II right censored order statistics from the generalized linear exponential distribution.

(3)

$$E(X) \simeq \phi(\mu) + \frac{1}{2} \sigma^2 \phi''(\mu),$$

$$D^2(X) \simeq [\phi''(\mu)]^2 \sigma^2,$$

and

$$X = \phi(U) = (1 - U^{\frac{1}{\theta}})^{-\frac{1}{\alpha}} - 1,$$

where

$$\phi' = \frac{d}{dx}.$$

(4) U_i and U_j are from the uniform distribution with

$$E(U_i) = \mu, E(U_j) = \nu, D^2(U_i) = \sigma^2,$$

$$D^2(U_j) = \tau^2, \text{ and } \rho(U_i, U_j) = \rho.$$

(5)

$$Z = \phi(u_i, u_j) = [(1 - u_i^{\frac{1}{\theta}})^{-\frac{1}{\alpha}} - 1][(1 - u_j^{\frac{1}{\theta}})^{-\frac{1}{\alpha}} - 1],$$

$$E(Z) \simeq \phi(\mu, \nu) + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial u_i^2} + \rho \sigma \tau \frac{\partial^2 \phi}{\partial u_i \partial u_j} + \frac{1}{2} \tau^2 \frac{\partial^2 \phi}{\partial u_j^2},$$

and

$$D^2(Z) \simeq \sigma^2 \left(\frac{\partial \phi}{\partial u_i} \right)^2 + 2\rho \sigma \tau \left(\frac{\partial \phi}{\partial u_i} \frac{\partial \phi}{\partial u_j} \right) + \tau^2 \left(\frac{\partial \phi}{\partial u_j} \right)^2.$$

Then we get the following expressions:

$$E(X) \simeq \left[\frac{-\lambda + \sqrt{2\theta(-\ln|1 - \mu|)^{1/\alpha} + \lambda^2}}{\theta} \right] + \frac{1}{2} \sigma^2 H, \quad (6)$$

where

$$H = \frac{1}{\alpha(1-\mu)^2} \left[(-\ln|1-\mu|)^{1/\alpha-1} \left(2\theta(-\ln|1-\mu|)^{1/\alpha} + \lambda^2 \right)^{-1/2} \right. \\ \left. + (1/\alpha - 1)(-\ln|1-\mu|)^{1/\alpha-2} \left(2\theta(-\ln|1-\mu|)^{1/\alpha} + \lambda^2 \right)^{-1/2} \right. \\ \left. + \frac{\theta}{\alpha} (-\ln|1-\mu|)^{2/\alpha-2} \left(2\theta(-\ln|1-\mu|)^{1/\alpha} + \lambda^2 \right)^{-3/2} \right],$$

and

$$D^2(X) \simeq \left(\frac{\sigma}{\alpha\mu} \right)^2 (-\ln \mu)^{2/\alpha-2} \left(2\theta(-\ln \mu)^{1/\alpha} + \lambda^2 \right)^{-1}, \quad (7) \\ E(Z) \simeq \left[\frac{-\lambda + \sqrt{2\theta(-\ln \mu)^{1/\alpha} + \lambda^2}}{\theta} \right] \left[\frac{-\lambda + \sqrt{2\theta(-\ln v)^{1/\alpha} + \lambda^2}}{\theta} \right] \\ + \frac{1}{2} \sigma^2 S + \rho \sigma_{\tau} Y + \frac{1}{2} \tau^2 T,$$

and

$$\text{Cov}(X_{i:m:n}, X_{j:m:n}) \simeq E(Z) - E(X_{i:m:n})E(X_{j:m:n}), \quad (8)$$

where

$$S = \frac{1}{\alpha\theta\mu^2} \left(-\lambda + \sqrt{2\theta(-\ln v)^{1/\alpha} + \lambda^2} \right) \left[(-\ln \mu)^{(1/\alpha)-1} \right. \\ \left. \left(2\theta(-\ln \mu)^{1/\alpha} + \lambda^2 \right)^{(-1/2)} + ((1/\alpha) - 1)(-\ln \mu)^{(1/\alpha)-2} \right. \\ \left. \left(2\theta(-\ln \mu)^{1/\alpha} + \lambda^2 \right)^{(-1/2)} - \frac{\theta}{\alpha} \left(2\theta(-\ln \mu)^{1/\alpha} + \lambda^2 \right)^{(-3/2)} \right. \\ \left. (-\ln \mu)^{(2/\alpha)-2} \right], \\ Y = \left(\frac{1}{\alpha^2\theta} \right) \left(\frac{1}{v\mu} \right) (-\ln \mu)^{(1/\alpha)-1} (-\ln v)^{(1/\alpha)-1} \\ \left(2\theta(-\ln \mu)^{1/\alpha} + \lambda^2 \right)^{(-1/2)} \left(2\theta(-\ln v)^{1/\alpha} + \lambda^2 \right)^{(-1/2)},$$

and

$$\begin{aligned}
 T = & \frac{1}{\alpha \theta v^2} \left(-\lambda + \sqrt{2\theta(-\ln \mu)^{1/\alpha} + \lambda^2} \right) \left[(-\ln v)^{(1/\alpha)-1} \right. \\
 & \left(2\theta(-\ln v)^{1/\alpha} + \lambda^2 \right)^{(-1/2)} + ((1/\alpha) - 1)(-\ln v)^{(1/\alpha)-2} \\
 & \left(2\theta(-\ln v)^{1/\alpha} + \lambda^2 \right)^{(-1/2)} - \frac{\theta}{\alpha} \left(2\theta(-\ln v)^{1/\alpha} + \lambda^2 \right)^{(-3/2)} \\
 & \left. (-\ln v)^{(2/\alpha)-2} \right].
 \end{aligned}$$

These expressions, will used to derive the best linear unbiased estimators BLUEs for the location (μ) and scale (σ) parameters of the generalized linear exponential distribution.

3. Estimations of Scale and Location Parameters

The best linear unbiased and maximum likelihood methods are used to obtain estimates of the location (μ) and scale (σ) parameters. Let the probability density function be:

$$f(x) = \alpha \left(\lambda \frac{(x-\mu)}{\sigma} + \frac{\theta}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right)^{\alpha-1} \left(\lambda + \theta \frac{(x-\mu)}{\sigma} \right) e^{-\left(\lambda \frac{(x-\mu)}{\sigma} + \frac{\theta}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right)^\alpha},$$

$\theta, \alpha > 0$ and $\lambda \geq 0$, (9)

the distribution function is:

$$F(x) = 1 - e^{-\left(\lambda \frac{(x-\mu)}{\sigma} + \frac{\theta}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right)^\alpha}. \quad (10)$$

3.1. Best linear unbiased estimates (BLUEs)

Suppose that $X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, X_{2:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)}$ is a progressively type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, \dots, R_m)$, drawn from the generalized linear exponential distribution whose pdf given in (9) and cumulative distribution function given in (10).

Notice that we may not be able to write down explicit expressions for the BLUEs of μ and σ however, so a simulation study is considered to obtain the BLUEs of μ and σ using

$$\mu^* = -\mu^T \Gamma Y = \sum_{i=1}^m A_i Y_{i:m:n}, \quad (11)$$

and

$$\sigma^* = \mathbf{1}^T \Gamma Y = \sum_{i=1}^m B_i Y_{i:m:n}, \quad (12)$$

where

$$\Gamma = \sum^{-1} (\mathbf{1}\mu^T - \mu\mathbf{1}^T) \sum^{-1} / \Delta,$$

and

$$\Delta = \left(\mathbf{1}^T \sum^{-1} \mathbf{1} \right) \left(\mu^T \sum^{-1} \mu \right) - \left(\mathbf{1}^T \sum^{-1} \mu \right)^2.$$

See [2].

3.2. Maximum likelihood estimates

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \dots, R_m)$ taken from the generalized linear exponential distribution whose probability function is given by (9) and the cumulative distribution function is given by (10).

The likelihood function can be written in the form:

$$L(\mu, \sigma) = A(n, m-1) \prod_{i=1}^m f(X_{i:m:n}) [1 - F(X_{i:m:n})]^{R_i},$$

where $A(n, m-1)$ is normalizing constant in (4), see [2].

The likelihood function to be maximized for estimators of μ and σ (which we will denoted by $\hat{\mu}$ and $\hat{\sigma}$) is given by:

$$L(\mu, \sigma) = (\text{const.}) \alpha^m \prod_{i=1}^m \left[\lambda + \theta \left(\frac{x_i - \mu}{\sigma} \right) \right] \left[\lambda \left(\frac{x_i - \mu}{\sigma} \right) + \frac{\theta}{2} \left(\frac{x_i - \mu}{\sigma} \right)^2 \right]^{\alpha-1} \\ \times e^{-\left(R_i+1\right) \left[\lambda \left(\frac{x_i - \mu}{\sigma} \right) + \frac{\theta}{2} \left(\frac{x_i - \mu}{\sigma} \right)^2 \right]^\alpha}.$$

For simplicity, we use X_i instead of $X_{i:m:n}$. The log-likelihood function can be written in form:

$$\ln L(\mu, \sigma) = \text{constant} + m \ln \alpha + \sum_{i=1}^m \ln \left[\lambda + \theta \left(\frac{x_i - \mu}{\sigma} \right) \right] \\ + (\alpha - 1) \sum_{i=1}^m \ln \left[\lambda \left(\frac{x_i - \mu}{\sigma} \right) + \frac{\theta}{2} \left(\frac{x_i - \mu}{\sigma} \right)^2 \right] \\ + \sum_{i=1}^m (1 + R_i) \left[\lambda \left(\frac{x_i - \mu}{\sigma} \right) + \frac{\theta}{2} \left(\frac{x_i - \mu}{\sigma} \right)^2 \right]^\alpha, \quad (13)$$

and hence we have the likelihood equations for μ and σ to be

$$\frac{-\theta}{\hat{\sigma}} \sum_{i=1}^m \frac{1}{\left(\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right)} - \frac{(\alpha - 1)}{\hat{\sigma}} \sum_{i=1}^m \frac{1}{\left(\lambda \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) + \frac{\theta}{2} \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2 \right)} \left(\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right) \\ + \frac{\alpha}{\hat{\sigma}} \sum_{i=1}^m (R_i + 1) \left(\lambda \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) + \frac{\theta}{2} \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2 \right)^{\alpha-1} \left(\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right) = 0, \quad (14)$$

and

$$\begin{aligned}
& \frac{-1}{\hat{\sigma}^2} \sum_{i=1}^m \frac{(x_i - \hat{\mu})}{\left(\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right)} + \frac{(\alpha - 1)}{\hat{\sigma}^2} \sum_{i=1}^m \frac{(x_i - \hat{\mu})}{\left(\lambda \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) + \frac{\theta}{2} \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2 \right)} \left(\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right) \\
& + \frac{\alpha}{\hat{\sigma}^2} \sum_{i=1}^m (R_i + 1) \left[\lambda \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) + \frac{\theta}{2} \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2 \right]^{\alpha-1} \left[\lambda + \theta \left(\frac{x_i - \hat{\mu}}{\hat{\sigma}} \right) \right] (x_i - \hat{\mu}) = 0.
\end{aligned}
\tag{15}$$

The MLEs of $\hat{\mu}$ and $\hat{\sigma}$ can be obtained by solving the likelihood equations. Since Equations (14) and (15) cannot be solved analytically, so we can use MATLAB program to solve these equations.

4. Simulation Study

A comparison between the MSEs of BLUEs and MLEs is made through a Monte Carlo simulation study. In the following, we present a simulation algorithm used to generate progressively type-II right censored samples from generalized linear exponential distribution.

4.1. Simulation algorithm

By using the algorithm given in [2], the following steps are used to generate progressively type-II right censored order statistics from generalized linear exponential distribution.

(1) Generate m independent uniform $U(0, 1)$, random variables $W_1, W_2, \dots, W_m$.

(2) For given values of the progressive censoring scheme $R_1, R_2, \dots, R_m$,

set $V_i = W_i^{1/\left(i + \sum_{j=m-i+1}^m R_j\right)}$ for $i = 1, 2, \dots, m$.

(3) Set $U_i = 1 - V_m V_{m-1} \dots V_{m-i+1}$ for $i = 1, 2, \dots, m$. Then $U_{1:m:n}, U_{2:m:n}, \dots, U_{m:m:n}$ is a progressively type-II right censored sample of size m from $U(0, 1)$.

(4) Finally, for given values of the two parameters μ and σ ,

$$X_i = \frac{-\lambda + \sqrt{2\theta(-\ln|1 - \mu|)^{1/\alpha} + \lambda^2}}{\theta}, \quad i = 1, 2, \dots, m \text{ is a progressively}$$

type-II right censored sample of size m from the generalized linear exponential distribution.

We generate a progressively type-II right censored samples from the generalized linear exponential distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 1$. For different values of α , θ , and λ , 5000 Monte Carlo runs are simulated based on different scheme given in Table 1.

By using the BLUEs presented in Equations (11), (12) and Table 1, the coefficient of the BLUEs A_i 's and B_i 's are tabulated in Tables 2-5. The MSEs of the BLUEs and MLEs are tabulated in Table 6.

4.2. Discussion of the results

In Table 1, n denotes the sample size and m denotes the number of removable during the different schemes considered in the table.

From the numerical results presented in Tables 2-5 we can conclude the following:

(1) As a check of the entries of Tables 2, 3, 4, and 5, we see that

$$\sum_{i=1}^n A_i \simeq 1, \quad \sum_{i=1}^n B_i \simeq 0.$$

(2) From Table 6, we see that as n increases, the mean square error $MSE(\mu^*)$ and $MSE(\sigma^*)$ decrease for all censoring schemes and all values of λ , θ , and α .

(3) From Table 6, we see that as n increases, the mean square error $MSE(\hat{\mu})$ and $MSE(\hat{\sigma})$ decrease for all censoring schemes and all values of λ and α .

(4) The MSEs of $\hat{\mu}$ and $\hat{\sigma}$ are better than MSEs of μ^* and σ^* for all schemes.

5. Numerical Examples

A progressively type-II censored sample of size $m = 6$ from a sample of size $n = 15$ from the generalized linear exponential distribution with $\mu = 0$, $\sigma = 1$, $\alpha = 0.5$, $\theta = 1$, $\lambda = 0.2$ with scheme $R_i = [1 \ 2 \ 0 \ 2 \ 3 \ 2 \ 0 \ 2]$, was simulated by using MATLAB program. The simulated progressively type-II right censored sample is given by:

$X_{i:8:20}$	0.0019	0.1486	0.1517	0.2614	0.9141	1.0858
R_i	1	2	0	2	3	2

By making use of Equations (11) and (12), and using the coefficients A_i and B_i given in Table 3 for $n = 15$ and $m = 6$, we get the BLUEs of the μ and σ as follows:

$$\begin{aligned}
 \mu^* &= (-0.1499 \times 0.0019) + (0.0362 \times 0.1486) + (0.2516 \times 0.1517) \\
 &\quad + (0.664 \times 0.2614) + (1.7695 \times 0.9141) + (-1.5713 \times 1.0858) \\
 &= 0.12821424, \\
 \sigma^* &= (0.6413 \times 0.0019) + (0.2873 \times 0.1486) + (-0.1333 \times 0.1517) \\
 &\quad + (-0.9257 \times 0.2614) + (-3.0024 \times 0.9141) + (3.1329 \times 1.0858) \\
 &= 0.43892064.
 \end{aligned}$$

The standard error of the estimates μ^* and σ^* are

$$SE(\mu^*) = \sigma^*(\text{Var}(\mu^*))^{\frac{1}{2}} = 0.43892064 \times (0.6620)^{\frac{1}{2}} = 0.357120679,$$

$$SE(\sigma^*) = \sigma^*(\text{Var}(\sigma^*))^{\frac{1}{2}} = 0.43892064 \times (1.13170)^{\frac{1}{2}} = 0.466929875.$$

Using the same data, we can get by simulation the MLEs of μ and σ as follows:

$$\hat{\mu} = 0.0789,$$

and

$$\hat{\sigma} = 1.0635.$$

6. Optimal Censoring Scheme

We aim in the following section to determine the optimal progressive censoring scheme for some practical choices of n and m when progressive type-II right censored samples are from generalized linear exponential distribution. Our focus on the variance-covariance matrix of the BLUEs for the location and scale parameter in the generalized linear exponential distribution. Choosing the optimal censoring scheme in different related problem has received considerable by [2], [14].

For various choices of m and n , possible censoring schemes are considered and the variances and covariance of the BLUE's arising from these censoring schemes are determined. The optimal censoring schemes and their efficiencies with respect to the minimum trace of the variance-covariance matrix are presented in Tables 7 and 8.

Tables 9 and 10 give coefficients of the progressive type-II right censored order statistics for BLUE's of the location and scale parameters for various schemes considered along with the variances and covariance of the BLUEs.

In Tables 7 and 8 efficiencies for the trace-optimal schemes (that is, the scheme for which minimum trace of the variance-covariance matrix of BLUE's is attained for some values of m and n) with respect to the conventional type-II right censoring schemes are shown on the first line of each box, and the actual traces corresponding to these optimal schemes are shown on the second line. From tables, as one would expect, as m increases (with n being held constant), the gain in efficiency over the conventional scheme decreases, that is, the smaller the proportion of failure times an experimenter would like to observe, the more precision can be gained by employing optimal progressive type-II right censoring over conventional type-II right censoring. Also, as n increases (with m being constant), the gain in trace-efficiency increases.

Looking at the trace-optimal schemes denoted with a + in Tables 9 and 10, it is quite evident that for small values of m , a scheme by which all but $m-1$ surviving units are randomly removed after the first observed failure (i.e., $R_1 = n - m$) will result in the best precision of BLUE's. Also, shows least precise censoring scheme with respect to the trace.

Tables 9 and 10 are given for selected values of m and n , the coefficient of the best linear unbiased estimators (BLUEs) for selected schemes and selected values of α , θ , and λ . We can see that the sum of coefficients of $\mu^* \simeq 1$ and the sum of coefficients of $\sigma^* \simeq 0$.

Table 1. Different schemes of progressively censored

n	m	Censoring scheme
10	4	[2 0 2 2]
15	6	[2 0 2 0 3 2]
20	8	[1 2 0 2 3 2 0 2]
25	10	[4 0 0 2 0 0 4 2 0 3]

Table 2. Coefficients of the BLUEs of μ and σ from the generalized linear exponential distribution using the first scheme when $\mu = 0$ and $\sigma = 1$

Scheme 1		$\alpha = 0.3, \theta = 1.5, \lambda = 0.5$		$\alpha = 0.5, \theta = 1, \lambda = 0.2$		$\alpha = 1, \theta = 2, \lambda = 1$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
10	4	0.2045	2.8444	0.1465	0.2791	0.1863	0.2625
		0.3471	- 3.5556	0.4932	- 0.3978	0.5141	- 0.5338
		0.6573	- 0.2142	1.4341	- 2.1568	1.3347	- 2.4407
		- 0.2089	0.9254	- 1.0738	1.3998	- 1.0351	2.7119
Sum		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 3. Coefficients of the BLUEs of μ and σ from the generalized linear exponential distribution using the second scheme when $\mu = 0$ and $\sigma = 1$

Scheme 2		$\alpha = 0.3, \theta = 1.5, \lambda = 0.5$		$\alpha = 0.5, \theta = 1, \lambda = 0.2$		$\alpha = 1, \theta = 2, \lambda = 1$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
15	6	0.0276	2.8256	- 0.1823	1.6358	- 0.1499	0.6413
		0.1280	- 3.6479	0.0248	0.7098	0.0362	0.2873
		0.2007	-1.0005	0.2662	- 0.3802	0.2516	- 0.1333
		0.3101	0.1374	0.7317	- 2.4431	0.6640	- 0.9257
		0.5875	0.6767	2.0024	- 7.9576	1.7695	- 3.0024
		- 0.2538	1.0088	- 1.8429	8.4343	- 1.5713	3.1329
Sum		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 4. Coefficients of the BLUEs of μ and σ from the generalized linear exponential distribution using the third scheme when $\mu = 0$ and $\sigma = 1$

Scheme 3		$\alpha = 0.3, \theta = 1.5, \lambda = 0.5$		$\alpha = 0.5, \theta = 1, \lambda = 0.2$		$\alpha = 1, \theta = 2, \lambda = 1$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
20	8	-0.0253	0.8201	-0.3221	1.9463	-0.2636	0.7253
		0.0329	0.7619	-0.1924	1.4055	-0.1544	0.5352
		0.0865	0.5714	-0.0387	0.7489	-0.0225	0.2953
		0.1283	0.3140	0.1268	0.0364	0.1195	0.0323
		0.1753	-0.0950	0.3579	-0.9568	0.3178	-0.3357
		0.2632	-0.9372	0.8414	-3.0104	0.7304	-1.0926
		0.7513	-4.7242	3.4048	-13.6974	2.9003	-4.9813
		-0.4122	3.2891	-3.1778	13.5275	-2.6277	4.8216
Sum		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 5. Coefficients of the BLUEs of μ and σ from the generalized linear exponential distribution using the fourth scheme when $\mu = 0$ and $\sigma = 1$

Scheme 4		$\alpha = 0.3, \theta = 1.5, \lambda = 0.5$		$\alpha = 0.5, \theta = 1, \lambda = 0.2$		$\alpha = 1, \theta = 2, \lambda = 1$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
25	10	-0.2769	2.2109	-3.0670	13.2478	-1.9005	3.5981
		-0.2129	2.1562	-2.4073	10.5060	-1.5065	2.9007
		-0.1371	1.8367	-1.7118	7.6015	-1.0764	2.1303
		-0.0556	1.3666	-0.9752	4.5203	-0.6134	1.2973
		0.0496	0.6604	-0.0491	0.6469	-0.0254	0.2385
		0.1590	-1.1464	1.0262	-3.8489	0.6551	-0.9869
		0.2782	-1.0746	2.3735	-9.4785	1.5011	-2.5093
		0.5420	-3.1523	5.5455	-22.7184	3.4835	-6.0713
		1.8458	-12.8109	22.4071	-92.9729	13.9259	-24.7759
		-1.1920	8.9533	-22.1417	92.4968	-13.4434	24.1789
Sum		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 6. MSE's of μ^* , σ^* , $\hat{\mu}$, and $\hat{\sigma}$ from the generalized linear exponential distribution

α	θ	λ	n	m	Scheme	$MSE(\mu^*)$	$MSE(\sigma^*)$	$MSE(\hat{\mu})$	$MSE(\hat{\sigma})$
0.3	1.5	0.5	10	4	1	0.6356	0.7120	2.0631×10^{-9}	2.0631×10^{-8}
			15	6	2	0.4736	0.5187	9.9817×10^{-10}	9.9817×10^{-9}
			20	8	3	0.3361	0.3103	2.0744×10^{-10}	3.3262×10^{-10}
			25	10	4	0.3282	0.1655	3.1821×10^{-11}	3.8367×10^{-11}
0.5	1	0.2	10	4	1	0.5102	0.4845	1.4478×10^{-9}	1.4477×10^{-8}
			15	6	2	0.4617	0.3943	1.1241×10^{-10}	1.1241×10^{-9}
			20	8	3	0.2689	0.1391	1.3831×10^{-11}	1.3831×10^{-10}
			25	10	4	0.0469	0.0176	1.4356×10^{-12}	1.4356×10^{-11}
1	2	1	10	4	1	0.2046	0.4588	3.0268×10^{-9}	1.0180×10^{-8}
			15	6	2	0.1943	0.2438	8.1524×10^{-10}	3.1242×10^{-9}
			20	8	3	0.1187	0.2201	1.9243×10^{-10}	7.4021×10^{-10}
			25	10	4	0.1168	0.1322	8.7645×10^{-11}	7.6305×10^{-11}

Table 7. Efficiencies and trace for trace-optimal censoring schemes: generalized linear exponential distribution (Line 1: Efficiency; Line 2: Trace) when $\lambda = 1.7$, $\alpha = 2$, and $\theta = 1$

n/m	2	3	4	5	6	7	$n-2$	$n-1$	n
10	2.4065	1.8097	1.5162	1.3472	1.2374	1.1659	1.1084	1.0598	1.0000
	0.0123	0.0184	0.0246	0.0311	0.0379	0.0452	0.0535	0.0635	0.078
15	4.0724	2.9215	2.3235	1.9881	1.7673	1.6059	1.0626	1.0384	1.0000
	0.0088	0.0131	0.0174	0.0218	0.0262	0.0307	0.0638	0.0724	0.0852
20	4.0724	2.9215	2.6544	1.9881	1.7673	1.6059	1.0626	1.0384	1.0000
	0.0069	0.0102	0.0136	0.0169	0.0202	0.0236	0.0702	0.078	0.0897
25	4.8596	3.4642	2.7567	2.326	2.0424	1.8385	1.0548	1.0354	1.0000
	0.0057	0.0084	0.0111	0.0138	0.0165	0.0192	0.0747	0.0818	0.0928

Table 8. Efficiencies and trace for trace-optimal censoring schemes: generalized linear exponential distribution (Line 1: Efficiency; Line 2: Trace) when $\lambda = 2$, $\alpha = 2$, and $\theta = 1$

n/m	2	3	4	5	6	7	$n-2$	$n-1$	n
10	2.5106	1.8591	1.5416	1.3703	1.2550	1.1731	1.1150	1.0648	1.0000
	0.0094	0.0142	0.0192	0.0243	0.0298	0.0358	0.0426	0.0509	0.0632
15	3.4029	2.4600	1.9850	1.7083	1.5270	1.4058	1.0802	1.0498	1.0000
	0.0067	0.01	0.0134	0.0168	0.0203	0.0239	0.0511	0.0582	0.0691
20	4.3076	3.0384	2.4368	2.0620	1.8193	1.6428	1.0674	1.0414	1.0000
	0.0052	0.0078	0.0103	0.0129	0.0155	0.0182	0.0563	0.0628	0.0728
25	5.1627	3.625	2.8928	2.4190	2.1023	1.8851	1.0583	1.0378	1.0000
	0.0043	0.0064	0.0084	0.0105	0.0127	0.0148	0.06	0.066	0.0754

Table 9. Coefficients of BLUE's for selected schemes for generalized linear exponential distribution when $\lambda = 1.7$, $\alpha = 2$, and $\theta = 1$

n	m	Scheme	Coefficient (μ^*)	Coefficient (σ^*)
10	2	$[0\ 8]^-$	1.9554, -0.9554	-17.3003, 17.3003
	2	$[8\ 0]^+$	1.1489, -0.1489	-2.6964, 2.6964
	3	$[0\ 0\ 7]^-$	0.4415, 1.9878, -1.4293	-0.5584, -15.2477, 15.8062
		$[0\ 7\ 0]^+$	0.5061, 0.7286, -0.2347	-0.4463, -2.2830, 2.7293
	4	$[0\ 0\ 0\ 6]^-$	0.0902, 0.5456, 2.3417, -1.9445	1.4611, -2.0516, -15.3424, 15.9330
		$[0\ 0\ 6\ 0]^+$	0.2638, 0.4243, 0.6524, -0.3405	0.2779, -0.7929, -2.3531, 2.8681
15	2	$[0\ 13]^-$	1.9727, -0.9727	-25.9006, 25.9006
		$[13\ 0]^+$	1.1001, -0.1001	-2.6656, 2.6656
	3	$[0\ 0\ 12]^-$	0.438, 2.0469, -1.4850	-0.8268, -23.4355, 24.2623
	3	$[0\ 12\ 0]^+$	0.5060, 0.6490, -0.1550	-0.4966, -2.1879, 2.6845
	4	$[0\ 0\ 0\ 11]^-$	0.0743, 0.5602, 2.4842, -2.1187	2.3179, -3.2348, -24.6986, 25.3855
	4	$[0\ 0\ 11\ 0]^+$	0.282, 0.4024, 0.5345, -0.2189	0.2541, -0.8416, -2.1853, 2.7728
20	2	$[0\ 18]^-$	1.9806, -0.9806	-34.4563, 34.4563
	2	$[18\ 0]^+$	1.0754, -0.0754	-2.6494, 2.6494
	3	$[0\ 0\ 17]^-$	0.4364, 2.0734, -1.5098	-1.0935, -31.5253, 32.6188
	3	$[0\ 17\ 0]^+$	0.5052, 0.6105, -0.1158	-0.5231, -2.1393, 2.6624
	4	$[0\ 0\ 0\ 16]^-$	0.0668, 0.5667, 2.5473, -2.1808	3.1633, -4.3968, -33.4106, 34.6441
	4	$[0\ 0\ 16\ 0]^+$	0.2916, 0.3897, 0.4801, -0.1613	0.2388, -0.8634, -2.1024, 2.7270

Table 10. Coefficients of BLUE's for selected schemes for generalized linear exponential distribution when $\lambda = 2$, $\alpha = 2$, and $\theta = 1$

n	m	Scheme	Coefficient (μ^*)	Coefficient (σ^*)
10	2	$[0\ 8]^-$	2.9761, -1.9761	-15.2425, 15.2425
	2	$[8\ 0]^+$	1.5144, -0.5144	-3.9678, 3.9678
	2	$[7\ 1]^-$	1.7477, -0.7477	-5.7671, 5.7671
	3	$[0\ 0\ 7]^-$	0.6237, 3.4173, -3.0410	-1.3375, -16.6374, 17.9749
	3	$[7\ 0\ 0]^+$	0.4641, 1.9041, -1.3681	-0.1858, -5.7629, 5.9487
	3	$[1\ 6\ 0]^+$	0.5066, 1.6054, -1.1120	-0.3737, -4.9900, 5.3637
	4	$[0\ 0\ 0\ 6]^-$	0.0995, 0.8865, 4.2312, -4.2172	0.9067, -3.0162, -19.0933, 21.2028
	4	$[6\ 0\ 0\ 0]^+$	-0.1388, 0.6246, 2.2710, -2.0345	0.5782, -1.2183, -6.8449, 7.4850
	4	$[0\ 6\ 0\ 0]^+$	0.2649, 0.4380, 1.7609, -1.4638	0.1628, -0.6218, -5.8734, 6.3324
15	2	$[0\ 13]^-$	2.9913, -1.9913	-18.6342, 18.6342
	2	$[13\ 0]^+$	1.3996, -0.3996	-3.7395, 3.7395
	2	$[12\ 1]^-$	1.5735, -0.5735	-5.3665, 5.3665
	3	$[0\ 0\ 12]^-$	0.6139, 3.5382, -3.1521	-1.5723, -21.0486, 22.6209
	3	$[12\ 0\ 0]^+$	0.4342, 1.8358, -1.2701	-0.0516, -5.8737, 5.9253
	3	$[2\ 0\ 10]^-$	0.5786, 3.3637, -2.9423	-1.2852, -19.2781, 20.5633
	4	$[0\ 0\ 0\ 11]^-$	0.0744, 0.9158, 4.5379, -4.5282	1.2465, -3.8602, -25.1188, 27.7325
	4	$[11\ 0\ 0\ 0]^+$	0.1183, 0.6105, 2.2337, -1.9625	0.6991, -1.2322, -7.0679, 7.6009
	4	$[9\ 0\ 2\ 0]^+$	0.1804, 0.6043, 1.5232, -1.3079	0.5313, -1.3779, -5.2410, 6.0875

Table 10. (Continued)

n	m	Scheme	Coefficient (μ^*)	Coefficient (σ^*)
20	2	$[0\ 18]^-$	2.9951, -1.9951	-21.4312, 21.4312
	2	$[18\ 0]^+$	1.3357, -0.3357	-3.6061, 3.6061
	2	$[16\ 2]^-$	1.6009, -0.6009	-6.4547, 6.4547
	3	$[0\ 0\ 17]^-$	0.6071, 3.5931, -3.2002	-1.7601, -24.6007, 26.3609
	3	$[17\ 0\ 0]^+$	0.4177, 1.7997, -1.2174	0.0345, -5.9683, 5.9339
	3	$[0\ 17\ 0]^+$	0.5836, 0.8796, -0.4632	-0.9572, -2.9755, 3.9327
	4	$[0\ 0\ 0\ 16]^-$	0.0603, 0.9279, 4.6813, -4.6695	1.5252, -4.5247, -29.8658, 32.8653
	4	$[16\ 0\ 0\ 0]^+$	0.1055, 0.6042, 2.2186, -1.9282	0.7810, -1.2491, -7.2362, 7.7043
	4	$[0\ 0\ 16\ 0]^+$	0.2998, 0.5195, 0.7802, -0.5995	0.0306, -1.3746, -2.9897, 4.3337

References

- [1] D. V. Lindley, Introduction to Probability and Statistics from a Bayesian Viewpoint, Camberidge, 1969.
- [2] N. Balakrishnan and R. Aggrawala, Progressive Censoring-Theory, Methods and Applications, Birkhauser, Boston, 2000.
- [3] N. Balakrishnan, A. Childs and B. Chandrasekar, An efficient computational method for moments of order statistics under progressive censoring, Statistics and Probability Letters 60(4) (2002), 359-365.
DOI: [https://doi.org/10.1016/S0167-7152\(02\)00267-5](https://doi.org/10.1016/S0167-7152(02)00267-5)
- [4] J. D. Gibbons and S. Chakraborti, Nonparametric Statistical Inference, Fourth Edition, Marcel Dekker, New York, 2003.
- [5] H. A. David and H. N. Nagaraja, Order Statistics, Third Edition, John Wiley & Sons, New York, 2003.
- [6] A. J. Fernandez, On estimating exponential parameters with general type II progressively censoring, Journal of Statistical Planning and Inference 121(1) (2004), 135-147.
DOI: [https://doi.org/10.1016/S0378-3758\(03\)00083-1](https://doi.org/10.1016/S0378-3758(03)00083-1)
- [7] N. Balakrishnan, Progressive censoring methodology: An appraisal, Test 16(2) (2007), 211-296.

- [8] A. Asgharzadeh, Point and interval estimation for a generalized logistic distribution under progressive type II censoring, *Communications in Statistics-Theory and Methods* 35(9) (2006), 1685-1702.
DOI: <https://doi.org/10.1080/03610920600683713>
- [9] N. Balakrishnan and C. R. Rao, Some efficiency properties of best linear unbiased estimators, *Journal of Statistical Planning and Inference* 113(2) (2003), 551-555.
DOI: [https://doi.org/10.1016/S0378-3758\(02\)00107-6](https://doi.org/10.1016/S0378-3758(02)00107-6)
- [10] M. A. W. Mahmoud and M. M. Mohie El-Din, The inverted linear exponential distribution as a life time distribution, *The Egyptian Statistical Journal* 50(1) (2006), 12-20.
- [11] L. Bordes, Non-parametric estimation under progressive censoring, *Journal of Statistical Planning and Inference* 119(1) (2004), 171-189.
DOI: [https://doi.org/10.1016/S0378-3758\(02\)00414-7](https://doi.org/10.1016/S0378-3758(02)00414-7)
- [12] M. Z. Raqab, A. Asgharzadeh and R. Valiollahi, Prediction for Pareto distribution based on progressively type-II censored samples, *Computational Statistics & Data Analysis* 54(7) (2010), 1732-1743.
DOI: <https://doi.org/10.1016/j.csda.2010.02.005>
- [13] M. A. W. Mahmoud, M. Moshref, N. M. Yhiea and N. M. Mohamed, Progressively censored data from the Weibull gamma distribution moments and estimation, *Journal of Statistics Applications & Probability* 3(1) (2014), 45-60.
DOI: <http://dx.doi.org/10.18576/jsap/030105>
- [14] M. Burkschat, E. Cramer and U. Kamps, Optimality criteria and optimal schemes in progressive censoring, *Communications in Statistics-Theory and Methods* 36(7) (2007), 1419-1431.
DOI: <https://doi.org/10.1080/03610920601077196>