GENERALIZED ORDER STATISTICS FROM KUMARASWAMY-BURR III DISTRIBUTION AND RELATED INFERENCE

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Abstract

In this paper, some recurrence relations for single and product moments of generalized order statistics arising from the Kumaraswamy-Burr III distribution are obtained. Further, various deductions and related results are discussed and identified some of these recurrence relations and other properties of generalized order statistics which characterize this distribution.

1. Introduction

The concept of generalized order statistics (*gos*) was given by Kamps [13], which is given as below:

Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with absolutely continuous

2010 Mathematics Subject Classification: 62G30, 62E10.

Keywords and phrases: order statistics, record values, generalized order statistics, Kumaraswamy-Burr III distribution, single moments, product moments, recurrence relations, conditional expectation and characterization.

Received December 26, 2017

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distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in \mathbb{N}$, $n \geq 2$, $k \geq 1$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathfrak{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be the parameters such that $\gamma_r = k + n - r + M_r \geq 1$, for all $r \in \{1, 2, \dots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$ are called *gos* if their joint *pdf* is given by

$$f_{X(1,n,\tilde{m},k), \dots, X(n,n,\tilde{m},k)}(x_1, x_2, \dots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i) \right) [\bar{F}(x_n)]^{k-1} f(x_n) \quad (1.1)$$

on the cone $F^{-1}(0) < x_1 \leq \dots \leq x_n < F^{-1}(1)$, where $\bar{F}(x) = 1 - F(x)$.

Here we may consider two cases:

Case. (i) $\gamma_i = \gamma_j$, i.e., $m_1 = m_2 = \dots = m_{n-1} = m$.

Case. (ii) $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \dots, n-1$.

For case I, *gos* will be denoted as $X(r, n, m, k)$ and its *pdf* is given by (Kamps [13])

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)), \alpha \leq x \leq \beta, \quad (1.2)$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k), X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad \alpha \leq x < y \leq \beta, \quad (1.3)$$

where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1}, & m \neq -1, \\ -\ln(1-x), & m = -1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1].$$

For case II, the *pdf* of $X(r, n, \tilde{m}, k)$ is (Kamps and Cramer [14])

$$f_{X(r, n, \tilde{m}, k)}(x) = C_{r-1} f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i - 1}, \quad (1.4)$$

and the joint *pdf* of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$, is

$$f_{X(r, n, \tilde{m}, k)X(s, n, \tilde{m}, k)}(x, y) = C_{s-1} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \\ \times \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \quad (1.5)$$

where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + n - i + M_i,$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^s \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \leq i \leq s \leq n.$$

It may be noted that $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$

$$a_i(r) = \frac{(-1)^{r-i}}{(r-1)!(m+1)^{r-1}} \binom{r-1}{r-i},$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(s-r-1)!(m+1)^{s-r-1}} \binom{s-r-1}{s-i}.$$

Consequently, the *pdf* of $X(r, n, \tilde{m}, k)$ and the joint *pdf* of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ reduce to the *pdf* of $X(r, n, m, k)$ and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, respectively.

Several models of ordered random variables such as order statistics and record value can be seen as special cases of *gos*. If $m = 0$ and $k = 1$, then $X(r, n, m, k)$ reduces to the r -th order statistic $X_{r:n}$ (David and Nagaraja [8]). If $m = -1$ and $k = 1$, then $X(r, n, m, k)$ is the r -th record values from an infinite sequence of *iid* *rv*'s (Ahsanullah [2]). Other special cases are k -th record values ($m = -1$, Dziubdziela and Kopociński [9]) and order statistics with non-integral sample size ($m = 0$, $k = \alpha - n + 1$, $\alpha = n$, Stigler [24], Rohatgi and Saleh [23]).

Many authors utilized the concept of *gos* in their studies. References may be made to Kamps and Gather [15], Keseling [16], Cramer and Kamps [6], Ahsanullah [3], Habibullah and Ahsanullah [10], Ahmad and Fawzy [1], Bieniek and Szynal [5], Cramer et al. [7], Khan and Alzaid [17], Khan et al. [18], Khan et al. [21], Khan and Zia [20], Khan and Khan [19] among others. In this paper, we mainly focus on the study of *gos* arising from the Kumaraswamy-Burr III distribution.

A random variable X is said to have Kumaraswamy-Burr III distribution (Huang and Oluyede [11]) if its *pdf* is of the form

$$f(x) = \alpha\beta\lambda x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}(1-(1+x^{-\beta})^{-\alpha})^{\lambda-1}, \quad x > 0, \quad \alpha, \beta, \lambda > 0, \quad (1.6)$$

with the corresponding df

$$F(x) = 1 - [1 - (1 + x^{-\beta})^{-\alpha}]^\lambda, \quad x > 0, \quad \alpha, \beta, \lambda > 0. \quad (1.7)$$

It is easy to see that

$$\alpha\beta\lambda\bar{F}(x) = \left\{ \alpha x + \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} x^{\beta(1-u)+1} \right\} f(x). \quad (1.8)$$

2. Relations for Single Moments

Theorem 2.1. *For the distribution given in (1.7) and $n \in N$, $\tilde{m} \in \mathfrak{R}$, $2 \leq r \leq n$, $j = 0, 1, 2, \dots$*

$$\begin{aligned} \left(1 - \frac{j}{\beta\lambda\gamma_r}\right) E[X^j(r, n, \tilde{m}, k)] &= E[X^j(r-1, n, \tilde{m}, k)] \\ &+ \frac{j}{\alpha\beta\lambda\gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^{j+\beta(1-u)}(r, n, \tilde{m}, k)]. \end{aligned} \quad (2.1)$$

Proof. For $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \dots$, we have Athar and Islam [4]

$$\begin{aligned} E[\xi(X(r, n, \tilde{m}, k))] &- E[\xi(X(r-1, n, \tilde{m}, k))] \\ &= C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx, \end{aligned} \quad (2.2)$$

where $\xi(x)$ is a measurable function of x and is differentiable.

Let $\xi(x) = x^j$, then

$$E[X^j(r, n, \tilde{m}, k)] - E[X^j(r-1, n, \tilde{m}, k)] = jC_{r-2} \int_0^{\infty} x^{j-1} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx. \quad (2.3)$$

On using (1.6) in (2.3), we find that

$$\begin{aligned} & E[X^j(r, n, \tilde{m}, k)] - E[X^j(r-1, n, \tilde{m}, k)] \\ &= \frac{j}{\alpha \beta \lambda \gamma_r} \left(\alpha C_{r-1} \int_0^\infty x^j f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} dx \right) \\ &+ \frac{j}{\alpha \beta \lambda \gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} \left(C_{r-1} \int_0^\infty x^{j+\beta(1-u)} f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} dx \right), \end{aligned}$$

which lead to (2.1).

Corollary 2.1. For $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$, the recurrence relation for single moments of gos from Kumaraswamy-Burr III distribution has the form

$$\begin{aligned} & \left(1 - \frac{j}{\beta \lambda \gamma_r} \right) E[X^j(r, n, m, k)] \\ &= E[X^j(r-1, n, m, k)] \\ &+ \frac{j}{\alpha \beta \lambda \gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^{j+\beta(1-u)}(r, n, m, k)]. \end{aligned} \quad (2.4)$$

Remark 2.1. Recurrence relation for single moments of order statistics (at $m = 0, k = 1$) from Kumaraswamy-Burr III distribution is

$$\left(1 - \frac{j}{\beta \lambda (n-r+1)} \right) E(X_{r:n}^j) = E(X_{r-1:n}^j) + \frac{j}{\alpha \beta \lambda (n-r+1)} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E(X_{r:n}^{j+\beta(1-u)}).$$

Remark 2.2. The recurrence relation for single moments of k -th record values will be

$$\begin{aligned} & \left(1 - \frac{j}{\beta \lambda k} \right) E[X^j(r, n, -1, k)] = E[X^j(r-1, n, -1, k)] \\ &+ \frac{j}{\alpha \beta \lambda k} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^{j+\beta(1-u)}(r, n, -1, k)], \end{aligned}$$

or

$$\left(1 - \frac{j}{\beta\lambda k}\right) E(X_{U_r}^{(k)})^j = E(X_{U_{r-1}}^{(k)})^j + \frac{j}{\alpha\beta\lambda k} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E(X_{U_r}^{(k)})^{j+\beta(1-u)},$$

where $X_{U_r}^{(k)}$, $r = 1, 2, \dots$ is r -th k records, which verify the result obtained by Kumar et al. [22].

For upper records ($k = 1$), the recurrence relation for single moments from Kumaraswamy-Burr III distribution is deduced as obtained by Kumar et al. [22].

3. Relations for Product Moments

Theorem 3.1. *For distribution given in (1.7) and for $k, n \in N$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathfrak{R}$, $1 \leq r < s \leq n-1$ and $i, j \geq 0$.*

$$\begin{aligned} & \left(1 - \frac{j}{\beta\lambda\gamma_s}\right) E[X^i(r, n, \tilde{m}, k) X^j(s, n, \tilde{m}, k)] \\ &= E[X^i(r, n, \tilde{m}, k) X^j(s-1, n, \tilde{m}, k)] \\ &+ \frac{j}{\alpha\beta\lambda\gamma_s} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^i(r, n, \tilde{m}, k) X^{j+\beta(1-u)}(s, n, \tilde{m}, k)]. \end{aligned} \quad (3.1)$$

Proof. In view of Athar and Islam [4], note that

$$\begin{aligned} & E[\xi\{X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)\}] - E[\xi\{X(r, n, \tilde{m}, k), X(s-1, n, \tilde{m}, k)\}] \\ &= C_{s-2} \int_0^\infty \int_x^\infty \xi'(x, y) \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i}\right) \frac{f(x)}{\bar{F}(x)} dy dx, \end{aligned} \quad (3.2)$$

Let

$\xi(x, y) = x^i y^j$, then

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= jC_{s-2} \int_0^\infty \int_x^\infty x^i y^{j-1} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x)}{\bar{F}(x)} dy dx. \end{aligned} \quad (3.3)$$

On substituting (1.8) in (3.3), we get

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= \frac{j}{\beta\lambda\gamma_s} \left[C_{s-1} \int_0^\infty \int_x^\infty x^i y^j \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)} dy dx \right. \\ & \quad + \frac{j}{\alpha\beta\lambda\gamma_s} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} C_{s-1} \int_0^\infty \int_x^\infty x^i y^{j+\beta(1-u)} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \\ & \quad \left. \times \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)} dy dx \right], \end{aligned} \quad (3.4)$$

which after simplification yields (3.1).

At $i = 0$ in (3.1), the recurrence relation for product moments reduces to relation for single moments as obtained in (2.1).

Corollary 3.1. *For $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$, the recurrence relation for product moments of gos from Kumaraswamy-Burr III distribution has the form*

$$\begin{aligned} & \left(1 - \frac{j}{\beta\lambda\gamma_s} \right) E[X^i(r, n, m, k)X^j(s, n, m, k)] \\ &= E[X^i(r, n, m, k)X^j(s-1, n, m, k)] \\ & \quad + \frac{j}{\alpha\beta\lambda\gamma_s} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^{j+\beta(1-u)}(r, n, m, k)]. \end{aligned} \quad (3.5)$$

Remark 3.1. Recurrence relation for product moments of order statistics (at $m = 0$, $k = 1$) from Kumaraswamy-Burr III distribution is

$$\begin{aligned} & \left(1 - \frac{j}{\beta\lambda(n-s+1)}\right) E(X_{r:n}^i X_{s:n}^j) \\ &= E(X_{r:n}^i X_{s-1:n}^j) + \frac{j}{\alpha\beta\lambda(n-s+1)} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E(X_{r:n}^i X_{s:n}^{j+\beta(1-u)}). \end{aligned}$$

Remark 3.2. Setting $m = -1$ and $k \geq 1$ in (3.1), we deduce the recurrence relation for product moments of k -th upper record values from the Kumaraswamy-Burr III distribution, established by Kumar et al. [22]).

4. Characterizations

Let $X(r, n, m, k)$, $r = 1, 2, \dots, n$ be gos, then the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \leq r < s \leq n$, is

$$\begin{aligned} & f_{X(s, n, m, k) | X(r, n, m, k)}(y | x) \\ &= \frac{C_{s-1}}{(s-r-1)! C_{r-1}} [\bar{F}(x)]^{m-\gamma_r+1} \\ & \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad x < y. \end{aligned} \quad (4.1)$$

Theorem 4.1. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \leq F(x) \leq 1$ for all $x > 0$, then

$$\begin{aligned} & E[\xi\{X(s, n, m, k)\} | X(l, n, m, k) = x] \\ &= [1 - (1 + x^{-\beta})^{-\alpha}]^\lambda \prod_{j=1}^{s-l} \left(\frac{\gamma_{l+j}}{\gamma_{l+j} + 1} \right), \quad l = r, r+1, \end{aligned} \quad (4.2)$$

if and only if

$$\bar{F}(x) = [1 - (1 + x^{-\beta})^{-\alpha}]^\lambda, \quad x > 0, \alpha, \beta, \lambda > 0,$$

where

$$\xi(y) = [1 - (1 + y^{-\beta})^{-\alpha}]^\lambda.$$

Proof. From (4.1) for $s > r + 1$, we have

$$\begin{aligned} & E[\xi\{X(s, n, m, k)\} \mid X(r, n, m, k) = x] \\ &= \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \\ & \times \int_x^\infty [1 - (1 + y^{-\beta})^{-\alpha}]^\lambda \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_{s-1}} \left\{ 1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right\}^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \end{aligned} \quad (4.3)$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{[1 - (1 + y^{-\beta})^{-\alpha}]^\lambda}{[1 - (1 + x^{-\beta})^{-\alpha}]^\lambda}$ from (1.5) in (4.3), we get

$$\begin{aligned} E[\xi\{X(s, n, m, k)\} \mid X(r, n, m, k) = x] &= \frac{C_{s-1}[1 - (1 + x^{-\beta})^{-\alpha}]^\lambda}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \\ & \times \int_0^1 u^{\gamma_s} (1 - u^{m+1})^{s-r-1} du. \end{aligned} \quad (4.4)$$

Again by setting $t = u^{m+1}$ in (4.4), we find that

$$\begin{aligned} & E[\xi\{X(s, n, m, k)\} \mid X(r, n, m, k) = x] \\ &= \frac{C_{s-1}[1 - (1 + x^{-\beta})^{-\alpha}]^\lambda}{(s-r-1)! C_{r-1}(m+1)^{s-r}} \int_0^1 t^{\frac{\gamma_s+1}{m+1}-1} (1-t)^{s-r-1} dt \\ &= \frac{C_{s-1}[1 - (1 + x^{-\beta})^{-\alpha}]^\lambda}{C_{r-1}(m+1)^{s-r}} \frac{\Gamma\left(\frac{k+1}{m+1} + n - s\right)}{\Gamma\left(\frac{k+1}{m+1} + n - r\right)}. \\ &= \frac{C_{s-1}[1 - (1 + x^{-\beta})^{-\alpha}]^\lambda}{C_{r-1}} \frac{1}{\prod_{j=1}^{s-r} (\gamma_{r+j} + 1)}, \end{aligned}$$

where

$$\frac{C_{s-1}}{C_{r-1}} = \prod_{j=1}^{s-r} \gamma_{r+j},$$

and hence the necessary part.

To prove the sufficient part, we have from (4.1) and (4.2)

$$\begin{aligned} & \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \int_x^\infty [1 - (1 + y^{-\beta})^{-\alpha}]^\lambda [(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1} \\ & \times [\bar{F}(y)]^{\gamma_{s-1}} f(y) dy = g_{s|r}(x) [\bar{F}(x)]^{\gamma_{r+1}}, \end{aligned} \quad (4.5)$$

where

$$g_{s|r}(x) = [1 - (1 + x^{-\beta})^{-\alpha}]^\lambda \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right).$$

Differentiating (4.5) both sides with respect to x , we get

$$\begin{aligned} & - \frac{C_{s-1} [\bar{F}(x)]^m f(x)}{(s-r-2)! C_{r-1} (m+1)^{s-r-2}} \int_x^\infty [1 - (1 + y^{-\beta})^{-\alpha}]^\lambda [(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2} \\ & \times [\bar{F}(y)]^{\gamma_{s-1}} f(y) dy = g'_{s|r}(x) [\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} g_{s|r}(x) [\bar{F}(x)]^{\gamma_{r+1}-1} f(x), \end{aligned}$$

or

$$\begin{aligned} & - \gamma_{r+1} g_{s|r+1}(x) [\bar{F}(x)]^{\gamma_{r+2}+m} f(x) \\ & = g'_{s|r}(x) [\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} g_{s|r}(x) [\bar{F}(x)]^{\gamma_{r+1}-1} f(x), \end{aligned}$$

where

$$g'_{s|r}(x) = -\alpha\beta\lambda x^{-\beta-1} (1 + x^{-\beta})^{-\alpha-1} [1 - (1 + x^{-\beta})^{-\alpha}]^{\lambda-1} \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right);$$

$$g_{s|r+1}(x) = [1 - (1 + x^{-\beta})^{-\alpha}]^\lambda \left(\frac{\gamma_{r+1} + 1}{\gamma_{r+1}} \right) \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right).$$

Therefore,

$$\begin{aligned} \frac{f(x)}{\bar{F}(x)} &= -\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} && \text{(Khan et al. [18])} \\ &= \frac{\alpha\beta\lambda x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}}{[1 - (1+x^{-\beta})^{-\alpha}]}. && (4.6) \end{aligned}$$

Integrating (4.6) on both the sides with respect to x between $(0, y)$, the sufficiency part is proved.

Remark 4.1. (i) As $m \rightarrow -1, k = 1$ in (4.2), we get the characterization results from the Kumaraswamy-Burr III distribution based on upper record values as obtained by Kumar et al. [22].

(ii) Setting $m = 0, k = 1$ in (4.2), we obtain the characterizing results of the Kumaraswamy-Burr III distribution based on order statistics.

Following theorem contains characterization of this distribution by a recurrence relation for the single moments of gos .

Theorem 4.2. For a positive integer k and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.4) is that

$$\begin{aligned} \left(1 - \frac{j}{\beta\lambda\gamma_r}\right)E[X^j(r, n, m, k)] &= E[X^j(r-1, n, m, k)] \\ &+ \frac{j}{\alpha\beta\lambda\gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} E[X^{j+\beta(1-u)}(r, n, m, k)]. \end{aligned} \quad (4.7)$$

Proof. The necessary part follows from (2.1). On the other hand, if the relation in (4.7) is satisfied, then on using (1.2), we have

$$\begin{aligned}
& \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx \\
&= \frac{C_{r-2}}{(r-2)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_{r-1}-1} f(x) g_m^{r-2}(F(x)) dx + \frac{j C_{r-1}}{(r-1)! \beta \lambda \gamma_r} \\
&\quad \times \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx \\
&\quad + \frac{j C_{r-1}}{(r-1)! \alpha \beta \lambda \gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} \int_0^\infty x^{j+\beta(1-u)} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx; \\
& \frac{C_{r-1}}{(r-1)! \gamma_r} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-2}(F(x)) \{ \gamma_r g_m(F(x)) - (r-1) [\bar{F}(x)]^{m+1} \} dx \\
&= \frac{j C_{r-1}}{(r-1)! \beta \lambda \gamma_r} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx + \frac{j C_{r-1}}{(r-1)! \alpha \beta \lambda \gamma_r} \\
&\quad \times \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} \int_0^\infty x^{j+\beta(1-u)} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx.
\end{aligned}$$

Let

$$h(x) = -\frac{1}{\gamma_r} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)). \quad (4.8)$$

Differentiating both the sides of (4.8), we get

$$h'(x) = \frac{1}{\gamma_r} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-2}(F(x)) \{ \gamma_r g_m(F(x)) - (r-1) [\bar{F}(x)]^{m+1} \}.$$

Thus,

$$\begin{aligned} \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j h'(x) dx &= \frac{jC_{r-1}}{(r-1)! \beta \lambda \gamma_r} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx \\ &+ \frac{jC_{r-1}}{(r-1)! \alpha \beta \lambda \gamma_r} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} \int_0^\infty x^{j+\beta(1-u)} \\ &\times [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx. \end{aligned} \quad (4.9)$$

Integrating left hand side in (4.9) by parts and using the value of $h(x)$ from (4.8), we find that

$$\begin{aligned} \frac{jC_{r-1}}{(r-1)! \gamma_r} \int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x)) \left\{ \bar{F}(x) - \frac{x}{\beta \lambda} f(x) \right. \\ \left. - \frac{1}{\alpha \beta \lambda} \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} x^{\beta(1-u)+1} f(x) \right\} dx = 0. \end{aligned} \quad (4.10)$$

Applying the extension of Müntz-Szász Theorem (see, for example, Hwang and Lin [12]), to (4.10), we get

$$\alpha \beta \lambda \bar{F}(x) = \left\{ \alpha x + \sum_{u=2}^{\alpha+1} \binom{\alpha+1}{u} x^{\beta(1-u)+1} \right\} f(x),$$

which proves that $f(x)$ has the form as in (1.8).

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