TESTING EXPONENTIALITY AGAINST $RNBUE_{mgf}$ CLASS OF LIFE DISTRIBUTION

L. S. DIAB

Department of Mathematics

Faculty of Science (Girls Branch)

AL-Azhar University

Nasr-city, 11884

Egypt

e-mail: lamiaa_deyab@yahoo.com

Abstract

The main objective of this study is to derive U-statistic test based on expected value method for testing exponentiality against renewal new better than used in expectation in moment generating function order ($RNBUE_{mgf}$). The critical values of this test are calculated and tabulated for sample size $n = 5(5)50$. The Pitman asymptotic efficiency (PAE), the Pitman asymptotic relative efficiency (PARE) are discussed and the power estimates of this test are also simulated for some commonly used distributions in reliability. The problem in case of right censored data is also handled and the power estimates of this test are also simulated for censored data for some commonly used distributions in reliability. Finally, real examples are presented to illustrate the application of this test statistic in reliability analysis.

2010 Mathematics Subject Classification: 62N01, 62N03, 62N05.

Keywords and phrases: renewal new better than used in expectation in moment generating function ($RNBUE_{mgf}$), U-statistic, life testing, efficiency, relative efficiency, Monte Carlo method, power and censored data.

Received October 5, 2016

1. Introduction

Because the accurate distribution of the life of an element or a system is often unavailable practically, nonparametric ageing properties are quite useful for modelling ageing or wear-out processes, and for constructing maintenance policies. Such ageing classes are derived via several notions of comparison between random variables. The most commonly used comparison include Muller and Stoyan [23].

In reliability theory, ageing life is usually characterized by a nonnegative random variable $X \geq 0$ with cumulative distribution function F and survival function $\bar{F} = 1 - F$. For any random variable X , let

$$X_t = [X - t | X > t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t , given that the device has survived up to time t . Its survival function is (see, for instance, Deshpande et al. [9])

$$\bar{F}_t(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}, \quad \bar{F}(t) > 0,$$

where $\bar{F}(x)$ is the survival function of X . It is well-known fact that when

$\bar{F}$ is an exponential distribution, then $X_t \stackrel{st}{=} X$ or $\bar{F}_t(x) = \bar{F}(x)$.

Comparing X and X_t in various forms and types create classes of ageing useful in many biomedical, engineering and statistical studies, cf. Barlow

and Proschan [7]. It is well known that the relation $X_t \stackrel{st}{\leq} X$ or $\bar{F}_t(x) \stackrel{st}{\leq} \bar{F}(x)$ defines the class of new better than used (NBU). On the other hand, the relation $E(X_t) \leq E(X) = \mu$ defines the class of new better than used in expectation (NBUE). Another notion associated with X is the weak limit

of X_t as $t \rightarrow \infty$. It is well known that, cf. Ross [28], X_t converges weakly to a nonnegative random variable $\tilde{X}$ with survival function

$$\bar{W}_F(x) = \frac{1}{\mu} \bar{v}(x), \text{ where } \bar{v}(x) = \int_x^\infty \bar{F}(u) du, \quad x \geq 0.$$

Define $\tilde{X}_t$ to be the random residual life of $\tilde{X}$ at age t . Thus, the survival function of $\tilde{X}_t$ is given by

$$\bar{W}_{F,t}(x) = \frac{\bar{W}_F(x+t)}{\bar{W}_F(t)}, \quad x, t \geq 0.$$

From the above discussion, we see that there are four random quantities related to life and these are the life itself X , the random residual life X_t , the equilibrium life $\tilde{X}$, and the residual equilibrium life $\tilde{X}_t$. It is also well known that stochastic or in moment comparisons between X and X_t define two of the most applicable ageing classes, namely, the NBU and the NBUE. These classes are useful to characterize ageing as well as in replacement policies. Hence, it would be of interest to compare a life X to its equilibrium form $\tilde{X}$ or to its residual equilibrium form $\tilde{X}_t$ or to compare the equilibrium life $\tilde{X}$ to the residual life X_t . This is precisely what we do in the current investigation. These comparisons produce new NBU type classes including “new better than renewal of used” (NBRU), “renewal new is better than used” (RNBUE), and (RNBRUE) “renewal new is better than renewal used” when comparing stochastically and comparing the residual equilibrium life at a certain age and its equilibrium (stationary) life in expectation, or similarly NBRUE and RNBUE when comparing in the mean. Other comparisons are also possible and some are addressed here. Some of the classes we discuss have been also developed by other authors including, Bhattacharjee et al. [8], Franco et al. [14], Li et al. [21], and Shaked and Shanthikumar [29]. Most of these authors address probabilistic properties of the ageing classes they study.

Abouammoh et al. [1] introduced the NRBUE, RNBUE, NRBUE, HNRBUE classes of life distributions and studied the relation between them. Abouammoh and Khalique [2] investigated a test statistic of NRBUE based on total time of test (TTT)-transform empirically. Diab et al. [10] investigated the test statistic for NRBUE based on U-statistic. Diab et al. [11] discussed a test statistic of RNBUE by using U-test. Testing exponentiality against NRBUE and NRBUE were studied by Abu-Youssef [5, 6]. While testing against RNBUE investigated by Mugdadi and Ahmad [24]. Mahmoud et al. [22] studied two test statistics for NRBUE and RNBUE classes of life distributions as alternatives based on the moment inequalities. Abdel-Aziz [4] investigated test statistics for testing exponentiality against RNBUE class of life distributions as alternatives based on the moment inequalities.

Finally, Diab [13] studied a new approach to moments inequalities for NRBUE and RNBUE classes of life distributions.

Precisely we have the following definitions:

Definition 1.1. (i) X is said to be renewal new is better than used (RNBUE) if

$$X_t \stackrel{st}{\leq} \tilde{X}, \text{ i.e., } \overline{F}_t(x) \leq \overline{F}_F(x), x \geq 0.$$

(ii) X is said to be renewal new is better than used in expectation (RNBUE) if

$$E(X_t) \leq E(\tilde{X}), \text{ i.e., } 2\mu \int_x^\infty \overline{F}(u) du \leq \mu_{(2)} \overline{F}(x),$$

where μ is the mean life and $\mu_{(2)}$ is the second moment, both assumed finite.

(iii) X is said to be renewal new is better than renewal used (RNBUR) if

$$\tilde{X}_t \stackrel{st}{\leq} \tilde{X}, \text{ i.e., } \overline{W}(x+t) \leq \overline{W}_F(x) \overline{W}_F(t), \quad x, t \geq 0,$$

$$\text{i.e., } \mu \int_{x+t}^{\infty} \overline{F}(u) du \leq \int_x^{\infty} \overline{F}(u) du \int_t^{\infty} \overline{F}(u) du, \quad x, t \geq 0.$$

(iv) X is said to be renewal new is better than renewal used in expectation (RNBRE) if

$$E(\tilde{X}_t) \stackrel{st}{\leq} E(\tilde{X}), \text{ i.e., } 2\mu \int_x^{\infty} \int_u^{\infty} \overline{F}(w) dw du \leq \mu_{(2)} \int_x^{\infty} \overline{F}(u) du.$$

Based on moment generating function, some stochastic order have been proposed in literature, cf. Diab [12].

In the current work, we focus on the formal definition of RNBUE_{mgf} class is as follows:

Definition 1.2. A random variable X or its distribution F is said to have renewal new better than used in expectation in moment generating function order denoted by RNBUE_{mgf} property, if $E(X_t) \leq_{mgf} E(\tilde{X})$.

This definition means that, Y is RNBUE_{mgf} , iff

$$2\mu \int_0^{\infty} e^{\lambda x} \int_x^{\infty} \overline{F}(u) du dx \leq \mu_{(2)} \int_0^{\infty} e^{\lambda x} \overline{F}(x) dx. \quad (1.2)$$

The corresponding dual class of life distribution is new renewal worse than used parent property, denoted by RNWUE_{mgf} , and is define by reversing the inequality sign of (1.2).

The rest of the article is structured as follow. In Section 2, I driven the U-statistic test based on expected value method for testing exponentiality against alternative that H_1 is $RNBUE_{mgf}$ and not exponential. This test statistic is simple to calculate and has exceptionally privet the Pitman asymptotic efficiency and the Pitman asymptotic relative efficiency with respect to Kango [17] and Mugdadi and Ahmad [24]. In Section 3, Monte Carlo null distribution critical points are simulated for sample size $n = 5(5)50$. The power estimates is also calculated and some examples using real data are given. Finally, in Section 4, we focus on the problem of dealing with right-censored data and selected critical values are tabulated with few applications and the power estimates is also simulated for censored data.

2. Testing Against $RNBUE_{mgf}$ Class for Non Censored Data

Consider the problem for testing the null $H_0 : \bar{F}(x) = e^{-x}, x \geq 0$, versus the alternative $H_1 : \bar{F}$ is $RNBUE_{mgf}$ and not exponential. This test based on a sample $X_1, X_2, \dots, X_n$ from the measure of departure from exponentiality is introduced in the following theorem:

Theorem 2.1. *If F is $RNBUE_{mgf}$, then the measure of deviation from the null hypothesis H_0 is $\delta(\lambda) > 0$, where*

$$\delta(\lambda) = (\lambda\mu_{(2)} - 2\mu) \int_0^{\infty} (\phi(\lambda) dF(x)) + (2\mu + 2\lambda\mu^2 - \lambda\mu_{(2)}) \int_0^{\infty} dF(x), \quad (2.1)$$

with $\phi(\lambda) = \int_0^{\infty} e^{\lambda x} dF(x)$ and $\mu = \int_0^{\infty} \bar{F}(x) dx$.

Proof. Obviously from (1.2) F is $RNBUE_{mgf}$

Then, we use the following measure of departure from H_0 .

$$\begin{aligned}
\delta(\lambda) &= E \left[\mu_{(2)} \int_0^\infty e^{\lambda x} \bar{F}(x) dx - 2\mu \int_0^\infty e^{\lambda x} \int_x^\infty \bar{F}(u) du dx \right] \geq 0 \\
&= \int_0^\infty \left[\mu_{(2)} \int_0^\infty e^{\lambda x} \bar{F}(x) dx - 2\mu \int_0^\infty e^{\lambda x} \int_x^\infty \bar{F}(u) du dx \right] dF(x). \quad (2.2)
\end{aligned}$$

But,

$$\int_0^\infty \left[\mu_{(2)} \int_0^\infty e^{\lambda x} \bar{F}(x) dx \right] dF(x) = \int_0^\infty \left[\frac{\mu_{(2)}}{\lambda} (\phi(\lambda) - 1) \right] dF(x), \quad (2.3)$$

and

$$\int_0^\infty \left[2\mu \int_0^\infty e^{\lambda x} \int_x^\infty \bar{F}(u) du dx \right] dF(x) = \int_0^\infty \left[\frac{2\mu}{\lambda^2} (\phi(\lambda) - \frac{2\mu^2}{\lambda}) \right] dF(x). \quad (2.4)$$

From (2.3) and (2.4), the result in (2.1) is now immediate.

Note that under $H_0 : \delta(\lambda) = 0$, while under $H_1 : \delta(\lambda) > 0$.

To estimate $\delta(\lambda)$, let $X_1, X_2, \dots, X_n$ be a random sample from F . So the empirical estimate form of $\delta(\lambda)$ in (2.1) is as follows:

$$\hat{\delta}(\lambda) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n [\lambda X_i^2 e^{\lambda X_j} - \lambda X_i^2 - 2X_i e^{\lambda X_j} + 2X_i + 2\lambda X_i X_j]. \quad (2.5)$$

To find the limiting distribution of $\hat{\delta}(\lambda)$, we resort the U-statistics theory (cf. Lee [20]).

Set,

$$\varphi_\lambda(X_1, X_2, X_3) = \lambda X_1^2 e^{\lambda X_2} - \lambda X_1^2 - 2X_1 e^{\lambda X_2} + 2X_1 + 2\lambda X_1 X_2, \quad (2.6)$$

and define, the symmetric kernel

$$\Psi(X_1, X_2, X_3) = \frac{1}{3!} \sum_R \varphi(X_{i1}, X_{i2}, X_{i3}),$$

where the summation is over all arrangements of X_{i1}, X_{i2}, X_{i3} , then $\hat{\delta}(\lambda)$ in (2.5) is corresponding to the U_n -statistic given by

$$U_n = \binom{n}{3}^{-1} \sum_{i < j < k} \varphi(X_i, X_j, X_k).$$

Thus,

$$\varphi_{1,\lambda}(X_1) = E[\varphi_\lambda(X_1, X_2, X_3)|X_1] = \frac{\lambda X_1^2 - 2X_1}{(1-\lambda)} - \lambda X_1^2 + 2\lambda X_1 + 2X_1,$$

$$\varphi_{2,\lambda}(X_1) = E[\varphi_\lambda(X_2, X_1, X_3)|X_1] = e^{\lambda X_1}(2\lambda - 2) + 2\lambda(X_1 - 1) + 2,$$

and

$$\varphi_{3,\lambda}(X_1) = E[\varphi_\lambda(X_3, X_1, X_2)|X_1] = 0.$$

Thus, set

$$\Psi_\lambda(X_1) = \varphi_{1,\lambda}(X_1) + \varphi_{2,\lambda}(X_1) + \varphi_{3,\lambda}(X_1).$$

$$\Psi_\lambda(X_1) = \frac{\lambda X_1^2 - 2X_1}{(1-\lambda)} - \lambda X_1^2 + 4\lambda X_1 + 2X_1 + 2e^{\lambda X_1}(\lambda - 1) + 2(1 - \lambda).$$

Thus the variance of $\hat{\delta}(\lambda)$ is

$$\sigma_\lambda^2 = \text{Var}(\Psi_\lambda(X_1)). \quad (2.7)$$

The next theorem summarizes the asymptotic normality of $\hat{\delta}(\lambda)$.

Theorem 2.2. (i) As $n \rightarrow \infty$, $\sqrt{n}(\hat{\delta}(\lambda) - \delta(\lambda))$, is asymptotically normal with mean zero and variance σ_λ^2 which is as in (2.4).

(ii) Under H_0 , the variance is reduces to

$$\sigma_0^2 = \left[\frac{-4\lambda(2\lambda^5 - 19\lambda^4 + 48\lambda^3 - 64\lambda^2 + 40\lambda - 9)}{(-1 + \lambda)^3(-1 + 2\lambda)} \right], \lambda > 0, \lambda \neq \frac{1}{2}, 1. \quad (2.8)$$

Proof. (i) and (ii) follow from the standard theorem of U-statistic (cf. Lee [20]) and direct calculations, respectively.

2.1. The Pitman asymptotic relative efficiency for RNBUE_{mgf}

In order to assess how good our proposed family of tests are relative to others in the literature we employ the concept of “Pitman’s asymptotic relative efficiency” (PARE) of proposed test. To do this, we need to evaluate the “Pitman’s asymptotic efficiency” (PAE) for our tests and then compare this (via taking ratios) to the PAEs of other tests to get the (PARE). Let us first evaluate the (PAE) for our proposed family of tests $\hat{\delta}_\theta(\lambda)$ which is defined in (2.1). It is known that Pitman’s asymptotic efficiency (PAE) which is defined as Pitman [27] is given by

$$PAE(\hat{\delta}_\theta(\lambda)) = \frac{1}{\sigma_0} \left| \frac{d}{d\theta} \hat{\delta}_\theta(\lambda) \right|_{\theta \rightarrow \theta_0}.$$

We calculate the Pitman asymptotic efficiency (PAE) of RNBUE_{mgf} test statistic. These calculations are done by using the following alternatives:

- (i) Linear failure rate family, $\bar{F}_1(x) = \exp(-x - \theta x^2 / 2)$, $x \geq 0$, $\theta \geq 0$.
- (ii) Makeham family, $\bar{F}_2(x) = \exp(-x - \theta(x + e^{-x} - 1))$, $x \geq 0$, $\theta \geq 0$.
- (iii) Weibull family, $\bar{F}_3(x) = \exp(-x^\theta)$, $x \geq 0$, $\theta \geq 0$.

Directly calculations for the above three cases most of the efficiencies of these families give:

- (i) Linear failure rate family:

$$PAE(\hat{\delta}_\theta(\lambda)) = \frac{1}{\sigma_0} \left| \frac{2\lambda^3}{(\lambda - 1)^2} \right|, \quad \lambda \neq 1.$$

- (ii) Makeham family:

$$PAE(\hat{\delta}_\theta(\lambda)) = \frac{1}{\sigma_0} \left| \frac{(9\lambda^2 - 20\lambda - 8)}{2(\lambda^2 - 3\lambda + 2)} \right|, \quad \lambda \neq 1, 2.$$

Using MATHEMATICA 9 program to calculate the Pitman asymptotic efficiency (PAE) of $RNBUE_{mgf}$ test statistic in case of Weibull family as alternatives and we get the following PAE values.

(iii) Weibull family:

$$PAE(\hat{\delta}_0(\lambda)) = \frac{1}{\sigma_0} |8.57007|.$$

The above results show that all these alternatives have increasing efficiency in $\lambda \geq 0$, except $\lambda = 1, 2$ and have maximum value at $\lambda = 5.0$, and this implies that our test has a very good efficiency.

As far as, no other tests have as yet been proposed for testing against $RNBUE_{mgf}$ alternatives. Thus compare it to others that may be useful for this problem. Here we choose the tests K^* and $\delta_{(3)}$ which represented by Kango [17] and Mugdadi and Ahmad [24], respectively.

Table 1. It shows that the asymptotic efficiencies $\hat{\delta}_0(\lambda)$ test

Distribution	$\hat{\delta}_0(\lambda)$	K^*	$\delta_{(3)}$
Linear failure rate	2.6085	0.433	0.408
Makeham	4.6258	0.144	0.039
Weibull	2.28170	0.132	0.170

Table 2. It shows that the asymptotic relative efficiencies $\hat{\delta}_0(\lambda)$ test

Distribution	$PARE(\hat{\delta}_0(\lambda), K^*)$	$PARE(\hat{\delta}_0(\lambda), \delta_{(3)})$
Linear failure	6.024	6.386
Makeham	32.123	118.6
Weibull	17.285	13.421

It is clear from Table 2, we can see that the statistic $\hat{\delta}_0(\lambda)$ for $RNBUE$ is more efficiently than both K^* and $\delta_{(3)}$ for all cases and also simpler.

3. Monte Carlo Null Distribution Critical Values

In this section, we have simulated the upper percentile values for 90%, 95%, 98%, and 99% for the statistic $\hat{\delta}(\lambda)$ that given in (2.2).

Table 3 presents these percentile values of the statistic $\hat{\delta}(\lambda)$ and the calculations are based on 10000 simulated samples of size $n = 5(5)50$, and 23, 43 sample sizes.

Table 3. The upper percentile of $\hat{\delta}(\lambda)$ with 10000 replications

n	90%	95%	98%	99%
5	0.0392	0.0622	0.1106	0.1594
10	0.0295	0.0432	0.0691	0.0960
15	0.0229	0.0316	0.0473	0.0625
20	0.0198	0.0256	0.0352	0.0476
23	0.0167	0.0213	0.0269	0.0315
25	0.0168	0.0209	0.0273	0.0326
30	0.0161	0.0198	0.0260	0.0308
35	0.0150	0.0189	0.0233	0.0273
40	0.0135	0.0158	0.01937	0.0240
43	0.0142	0.0168	0.0209	0.0242
45	0.0132	0.01588	0.0198	0.0227
50	0.0131	0.0152	0.0178	0.0197

It can be noticed from Table 3 that the critical values are increasing as the confidence level increasing and decreasing as the sample size increasing.

3.1. The power of the proposed test

The power of the proposed test statistic $\hat{\delta}(\lambda)$ is calculated at significance level $\alpha = 0.05$ and for commonly used distributions in reliability. These distributions are:

(i) Linear failure rate family, $\bar{F}_1(x) = \exp(-x - \theta x^2 / 2)$, $x \geq 0, \theta \geq 0$.

(ii) Weibull family, $\bar{F}_2(x) = \exp(-x^\theta)$, $x \geq 0, \theta \geq 0$.

(iii) Gamma family, $\bar{F}_3(x) = \int_x^\infty e^{-u} u^{\theta-1} du / \Gamma(\theta)$, $x \geq 0, \theta \geq 0$.

Table 4, using simulated number of sample 10000 for sample sizes $n = 10, 20$, and 30 and θ values $1, 2$, and 3 .

Table 4. Alternative distributions: LFR, Weibull and Gamma

n	θ	LFR	Weibull	Gamma
10	1	0.276	0.959	0.956
10	2	0.518	0.982	1.000
10	3	0.764	0.995	1.000
20	1	0.997	0.948	0.945
20	2	0.999	1.000	0.992
20	3	1.000	1.000	1.000
30	1	1.000	0.947	0.947
30	2	1.000	1.000	0.99
30	3	1.000	1.000	0.998

It is clear from Table 4 that our test has a very good powers for all alternatives and the power increases as the sample size increases. The power is getting as smaller as the $RNBUE_{mgf}$ approaches the exponential distribution.

3.2. Applications using uncensored data

Here, we present in this section, some of a good real examples to elucidate the applications of our test in the case of non censored data at 95% confidence level.

Example 1. In an experiment at Florida state university to study the effect of methyl mercury poisoning on the life lengths of fish goldfish were subjected to various dosages of methyl mercury Kocher [19], at one dosage level the ordered times to death in week where

6.000, 6.143, 7.286, 8.714, 9.429, 9.857, 10.143, 11.571, 11.714, 11.714.

It was found that, $\hat{\delta}(\lambda) = 0.9862$ which is greater than the critical value of the Table 3. There is enough evidence to accept H_1 which states that the data set have RNBUE_{mgf} property and not exponential.

Example 2. The following data was considered by Pavur et al. [25]. The results recorded as the following which are the number of revolutions (in the millions) to failure of 23 ball bearings in a life test study.

1.788	2.892	3.300	4.152	4.212	4.560	4.848	5.184
5.196	5.412	5.556	6.78	6.864	6.864	6.988	8.412
9.312	9.864	10.512	10.584	12.792	12.804	17.340.	

It was found that, $\hat{\delta}(\lambda) = 0.8532$ and this value is greater than the critical value of the Table 3. Then we accept H_1 which states that the data set have RNBUE_{mgf} property and not exponential.

Example 3. Using the data set given in Grubbs [15]. This data set gives the times between arrivals of 25 customers at a facility:

1.80	2.89	2.93	3.03	3.15	3.43	3.48	3.57	3.85	3.92
3.98	4.06	4.11	4.13	4.16	4.23	4.34	4.37	4.53	4.62
4.65	4.84	4.91	4.99	5.17.					

It is easily to show that, $\hat{\delta}(\lambda) = 0.0732$ which is greater than the critical value of the Table 3. Then we accept H_1 which states that the data set have RNBUE_{mgf} property and not exponential.

Example 4. Consider the data in Abouammoh et al. [3], these data represent set of 40 patients suffering from blood cancer (leukemia) from one of ministry of health hospital in Saudi Arabia and the ordered values in years are:

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036
2.162	2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192
3.263	3.348	3.348	3.427	3.499	3.534	3.767	3.751	3.858
3.986	4.049	4.244	4.323	4.381	4.392	4.397	4.647	4.753
4.929	4.973	5.074	5.381.					

It was found that, $\hat{\delta}(\lambda) = 0.0508$ and this value is greater than the critical value of the Table 3. Then we accept H_1 which states that the data set have $RNBUE_{mgf}$ property and not exponential

Example 5. Consider the following data set is from Kotz and Johnson [18] and represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia.

0.019	0.129	0.159	0.203	0.485	0.636	0.748	0.781	0.869
1.175	1.206	1.219	1.219	1.282	1.356	1.362	1.458	1.564
1.586	1.592	1.781	1.923	1.959	2.134	2.413	2.466	2.548
2.652	2.951	3.038	3.6	3.655	3.745	4.203	4.690	4.888
5.143	5.167	5.603	5.633	6.192	6.655	6.874.		

It was found that, $\hat{\delta}(\lambda) = 0.0484$ which is greater than the critical value of the Table 3. Then we accept H_1 which states that the data set have $RNBUE_{mgf}$ property and not exponential.

4. Testing Against $RNBUE_{mgf}$ Class for Censored Data

In this section, a test statistic is proposed to test H_0 versus H_1 with randomly right-censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This

experimental situation can formally be modelled as follows. Suppose n objects are put on test, and $X_1, X_2, \dots, X_n$ denoted their true life time. We assume that $X_1, X_2, \dots, X_n$ be independent identically distributed (i.i.d.) according to a continuous life distribution F . Let $Y_1, Y_2, \dots, Y_n$ be (i.i.d.) according to a continuous life distribution G . Also assume that X 's and Y 's are independent. In the randomly right-censored model, we observe the pairs (Z_j, δ_j) , $j = 1, \dots, n$, where $Z_j = \min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1 & \text{if } Z_j = X_j \text{ (} j\text{-th observation is uncensored),} \\ 0 & \text{if } Z_j = Y_j \text{ (} j\text{-th observation is censored).} \end{cases}$$

Let $Z(0) = 0 < Z(1) < Z(2) < \dots < Z(n)$ denote the ordered Z 's and $\delta(j)$ is the δ_j the corresponding to $Z_{(j)}$, respectively. Using the censored data (Z_j, δ_j) , $j = 1, \dots, n$, Kaplan and Meier [16] proposed the product limit estimator,

$$\bar{F}_n(X) = 1 - F_n(x) = \prod_{[j: Z_{(j)} \leq X]} \{(n-j)/(n-j+1)\}^{\delta(j)}, \quad X \in [0, Z_{(n)}].$$

Now, for testing $H_0 : \delta(\lambda) = 0$ against $H_1 : \delta(\lambda) > 0$, using the randomly right-censored data.

We proposed the following test statistic:

$$\hat{\delta}^C(\lambda) = (\lambda\mu_{(2)} - 2\mu) \int_0^\infty (\phi(\lambda) dF_n(x) + (2\mu + 2\lambda\mu^2 - \lambda\mu_{(2)}) \int_0^\infty dF_n(x)),$$

where $\phi(\lambda) = \int_0^\infty e^{\lambda x} dF_n(x)$ and $\mu_{(2)} = \int_0^\infty x^2 dF_n(x)$. For computational

purposes, $\hat{\delta}^C(\lambda)$ may be rewritten as

$$\hat{\delta}^C(\lambda) = [(\lambda\Omega - 2\eta)\Psi + (2\eta + 2\lambda\eta^2 - \lambda\Omega)\zeta], \quad (4.1)$$

where

$$\Omega = \sum_{i=1}^n (Z_{(i)})^2 \left[\prod_{P=1}^{i-2} C_P^{\delta(P)} - \prod_{P=1}^{i-1} C_P^{\delta(P)} \right], \quad \eta = \sum_{j=1}^n \left[\prod_{M=1}^{j-1} C_M^{\delta(M)} (Z_{(j)} - Z_{(j-1)}) \right],$$

$$\Psi = \sum_{K=1}^n \sum_{L=1}^n e^{\lambda Z_{(L)}} \left[\prod_{Q=1}^{L-2} C_Q^{\delta(Q)} - \prod_{Q=1}^{L-1} C_Q^{\delta(Q)} \right]^2, \quad \zeta = \sum_{r=1}^n \left[\prod_{M=1}^{r-1} C_M^{\delta(M)} (Z_{(r)} - Z_{(r-1)}) \right],$$

$$\text{and } dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), \quad C_k = [n - k] [n - k + 1]^{-1}.$$

Table 5 gives the critical values percentiles of $\hat{\delta}^C(\lambda)$ test for sample sizes $n = 5(5)30(10)70, 81$ and 86 :

Table 5. The upper percentile of $\hat{\delta}^C(\lambda)$ with 10000 replications

n	90%	95%	98%	99%
5	0.2150	0.2680	0.3149	0.3393
10	0.1301	0.2567	0.3823	0.4221
15	0.1595	0.3778	0.4617	0.4977
20	0.1897	0.4204	0.4965	0.5138
25	0.3245	0.4626	0.5127	0.5300
30	0.3421	0.4780	0.5227	0.5359
40	0.4238	0.5075	0.5373	0.5482
50	0.4534	0.5193	0.5438	0.5533
60	0.4780	0.5255	0.5514	0.5573
70	0.4736	0.5314	0.5542	0.5626
81	0.5091	0.5412	0.5611	0.5693
86	0.5168	0.5451	0.5612	0.5665

It can noticed from Table 5 that, the critical values are increasing as the confidence level increasing and almost increasing as the sample size increasing.

4.1. The power estimates

In this section, we present an estimation of the power for testing exponentiality versus RNBUE_{mgf} . Using significance level $\alpha = 0.05$ with suitable parameter values of θ at $n = 10, 20$, and 30 . The estimate based on 10000 simulated, and for commonly used distributions in reliability such as Linear failure rate family, Weibull and Gamma family alternatives.

(i) Linear failure rate family, $\bar{F}_1(x) = \exp(-x - \theta x^2 / 2)$, $x \geq 0, \theta \geq 0$.

(ii) Weibull family, $\bar{F}_2(x) = \exp(-x^\theta)$, $x \geq 0, \theta \geq 0$.

(iii) Gamma family, $\bar{F}_3(x) = \int_x^\infty e^{-u} u^{\theta-1} du / \Gamma(\theta)$, $x \geq 0, \theta \geq 0$.

Which include in Table 6.

Table 6. Alternative distributions: LFR, Weibull and Gamma

n	θ	LFR	Weibull	Gamma
10	1	0.9532	0.9426	0.9486
10	2	0.9616	0.9464	0.9656
10	3	0.9654	0.9496	0.9926
20	1	0.9532	0.9504	0.9427
20	2	0.9642	0.9556	0.972
20	3	0.9684	0.9596	0.9976
30	1	0.9686	0.9508	0.9468
30	2	0.957	0.9606	0.9762
30	3	0.97	0.9692	0.9992

Table 6 shows that our test has a very good power for all alternatives and the power increases as the sample size increases. The power is getting as smaller as the RNBUE_{mgf} approaches the exponential distribution.

4.2. Applications using censored data

Finally, we present two real examples in medicine to illustrate the applications of our test in censored data at 95% confidence level.

Example 6. Consider the data from Susarla and Vanryzin [30], which represent 81 survival times (in months) of patients of melanoma. Out of these 46 represents non-censored data, and order values:

3.25	3.5	4.75	4.75	5	5.25	5.75	5.75	6.25
6.5	6.5	6.75	6.75	7.78	8	8.5	8.5	9.25
9.5	9.5	10	11.5	12.5	13.25	13.5	14.25	14.5
14.75	1.5	16.25	16.25	16.5	17.5	21.75	22.5	24.5
25.5	25.75	27.5	29.5	31	32.5	34	34.5	35.25
58.5								

The order censored data are

4	5.25	11	12.5	13.75	16.75	18.25	19	20
20.25	21.5	23.25	25	27	28.5	30	31	31.25
32.25	32.5	33	33.5	35	36.75	37	37.75	38
38	39.5	45.25	47.5	48.25	48.5	53.25	53.75	

Now taking into account the whole set of survival data (both censored and uncensored), and computing the statistic from (4.1) censored data, we get $\hat{\delta}^C(\lambda) = 0.3407$ which is less than the critical value of the Table 5 at 95% upper percentile. Then we conclude that this data set have exponential property and not $RNBUE_{mgf}$ property at 95% upper percentile.

Example 7. On the basis of right censored data for lung cancer patients from Pena [26]. These data consists of 86 survival times (in month) with 22 right censored. The whole life times (non-censored data) are:

0.99	1.28	1.77	1.97	2.17	2.63	2.66	2.76	2.79
2.86	2.99	3.06	3.15	3.45	3.71	3.75	3.81	4.11
4.27	4.34	4.40	4.63	4.73	4.93	4.93	5.03	5.16
5.17	5.49	5.68	5.72	5.85	5.98	8.15	8.62	8.48
8.61	9.46	9.53	10.05	10.15	10.94	10.94	11.24	11.63
12.26	12.65	12.78	13.18	13.47	13.96	14.88	15.05	15.31
16.13	16.46	17.45	17.61	18.20	18.37	19.06	20.70	22.54
23.36								

The ordered censored observations are:

11.04	13.53	14.23	14.65	14.91	15.47	15.47	17.05	17.28
17.88	17.97	18.83	19.55	19.55	19.75	19.78	19.95	20.04
20.24	20.73	21.55	21.98					

Now taking into account the whole set of survival data (both censored and uncensored), and computing the statistic from (4.1) censored data, we get $\hat{\delta}^C(\lambda) = 0.5507$ which is greater than the critical value of the Table 5 at 95% upper percentile. Then, we accept H_1 which states that the data set have RNBUE_{mgf} property and not exponential.

5. Conclusion

In this paper, we derived U-statistic test based on expected value method for testing exponentiality versus renewal new better than used in expectation in moment generating function class (RNBUE_{mgf}). The critical values of this test are calculated and tabulated for uncensored and censored data. The Pitman asymptotic efficiency is discussed and the Pitman asymptotic relative efficiency with respect to Kango [17] and Mugdadi and Ahmad [24] the power estimates of this test is simulated for

most used of life time distributions families in reliability for uncensored and censored data. Finally, we have used some sets of data to elucidate the applications of the (RNBUE_{mgf}) property to real data.

References

- [1] A. M. Abouammoh, R. Ahmad and A. Khalique, On new renewal better than used classes of life distributions, *Statist. Prob. Letters* 48 (2000), 189-194.
- [2] A. M. Abouammoh and A. Khalique, On tests of new renewal better than used classes of life distributions, *Parisankhyan Samikkha* 5 (1998), 21-32.
- [3] A. M. Abouammoh, S. A. Abdulghani and I. S. Qamber, On partial orderings and testing of new better than renewal used classes, *Reliability Eng. Syst. Safety* 43 (1994), 37-41.
- [4] A. A. Abdel-Aziz, On testing exponentiality against RNBRUE alternatives, *Applied Mathematical Sciences* 1(35) (2007), 1725-1736.
- [5] S. E. Abu-Youssef, Amoment inequality on new renewal better than used in expectation class of life distributions with hypothesis testing application, *International Journal of Reliability and Applications* 4 (2003), 191-199.
- [6] S. E. Abu-Youssef, Moment inequality on new renewal better than used class of life distributions with hypothesis testing application, *Applied Mathematics and Computation* 149 (2004), 651-659.
- [7] R. E. Barlow and F. Proschan, *Statistical theory of reliability and life testing, To begin with*, Sliver Spring, MD. 13, (1981).
- [8] M. C. Bhattacharjee, A. M. Abouammoh, A. N. Ahmed and A. M. Barry, Preservation results for life distributions based on comparisons with asymptotic remaining life under replacements, *J. Appl. Probab.* 37 (2000), 999-1009.
- [9] J. V. Deshpande, S. C. Kochar and H. Singh, Aspects of positive ageing, *J. Appl. Probab.* 23 (1986), 748-758.
- [10] L. S. Diab, M. A. W. Mahmoud and S. M. EL-Arishy, A non-parametric test of new renewal better than used class of life distributions, *Proceeding of International Conference on Mathematics trends and developments, Cairo (EGYPT)* 4 (2002), 191-203.
- [11] L. S. Diab, M. A. W. Mahmoud and S. M. EL-Arishy, Testing renewal new better than used life distributions based on U-test, *Applied Mathematical Modelling* 29 (2005), 784-796.
- [12] L. S. Diab, Non-parametric test of NBUE_{mgf} class of life distributions, *Journal of Statistical Theory and Applications* 8(3) (2009), 253-267.
- [13] L. S. Diab, A new approach to moments inequalities for NRBUE and RNBUE classes with hypothesis testing applications, *International Journal of Basic & Applied Sciences IJBAS-IJENS* 13(06) (2013), 7-13.

- [14] M. Franco, J. M. Ruiz and M. C. Ruiz, On closure of the IFR(2) and NBU(2) classes, *J. Appl. Probab.* 38 (2001), 235-241.
- [15] F. E. Grubbs, Fiducial bounds on reliability for the two parameter negative exponential distribution, *Technomet.* 13 (1971), 873-876.
- [16] E. L. Kaplan and P. Meier, Nonparametric estimation from incomplete observation, *J. Amer. Statist. Assoc.* 53 (1958), 457-481.
- [17] A. I. Kango, Testing for new is better than used, *Communication in Statistics Theor. Method.* 12 (1993), 311-321.
- [18] S. Kotz and N. L. Johnson, *Encyclopedia of Statistical Sciences*, 3, Wiley, New York, 1983.
- [19] S. C. Kocher, Testing exponentiality against monotone failure rate average, *Comm. Statist. Theor. Meth.* 14 (1985), 381-392.
- [20] A. J. Lee, *U-Statistics*, Marcel Dekker, New York, 1989.
- [21] X. Li, Z. Li and B. Jing, Some results about the NBUC class of life distributions, *Statist. Probab. Lett.* 46 (2000), 229-237.
- [22] M. A. W. Mahmoud, S. M. EL-Arshy and L. S. Diab, Moment inequalities for testing new renewal better than used and renewal new better than used classes of life distributions, *International Journal of Reliability and Applications* 4(3) (2003), 97-123.
- [23] A. Muller and D. Stoyan, *Comparison Methods for Stochastic Models and Risks*, Wiley and Sons, New York, NY, 2002.
- [24] A. R. Mugdadi and I. A. Ahmad, Moment inequalities derived from comparing life with its equilibrium form, *Journal of Statistical Planning and Inference* 134 (2005), 303-317.
- [25] R. J. Pavur, R. L. Edgeman and R. C. Scott, Quadratic statistics for the goodness of fit test of inverse Gaussian distribution, *IEEE Trans. Reli.* 41 (1992), 118-123.
- [26] A. E. Pena, Goodness of fit tests with censored data, (2002).
<http://statmanStat.sc.edu/penajtajkspresentedjtalkactronel>
- [27] E. J. G. Pitman, *Some Basic Theory for Statistical Inference*, Chapman & Hall, London, 1979.
- [28] S. M. Ross, *Introduction to Probability Models*, 8th Edition, Academic Press, New York, NY, 2003.
- [29] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and their Applications*, Academic Press, New York, NY, 1994.
- [30] V. Susarla and J. Vanryzin, Empirical bayes estimations of a survival function right censored observation, *Ann. Statist.* 6 (1978), 710-755.