ON DERIVATIONS IN SEMISIMPLE BANACH ALGEBRAS

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Abstract

The objective of the present paper is to prove the following result: Let A be a unital semisimple Banach algebra and G_1 and G_2 be open subsets of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$. If A admits a linear derivation $d : A \rightarrow A$ such that either $d((xy)^n) + d(x^n)d(y^n) \in Z(A)$ or $d((xy)^n) - d(x^n)d(y^n) \in Z(A)$, then $d(A) \subseteq Z(A)$. Moreover, other related results are obtained.

1. Introduction and Main Results

This research has been motivated by the work of Yood [9]. Throughout this manuscript, A denotes a Banach algebra with identity over $\mathbb{C}$, $Z(A)$ denotes the center of A and M be a closed linear subspace of A . Recall that an algebra A is said to be prime if for any $\alpha, b \in A$,

2010 Mathematics Subject Classification: 16W25, 46J10.

Keywords and phrases: semisimple Banach algebra, linear derivation.

Received December 14, 2017; Revised February 12, 2018

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$aAb = (0)$ implies $a = 0$ or $b = 0$ and A is semiprime if for any $a \in A$, $aAa = (0)$ implies $a = 0$. Radical of an algebra is the intersection of all primitive ideals of A and is denoted as $rad(A)$. An algebra A is called semisimple if $rad(A) = (0)$. For any $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$.

A linear mapping $d : A \rightarrow A$ is said to be a derivation on A if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in A$. The purpose of this paper is to prove some theorems in this spirit for semisimple Banach algebra involving linear derivation. In [7, 8], Herstein proved that a ring R is commutative if it has no non-zero nilpotent ideal and there is a fixed integer $n > 1$ such that $(xy)^n = x^n y^n$ for all $x, y \in R$ (see also [3]). Also it is proved in [2] that a ring is a commutative if $(xy)^n = y^n x^n$ for all $x, y \in R$ and R has the property that $n(n+1)[x, y] = 0$ implies $[x, y] = 0$ for all $x, y \in R$. In the case of Banach algebra Yood [9] sharpened these results. More precisely, he proved the following result:

Theorem 1.1 ([9], Theorem). *Suppose that there are non-empty open subsets G_1 and G_2 of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$, where either $(xy)^n - x^n y^n$ or $(xy)^n - y^n x^n$ lies in M . Then $[x, y] \in M$ for all $x, y \in A$.*

This result motivated us to prove following theorem:

Theorem 1.2. *Let A be a unital semisimple Banach algebra and G_1 and G_2 be open subsets of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$. If A admits a linear derivation $d : A \rightarrow A$ such that either $d((xy)^n) + d(x^n)d(y^n) \in Z(A)$ or $d((xy)^n) - d(x^n)d(y^n) \in Z(A)$, then $d(A) \subseteq Z(A)$.*

Proof. Fix $x \in G_1$ and for each $n > 1$ we define the set $U_n = \{y \in A \mid d((xy)^n) + d(x^n)d(y^n) \notin Z(A) \text{ and } y \in A \mid d((xy)^n) - d(x^n)d(y^n) \notin Z(A)\}$. It is easy to see that U_n is open. Thus, by Baire category theorem, if every U_n is dense then their intersection is also dense, which contradicts the existence of G_2 . Hence there exists a positive integer $r > 1$ such that U_r is not dense. Therefore, there exists a non-empty open set G_3 in the complement of U_r such that either $d((xy)^r) + d(x^r)d(y^r) \in Z(A)$ or $d((xy)^r) - d(x^r)d(y^r) \in Z(A)$ for all $y \in G_3$. Let $v_0 \in G_3$ and $w \in A$. Then $v_0 + tw \in G_3$ for all sufficiently small real t . Then, for each such t

$$d((x(v_0 + tw))^r) + d(x^r)d((v_0 + tw)^r) \in Z(A), \quad (1.1)$$

or

$$d((x(v_0 + tw))^r) - d(x^r)d((v_0 + tw)^r) \in Z(A), \quad (1.2)$$

for $v_0 \in G_3$ and $w \in A$. Thus at least one of (1.1) and (1.2) is valid for infinitely many t . Suppose (1.1) holds for these t . Then the expression $d((x(v_0 + tw))^r) + d(x^r)d((v_0 + tw)^r)$ can be written as

$$\begin{aligned} & d(A_{r,0}(x, v_0, w)) + d(x^r)B_{r,0}(v_0, w) \\ & + d(A_{r-1,1}(x, v_0, w)) + d(x^r)B_{r-1,1}(v_0, w)t \\ & + \dots \\ & + d(A_{1,r-1}(x, v_0, w)) + d(x^r)B_{1,r-1}(v_0, w)t^{r-1} \\ & + d(A_{0,r}(x, v_0, w)) + d(x^r)B_{0,r}(v_0, w)t^r, \end{aligned}$$

where $A_{i,j}(x, v_0, w)$ denotes the sum of all terms in which xv_0 appears exactly i times and xw appears exactly j times in the expansion of $d(x(v_0 + tw)^r)$, where i and j are non-negative integers such that

$i + j = r$. Similarly, $B_{i,j}(v_0, w)$ is sum of all terms in which v_0 appears exactly i times and w appears exactly j times in the expansion of $d((v_0 + tw)^r)$, where i and j are nonnegative integers such that $i + j = r$. The above expression is a polynomial in t and the coefficient of t^r in this polynomial is $d((xw)^r) + d(x^r)d(w^r)$. Hence $d((xw)^r) + d(x^r)d(w^r) \in Z(A)$. If (1.2) holds for these t then, we are forced to conclude that $d((xw)^r) - d(x^r)d(w^r) \in Z(A)$. Thus, given $x \in G_1$ there is a positive integer $r > 1$ depending on w such that for each $w \in A$ either $d((xw)^r) + d(x^r)d(w^r) \in Z(A)$ or $d((xw)^r) - d(x^r)d(w^r) \in Z(A)$. Next, fix $y \in A$ and for each positive integer $k > 1$, set $V_k = \{v \in A \mid d((vy)^k) + d(v^k)d(y^k) \notin Z(A) \text{ and } d((vy)^k) - d(v^k)d(y^k) \notin Z(A)\}$. It is easy to see that each V_k is open. If each V_k is dense, then by Baire category theorem so is intersection also but this is contrary to what was shown earlier concerning the open set G_1 . Hence there is an integer $m > 1$ and a non-empty open subset G_4 in the complement of V_m . If $x_0 \in G_4$ and $u \in A$, then $x_0 + tu \in G_4$ for all sufficiently small real t . Hence for positive integer $m > 1$ either $d(((x_0 + tu)y)^m) + d((x_0 + tu)^m)d(y^m) \in Z(A)$ or $d(((x_0 + tu)y)^m) - d((x_0 + tu)^m)d(y^m) \in Z(A)$ for each $u \in A$ and $x_0 \in G_4$. Arguing as above we see that either $d((uy)^m) + d(u^m)d(y^m) \in Z(A)$ or $d((uy)^m) - d(u^m)d(y^m) \in Z(A)$ for each $u \in A$.

Now S_k , $k > 1$, be the set of $y \in A$ such that for each $w \in A$ either $d((wy)^k) + d(w^k)d(y^k) \in Z(A)$ or $d((wy)^k) - d(w^k)d(y^k) \in Z(A)$, then the union of S_k will be A . It can be easily proved that each S_k is closed. Hence again by Baire category theorem some S_l must have a non empty open subset G_5 . Let $y_0 \in G_5$ and $z \in A$, for all sufficiently small real t

and each $w \in A$ either $d((w(y_0 + tz))^l) + d(w^l)d((y_0 + tz)^l) \in Z(A)$ or $d((w(y_0 + tz))^l) - d(w^l)d((y_0 + tz)^l) \in Z(A)$. By earlier arguments, we have for each $w, z \in A$ either $d((wz)^l) + d(w^l)d(z^l) \in Z(A)$ or $d((wz)^l) - d(w^l)d(z^l) \in Z(A)$. Next, since A is unital then, for all real t either $d(((e + tx)y)^n) + d((e + tx)^n)d(y^n) \in Z(A)$ or $d(((e + tx)y)^n) - d((e + tx)^n)d(y^n) \in Z(A)$ for all $x, y \in A$. Hence taking coefficient of t in the expansion of above equations, we get either

$$d(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nd(x)d(y^n) \in Z(A), \quad (1.3)$$

or

$$d(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) - nd(x)d(y^n) \in Z(A), \quad (1.4)$$

for all $x, y \in A$. Now, taking $d[(y(e + tx))^n]$ in instead of $d[((e + tx)y)^n]$, we see that either

$$d(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nd(y^n)d(x) \in Z(A), \quad (1.5)$$

or

$$d(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) - nd(y^n)d(x) \in Z(A), \quad (1.6)$$

for all $x, y \in A$. Then at least one of pair of equations $\{(1.3), (1.5)\}$, $\{(1.3), (1.6)\}$, $\{(1.4), (1.5)\}$, and $\{(1.4), (1.6)\}$ must hold. Then, we get

$$d([x, y^n]) \pm n[d(x), d(y^n)] \in Z(A), \quad (1.7)$$

for all $x, y \in A$. Replace y by $e + ty$ in above equation and using same arguments as we have used above, we get

$$d([x, y]) \pm n[d(x), d(y)] \in Z(A), \quad (1.8)$$

for all $x, y \in A$. First we assume that

$$d([x, y]) + n[d(x), d(y)] \in Z(A), \quad (1.9)$$

for all $x, y \in A$. Replace y by yw in above expression, we obtain

$$\begin{aligned} d([x, y])w + [x, y]d(w) + nd(y)[d(x), w] \\ + n[d(x), d(y)]w + n[d(x), y]d(w) \\ + ny[d(x), d(w)] \in Z(A), \end{aligned} \quad (1.10)$$

for all $x, y, w \in A$. Taking $y = e + y$ in (1.10), we obtain

$$n[d(x), d(w)] \in Z(A), \quad (1.11)$$

for all $x, w \in A$. In particular, we have

$$[[d(x), d(w)], d(w)] = 0, \quad (1.12)$$

for all $x, w \in A$. Then, by Kleinecke-Shirokov theorem ([4], Proposition 13, p. 91) each $z = [d(x), d(w)]$ is a generalized nilpotent element of A , i.e., $\rho(z) = \|z^n\|^{1/n} = 0$. Therefore, $[d(x), d(w)]$ is a generalized nilpotent element in $Z(A)$ and so is in radical of $Z(A)$. Since A is semisimple, so $Z(A)$ will also semisimple. Therefore, we conclude that

$$[d(x), d(w)] = 0, \quad (1.13)$$

for all $x, w \in A$. Next, replace w by ya in (1.13), we get

$$d(y)[d(x), a] + [d(x), y]d(a) = 0, \quad (1.14)$$

for all $a, x, y \in A$. Taking $a = x$ in above expression, we get

$$d(y)[d(x), x] + [d(x), y]d(x) = 0, \quad (1.15)$$

for all $x, y \in A$. Again, replace y by xy in (1.15), we obtain

$$d(x)y[d(x), x] + [d(x), x]yd(x) = 0, \quad (1.16)$$

for all $x, y \in A$. In view of ([5], Lemma 4), we have

$$[d(x), x]yd(x) = 0, \quad (1.17)$$

for all $x, y \in A$. In particular, it follows that $[d(x), x]y[d(x), x] = 0$ for all $x \in A$. Since A is semisimple, the last expression yields that $[d(x), x] = 0$. Hence, $d(A) \subseteq Z(A)$.

In a similar manner, we can prove that the same conclusion holds for the case $d([x, y]) - [d(x), d(y)] \in Z(A)$ for all $x, y \in A$. Thereby the proof of the theorem is completed. $\square$

For unital prime Banach algebra, we can write the following:

Corollary 1.1 ([1], Theorem 1.3). *Let A be a unital prime Banach algebra and G_1 and G_2 be open subsets of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$. If A admits a continuous linear derivation $d : A \rightarrow A$ such that either $d((xy)^n) + d(x^n)d(y^n) \in Z(A)$ or $d((xy)^n) - d(x^n)d(y^n) \in Z(A)$, then A is commutative.*

Theorem 1.3. *Let A be a unital semisimple Banach algebra and G_1 and G_2 be open subsets of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$. If A admits a linear derivation $d : A \rightarrow A$ such that either $d((xy)^n) + x^n y^n \in Z(A)$ or $d((xy)^n) - x^n y^n \in Z(A)$, then $d(A) \subseteq Z(A)$.*

Proof. Proceeding on same lines as in case of Theorem 1.1, we arrive at either $d((xy)^n) + x^n y^n \in Z(A)$ or $d((xy)^n) - x^n y^n \in Z(A)$ for each $x, y \in A$. Next, since A is unital then, for all real t either $d(((e + tx)y)^n) + (e + tx)^n y^n \in Z(A)$ or $d(((e + tx)y)^n) - (e + tx)^n y^n \in Z(A)$ for all $x, y \in A$. Hence taking coefficient of t in the expansion of above equations, we get either

$$d(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nxy^n \in Z(A), \quad (1.18)$$

or

$$d(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nxy^n \in Z(A), \quad (1.19)$$

for all $x, y \in A$. Now, taking $d[(y(e + tx))^n]$ in instead of $d[((e + tx)y)^n]$, we see that either

$$d(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) + ny^n x \in Z(A), \quad (1.20)$$

or

$$d(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) - ny^n x \in Z(A), \quad (1.21)$$

for all $x, y \in A$. Then at least one of pair of equations $\{(1.18), (1.20)\}$, $\{(1.18), (1.21)\}$, $\{(1.19), (1.20)\}$, and $\{(1.19), (1.21)\}$ must hold. Then we obtain

$$d([x, y^n]) \pm n[x, y^n] \in Z(A), \quad (1.22)$$

for all $x, y \in A$. Replace y by $e + ty$ in above equation and using same arguments as we have used above, we get

$$d([x, y]) \pm n[x, y] \in Z(A), \quad (1.23)$$

for all $x, y \in A$. Suppose

$$d([x, y]) + n[x, y] \in Z(A), \quad (1.24)$$

for all $x, y \in A$. That is,

$$[[d(x), y] + [x, d(y)] + n[x, y], x] = 0, \quad (1.25)$$

for all $x, y \in A$. Replace y by $[y, z]$ in above expression, we get

$$[d(x), [y, z]] + [x, d[y, z]] + n[x, [y, z]] \in Z(A), \quad (1.26)$$

for all $x, y \in A$. That is,

$$[d(x), [y, z]] + [x, d[y, z]] + n[y, z]] \in Z(A), \quad (1.27)$$

for all $x, y \in A$. Application of (1.24) yields that

$$[d(x), [y, z]] \in Z(A), \quad (1.28)$$

for all $x, y \in A$. By ([6], Lemma 1), $d(x) \in Z(A)$ for all $x \in A$, i.e., $d(A) \subseteq Z(A)$. Similar conclusion holds for the case $d([x, y]) - [x, y] \in Z(A)$ for all $x, y \in A$. This proves the theorem. $\square$

For unital prime Banach algebra, we can write the following:

Corollary 1.2 ([1], Theorem 1.2). *Let A be a unital prime Banach algebra and G_1 and G_2 be open subsets of A such that for each $x \in G_1$ and $y \in G_2$, there is an integer $n = n(x, y) > 1$. If A admits a continuous linear derivation $d : A \rightarrow A$ such that either $d((xy)^n) + x^n y^n \in Z(A)$ or $d((xy)^n) + x^n y^n \in Z(A)$, then A is commutative.*

Acknowledgement

The authors wishes to thank the referee for his/her their valuable comments and suggestions.

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