ON A NEUMANN EQUILIBRIUM STATES IN ONE MODEL OF ECONOMIC DYNAMICS

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Abstract

The model of economic dynamics of Neumann type is considered. With the help of the equilibrium mechanism without loss a trajectory of the model is constructed. The necessary and sufficient conditions for the existence of the equilibrium state without loss is derived.

1. Introduction

Let us introduce the following denotations: $B^k = (\nu^{k1}, \dots, \nu^{kn})$, $\nu^{ki} \in [0, 1]$, $k, i \in I = \{1 \dots n\}$ is a preservation matrix of the k -th branch, $C = (C^{ij})_{i,j=1}^n$, $C^{ij} > 0$, $i, j \in I = \{1 \dots n\}$ is a cost matrix coefficients, where C^{ij} is a number of product of the i -th branch, necessary for the production of the unit product of the j -th branch.

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$F^i = \min \frac{x^{ij}}{C^{ji}}$, $i, j \in I = \{1 \dots n\}$ is a productive function of the i -th branch, $\ell = (\ell_1, \dots, \ell_2)$ – price vector, μ^k – maximal growth rate of the total wealth of the k -th branch.

Then we can define the model Z by the productive [1-4]:

$$a(X) = \left\{ \tilde{X} = (\tilde{x}^1, \dots, \tilde{x}^{n'}) \in (R_+^n)^n \mid \sum_{k \in I} \tilde{x}^{ki} \leq \sum_{k \in I} \nu^{ki} \cdot x^{ki} + \min_{j \in I} \frac{x^{ij}}{c^{ji}}, \right. \\ \left. x^{k\cdot} = (x^{k1}, \dots, x^{kn}), k \in I, \nu^{ki} \in [0, 1], c^{ij} > 0 (i, j \in I) \right\}.$$

Here we describe the construction of the trajectory of this model by the help of equilibrium mechanism without loss [4].

The working of the economy as a dynamic system, is given by the state y at time t .

Note that equilibrium is a set $(P, x^1, \dots, x^{n'}, y)$, and if the equilibrium exists then the vector $x^{k\cdot}$ is definitely proportional to $c^{k\cdot} (k \in I)$. If $y \notin \text{cone} \{c^i \mid i \in I\}$, the equilibrium obviously does not exist.

(1) If $y \in \text{cone} \{c^i \mid i \in I\}$, then expanding it on the basis, we obtain $\sum_{i \in I} \gamma^i \cdot c^i$, moreover the vector type of $\alpha^i \cdot c^i (i \in I)$ are solutions of the consumer problem.

(2) If one may choose $\ell^i, \mu^j (i, j \in I)$ by the way that the following inequality would be satisfied:

$$\ell^i \cdot \nu^{ji} + \frac{1}{|C|} \cdot \sum_{k \in I} (-1)^{i+k+1} \cdot \frac{\mu^j}{\mu^k} \left(\ell^k + \sum_{m \in I} \ell^m \cdot \nu^{km} \cdot c^{mk} \right) |C_i^k| \leq 0,$$

where C_i^k is $(n-1) \times (n-1)$ matrix obtained from the matrix C by removing the k -th column and i -th row, then equilibrium without loss exists.

Thereafter, according to $\bar{x}^i (i \in I)$ via preservation matrix B^i and the production function F^i yields the following vector of distributed resources $\tilde{y}$ at time $t+1$, and then the process is repeated.

2. Main Part

In [7], necessary and sufficient conditions are obtained for the existence of an equilibrium state without loss (see Theorems 3 and 4). But verification of necessary and sufficient conditions for the existence of equilibrium prices without loss is complicated enough - a system of n^2 linear inequalities (24) or system of n superlinear inequalities (29) [7]. Therefore, it is advisable to derive the necessary conditions for the existence of equilibrium prices without loss.

Consider the special case when all $\mu^j (j \in I)$ are the same, that is, the branches grow with the same maximal rate of the total wealth.

Equilibrium prices without loss P are solution of (12) [7]. By Theorem Fan Ji [8], the necessary conditions of absence of equilibrium prices without loss can be formulated as the following problem:

By which $\ell^i, \nu^{ji}, c^{ij} (i, j \in I)$ there exist $\nu^{ji} \geq 0$ and $u^j (i, j \in I)$, satisfying the inequality

$$\sum_{j \in I} \nu^{ji} = - \sum_{j \in I} u^j \cdot c^{ij}, \quad \forall i \in I, \quad (1)$$

for which is valid

$$\sum_{j \in I} \left(\sum_{i \in I} \nu^{ji} \cdot \ell^i \cdot \nu^{ji} + u^j \cdot \left(\ell^j + \sum_{i \in I} \ell^i \cdot \nu^{ji} \cdot c^{ij} \right) \right) > 0. \quad (2)$$

Let the prices $\ell = (\ell^1, \dots, \ell^n)$ are normed by the relation

$$\sum_{i \in I} \ell^i = 1, \quad \ell^i > 0 \ (i \in I). \quad (3)$$

Consider the left hand side of (2) taking into account (3):

$$\begin{aligned} & \sum_{j \in I} \left(\sum_{i \in I} v^{ji} \cdot \ell^i \cdot v^{ji} + u^j \cdot \left(\ell^j + \sum_{i \in I} \ell^i \cdot v^{ji} \cdot c^{ij} \right) \right) \\ &= \sum_{j \in I} \left(\sum_{i \in I} \ell^i \cdot v^{ji} \cdot v^{ji} + u^j \cdot \left(\ell^j \sum_{i \in I} \ell^i + \sum_{i \in I} \ell^i \cdot v^{ji} \cdot c^{ij} \right) \right) \\ &= \sum_{i \in I} \ell^i \cdot \left(\sum_{j \in I} [v^{ji} \cdot v^{ji} + u^j \cdot (\ell^j + v^{ji} \cdot c^{ij})] \right). \end{aligned}$$

Thus the inequality (2) takes the form

$$\sum_{i \in I} \ell^i \cdot \left(\sum_{j \in I} [v^{ji} \cdot v^{ji} + u^j \cdot (\ell^j + v^{ji} \cdot c^{ij})] \right) > 0. \quad (4)$$

Let's introduce the set on indexes

$$\begin{aligned} J_1(i) &= \left\{ j \in I \mid v^{ji} = \max_{k \in I} v^{ki} \right\}, \quad \forall i \in I, \\ J_2(i) &= \left\{ j \in I \mid \ell^j + v^{ji} \cdot c^{ij} = \min_{k \in I} (\ell^k + v^{ki} \cdot c^{ik}) \right\}, \quad \forall i \in I. \end{aligned} \quad (5)$$

It takes place

Lemma 1. *Let the conditions (3) and the inequality*

$$\frac{\max_{j \in I} v^{ji}}{|J_1(i)|} \cdot \sum_{j \in J_1(i)} c^{ij} > \min_{j \in I} (\ell^j + v^{ji} \cdot c^{ij}), \quad i \in I, \quad (6)$$

are valid, where $|J_1(i)|$ is a number of elements in the set of indexes $J_1(i)$.

In this case there exist the numbers $v^{ji} \geq 0$ and u^j ($i, j \in I$), satisfying (1), for which (2) or (3) is fulfilled.

Proof. Introduce the sets $J_1(i)$ and $J_2(i)$ by the formulas (5). Let the conditions (3) and (1) be satisfied. Then define ν^{ji} and u^j ($i, j \in I$) as follows:

$$u^j = \begin{cases} -\varepsilon^i, & \text{if } j \in J_2(i), \\ 0, & \text{if } j \notin J_2(i), \end{cases} \quad (i \in I),$$

$$\nu^{ji} = \begin{cases} \frac{\varepsilon^i}{|J_1(i)|} \cdot \sum_{j \in J_1(i)} c^{ij}, & \text{if } j \in J_1(i), \\ 0, & \text{if } j \notin J_1(i), \end{cases} \quad (i \in I),$$

where $\varepsilon^i > 0$ is some number ($i \in I$).

Substituting the values u^j and ν^{ji} ($i, j \in I$) into the condition (1), it is easy to see that it is satisfied.

Multiplying both sides of the inequality (6) by $\ell^i \cdot \varepsilon^i (\varepsilon^i > 0)$ and summing over $i \in I$, we get

$$\sum_{i \in I} \ell^i \cdot \varepsilon^i \cdot \max_{j \in I} \nu^{ji} \cdot \frac{1}{J_1(i)} \cdot \sum_{j \in J_1(i)} c^{ij} > \sum_{i \in I} \ell^i \cdot \varepsilon^i \cdot \min_{j \in I} (\ell^j + \nu^{ji} \cdot c^{ij}). \quad (7)$$

From the other hand substituting the values u^j and ν^{ji} ($i, j \in I$) into (4) (or (2)), considering (3), we get (7).

Lemma 1 is proved.

Theorem 1. *Let the numbers $\nu^{ji} \geq 0$, $\ell^i > 0$ ($i, j \in I$) be such that $\max_{j \in I} \nu^{ji} > 0$, $\sum_{i \in I} \ell^i = 1$ and $\mu^j = 1$ ($j \in I$). If equilibrium prices without loss by given ν^{ji} , ℓ^i and some $c^{ij} > 0$ ($i, j \in I$) exist, then the following type inequality turns true:*

$$\min_{i \in I} \left(\frac{\max_{j \in I} \nu^{ji}}{|J_1(i)|} \cdot \sum_{j \in J_1(i)} c^{ij} - \min_{j \in I} (\ell^j + \nu^{ji} \cdot c^{ij}) \right) \leq 0.$$

The proof immediately follows from Fan Ji [8] and Lemma 1.

Consider the Neumann-Gale model, given by production mapping α :

$$\begin{aligned} a(X) = \left\{ \tilde{X} = (\tilde{x}^{1\cdot}, \dots, \tilde{x}^{n\cdot}) \in (R_+^n)^n \mid 0 \leq \sum_{k \in I} \tilde{x}^{i\cdot} \leq \sum_{k \in I} B^k \cdot x^{k\cdot} \right. \\ \left. + (F^1(x^{1\cdot}), \dots, F^n(x^{n\cdot})), x^{k\cdot} = (x^{k1}, \dots, x^{kn}), k \in I \equiv \{1, 2, \dots, n\} \right\}, \end{aligned}$$

where B^k is a diagonal matrix the main diagonal of which has a form

$$(\nu^{k1}, \dots, \nu^{kn}), \quad \nu^{ki} \in [0, 1], \quad k, i \in I;$$

$$F^i(x^{i\cdot}) = \min_{j \in I} \frac{x^{ij}}{c^{ji}}, \quad c^{ji} > 0, \quad i, j \in I.$$

We define the Neumann equilibrium state in this model Z [1, 5, 6]. Recall that the set $(\alpha, \bar{X}, \bar{P})$ is a Neumann equilibrium state, if the following conditions are satisfied:

$$\alpha \cdot \bar{X} \in a(\bar{X}), \tag{8}$$

$$[\bar{P}, Y] \leq \alpha \cdot [\bar{P}, X] \quad (Y \in a(X)), \tag{9}$$

$$[\bar{P}, \bar{X}] > 0, \tag{10}$$

where α is a production mapping of the considered model Z .

Note that the prices of the products do not depend on the branches where they are considered, i.e., $\bar{P} = (P, \dots, P)$, where $P = (P^1, \dots, P^n)$.

Note that nondegenerating case means that

$$\alpha \cdot \sum_{k \in I} \bar{x}^{ki} = \sum_{k \in I} \nu^{ki} \cdot \bar{x}^{ki} + \min_{j \in I} \frac{\bar{x}^{ij}}{c^{ji}}, \quad \forall i \in I. \quad (11)$$

Due to the fact that in the system (11) only strict equalities should be satisfied, therefore, we are dealing with equilibrium without a loss, i.e.,

$$\frac{\bar{x}^{k1}}{c^{1k}} = \dots = \frac{\bar{x}^{kn}}{c^{nk}}, \quad \forall k \in I.$$

It follows from this

$$\bar{x}^{k\cdot} = \sigma^k \cdot c^{\cdot k}, \quad k \in I. \quad (12)$$

Assume that

$$\ell = \frac{1}{\alpha} \cdot P, \quad \mu^k = 1. \quad (13)$$

Note that μ^k is a maximal growth rate of the total wealth of k -th branch [4, 9, 10]:

$$\mu^k = \max_{x \geq 0} \frac{U^k(\ell, x)}{[P, x]}, \quad k \in I,$$

where

$$U^k(\ell, x^{k\cdot}) = \sum_{j \in I} \ell^i \cdot \nu^{ki} \cdot x^{ki} + \ell^k \cdot \min_{j \in I} \frac{x^{ki}}{c^{ik}}, \quad k \in I.$$

Due (11) and (13) we have

$$\alpha = \frac{U^1(P, \bar{x}^{1\cdot})}{[P, \bar{x}^{1\cdot}]} = \dots = \frac{U^n(P, \bar{x}^{n\cdot})}{[P, \bar{x}^{n\cdot}]}, \quad (14)$$

where $\bar{X} = (\bar{x}^{1\cdot}, \dots, \bar{x}^{n\cdot})$, $\bar{P} = (P, \dots, P)$.

The system of relations (14) one may obtain also from the condition (9) taking $X = \bar{X}$.

Neumann equilibrium state $(\alpha, \bar{X}, \bar{P})$ is determined up to a factor, as the multiplication of any of the relations (8)-(10) by a positive factor does not change these relations (recall that α is a superlinear mapping). Therefore it is suitable to take $P^1 = 1, \sigma^1 = 1$ in the formula (12). From (12) follows, Neumann equilibrium vector $\bar{X}$ is defined by the parameters $\sigma^1, \dots, \sigma^n$. Due to above considerations, we get that Neumann equilibrium state $(\alpha, \bar{X}, \bar{P})$ is defined by $2n - 1$ variables $\alpha, P^2, \dots, P^n, \sigma^2, \dots, \sigma^n$. Note that

$$\min_{j \in I} \frac{\bar{x}^{ij}}{c^{ji}} = \sigma^i, \quad i \in I.$$

Then substituting (12) into (11) and (14), we obtain a system of $2n - 1$ equations with respect to $\alpha, P^2, \dots, P^n, \sigma^2, \dots, \sigma^n$:

$$\left\{ \begin{array}{l} \sum_{k \in I} \sigma^k \cdot c^{ik} = \sum_{k \in I} \sigma^k \cdot v^{ki} \cdot c^{ik} + \sigma^i, \quad (i \in I); \\ \frac{P^1 \cdot \sigma^1 + \sum_{i \in I} P^i \cdot \sigma^i \cdot v^{1i} \cdot c^{i1}}{\sigma^1 \cdot \sum_{i \in I} P^i \cdot c^{i1}} = \dots = \frac{P^n \cdot \sigma^n + \sum_{i \in I} P^i \cdot \sigma^i \cdot v^{ni} \cdot c^{in}}{\sigma^n \cdot \sum_{i \in I} P^i \cdot c^{in}} = \alpha. \end{array} \right. \quad (15)$$

In addition to this system these variables should satisfy the inequality (10), which due to (12) takes the form

$$\sum_{j \in I} P^j \cdot \sum_{i \in I} \sigma^i \cdot c^{ji} > 0. \quad (16)$$

Thus in fact the following theorem is proved.

Theorem 2. *Neumann equilibrium state that satisfies the Equation (11) can be constructed using the equilibrium model without loss by $\ell = \frac{1}{\alpha} \cdot P, \mu^k = 1$ ($k \in I$), wherein the equilibrium can be found from the system (15) and (16).*

Note: Let Neumann prices $P = (P^1, \dots, P^n)$ are normed by the condition $[P, \bar{x}^{k^*}] = 1$ ($k \in I$).

In this case,

$$\sigma^k = \frac{1}{[P, c^{k^*}]}, \quad k \in I. \quad (17)$$

Substituting (17) into (15) we obtain a system with respect to $n + 1$ variables $\alpha, P^1, \dots, P^n$.

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