MULTIPLICITY OF SOLUTIONS FOR $2p$ -ORDER ORDINARY DIFFERENTIAL EQUATION WITH PERIODIC NONLINEARITY

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Abstract

In this paper, a class of $2p$ -order ordinary differential equation with periodic nonlinearity are concerned. Several assumptions are made by an index theory for the corresponding linear homogeneous equation. Under these assumptions, a multiplicity result of solutions for the concerned equation is obtained by applying Lyapunov-Schmidt reduction and a careful analysis of the oscillatory behaviour of the bifurcation equation induced by the concerned equation.

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1. Introduction and Main Result

In this paper, we are concerned with the boundary-value problem

$$\begin{cases} (-1)^{p+1}x^{(2p)}(t) + q(t)x(t) + g(x(t)) = h(t), & t \in [0, 1], \\ x^{(2k-2)}(0) = 0 = x^{(2k-2)}(1), & k = 1, 2, \dots, p, \end{cases} \quad (1.1)$$

where p is a positive integer, $q \in L^\infty[0, 1]$, g is a continuous and T -periodic function with zero mean value, not identically zero, $h \in C[0, 1]$.

Clearly, if $g \equiv 0 \equiv h$, then (1.1) will become the linear homogeneous equation

$$\begin{cases} (-1)^{p+1}x^{(2p)}(t) + q(t)x(t) = 0, & t \in [0, 1], \\ x^{(2k-2)}(0) = 0 = x^{(2k-2)}(1), & k = 1, 2, \dots, p. \end{cases} \quad (1.2)$$

Some special cases of (1.1) have been discussed in many papers. Specially, the existence results of solutions for (1.1) where $p = 1$ and q is identically equal to a real number (including its generalizations with the form of partial differential equations) are found in [17, 18, 20, 21]. The multiplicity result of solutions for (1.1) where $p = 1$ and $q \in C^1([0, 1]; \mathbf{R}^+)$ was first studied in [19], and the technique from bifurcation theory has been successfully employed in the course of its proof. Later on, (1.1) where $p = 1$ and $q \equiv 1$ was discussed in [3], and the proof of its main result combined Lyapunov-Schmidt reduction together with a careful analysis of the oscillatory behaviour of the so-called bifurcation equation induced by (1.1). This method was employed to several kinds of equations (see [5-9]) and the similar results were obtained correspondingly. Meanwhile, another multiplicity result of solutions for the same problem was proved in [4] by studying the restriction of the corresponding functional and using Lagrange multipliers method. In addition, a generalization of the results in [3, 19] where the corresponding nontrivial solution to the linear part changes sign was given in [15] and its proof was used Lyapunov-Schmidt

reduction and asymptotic methods based on the stationary phase argument. In [16], the so-called dynamical version of the classical Lyapunov-Schmidt procedure was employed and the multiplicity results of solutions for the same problem was also obtained. Multiplicity questions for problems similar to (1.1) where $p = 1$ (including partial differential equations) have been also discussed in [2, 10, 11]. In this paper, we consider the multiplicity of solutions for (1.1). To the best of our knowledge, (1.1) has not been previously considered in the literature.

This paper is organized as follows. In Section 2, we first recall a classification theory for linear $2p$ -order Hamiltonian systems which is the higher dimensional case of (1.2) in [14] (especially, the case of $p = 1$ is seen in [12, 13]). Thanks to the classification theory, we make some assumptions of q in (1.1). Then, we quote an important lemma in [1] which can help us find the nontrivial solutions of (1.1). In addition, we concern a decomposition of a closed subspace of the Banach space $C[0, 1]$ which depends on the nontrivial solution of (1.2), and rewrite (1.1) in another form. In Section 3, we state our main result and give its proof by using Lyapunov-Schmidt reduction and a careful analysis of the oscillatory behaviour of the bifurcation equation induced by (1.1).

2. Preliminaries

First, let us recall several definitions and propositions of an index classification theory for linear $2p$ -order Hamiltonian systems in [14]. For the sake of our need, we state them only in one dimensional case. According to these results we can make some assumptions of q in (1.1).

Definition 2.1 ([14]). For any $A \in L^\infty[0, 1]$, we define

$$q_A(x, y) = \int_0^1 [x^{(p)}(t) \cdot y^{(p)}(t) - A(t)x(t) \cdot y(t)]dt, \quad \forall x, y \in E,$$

where $E \equiv \{x \in H^p[0, 1] | x^{(2j)}(0) = 0 = x^{(2j)}(1), 0 \leq j < \frac{p}{2}\}$ is a Hilbert

space with the norm $\|x\| = \left(\int_0^1 (|x^{(p)}(t)|^2 + |x(t)|^2)dt \right)^{\frac{1}{2}}$. For any $x, y \in E$,

if $q_A(x, y) = 0$, we say that x and y q_A -orthogonal. For any $x \in E_1$ and $y \in E_2$, where E_1 and E_2 are the two subspaces of E , if $q_A(x, y) = 0$, we say that E_1 and E_2 q_A -orthogonal.

Proposition 2.2 ([14]). *For any $A \in L^\infty[0, 1]$, E has a q_A -orthogonal decomposition*

$$E = E^+(A) \oplus E^0(A) \oplus E^-(A),$$

such that

$$(i) \quad q_A(x, x) > 0, \forall x \in E^+(A), x \neq 0;$$

$$(ii) \quad q_A(x, x) = 0, \forall x \in E^0(A);$$

$$(iii) \quad q_A(x, x) < 0, \forall x \in E^-(A), x \neq 0.$$

Moreover, $E^0(A)$ and $E^-(A)$ are finitely dimensional.

Definition 2.3 ([14]). For any $A \in L^\infty[0, 1]$, we define $i_p(A) = \dim E^-(A)$, $\nu_p(A) = \dim E^0(A)$. We call $i_p(A)$ and $\nu_p(A)$ the index and nullity of A , respectively.

Proposition 2.4 ([14]). *For any $A \in L^\infty[0, 1]$, $\nu_p(A)$ is the dimension of the solution space of $(-1)^{p+1}x^{(2p)}(t) + A(t)x(t) = 0$, $x^{(2k-2)}(0) = 0 = x^{(2k-2)}(1)$, $k = 1, 2, \dots, p$, and $i_p(A) = \sum_{\lambda < 0} \nu_p(A + \lambda)$.*

The following proposition is a direct corollary of Proposition 5.1.2, Definition 2.2.1 and Remark 2.2.1 in [13].

Proposition 2.5 ([13]). *For any $A \in L^\infty[0, 1]$, $i_1(A) = k$, $\nu_1(A) = 1$ if and only if $A \in H_k \equiv \{A \in L^\infty[0, 1] \mid \text{the nontrivial solution of (1.2) where } p = 1 \text{ has exactly } k \text{ zeros in } (0, 1)\}$.*

Proposition 2.6. *For some $A \in L^\infty[0, 1]$, $\nu_1(A) = 1$, $A(t) > 0$ for any $t \in [0, 1]$, and $e(t)$ is a nontrivial solution of (1.2), then e' has only finite zeros in $(0, 1)$ with $e'(t)^2 - e(t) \cdot e''(t) > 0$.*

Proof. We introduce the Prüfer transformation

$$\rho(t) = (e(t)^2 + e'(t)^2)^{\frac{1}{2}} \text{ and } \phi(t) = \arccot \frac{e'(t)}{e(t)},$$

which is a continuous function and satisfies $\phi(0) \in [0, 2\pi)$ with $e(0) = \rho(0) \sin \phi(0)$, $\dot{e}(0) = \rho(0) \cos \phi(0)$. Then,

$$e(t) = \rho(t) \sin \phi(t), \quad e'(t) = \rho(t) \cos \phi(t), \quad (2.1)$$

$$\phi'(t) = \cos^2 \phi(t) + A(t) \sin^2 \phi(t). \quad (2.2)$$

We know that $e' = 0$ if and only if $\phi(t) = \frac{\pi}{2} + m\pi$, $m \in \mathbf{Z}$ from (2.1). On the other hand, thanks to $A(t) > 0$, we find that $\phi(t)$ is a strictly monotone increasing function from (2.2). Moreover, the continuity of ϕ implies that ϕ is also bounded on $[0, 1]$. Hence, e' has only finite zeros. In addition, (2.1) implies $e'(t)^2 - e(t) \cdot e''(t) > 0$.

Thanks to the results of the index classification theory mentioned above, we can make some assumptions of q so as to discuss the multiplicity of solutions for (1.1). The first one is that

$$(Q1) \quad i_p(q) = 0, \quad \nu_p(q) = 1, \text{ where } q \in L^\infty[0, 1].$$

Remark 2.7. Assumption (Q1) is natural. Moreover, $\nu_p(q) = 1$ shows that the solution space of (1.2) is one dimensional via Proposition 2.4, and any nontrivial solution of (1.2) where $p = 1$ has no zero in $(0, 1)$ via Proposition 2.5. In other words, any nontrivial solution of (1.2) does not change its sign in $(0, 1)$. Hence, without loss of generality, we further assume that

(Q2) there exists a nontrivial solution of (1.2) denoted by $\bar{x}$ such that $\bar{x}(t) > 0$ for any $t \in (0, 1)$ and $\int_0^1 \bar{x}^2(t) dt = 1$.

Remark 2.8. We can illustrate the reasonability of Assumptions (Q2) and (Q1) further. For any given p , set $q_1(t) \equiv \pi^{2p}$ on $[0, 1]$. Then, $i_p(q_1) = 0$, $\nu_p(q_1) = 1$ and $x = \sqrt{2} \sin \pi t$ is a nontrivial solution of (1.2) where $q = q_1$. Clearly, $x > 0$ in $(0, 1)$ at the moment. However, if we set $q_2(t) \equiv (2\pi)^{2p}$ on $[0, 1]$, then $i_p(q_2) = 1$, $\nu_p(q_2) = 1$ and $x = \sqrt{2} \sin 2\pi t$ is also a nontrivial solution of (1.2) where $q = q_1$. Now we find that x changes its sign in $(0, 1)$. In addition, we have $\int_0^1 x^2(t) dt = 1$.

Remark 2.8 shows that for any given p , the derivative of the nontrivial solution $x = \sqrt{2} \sin 2\pi t$ of (1.2) where $q(t) \equiv \pi^{2p}$ has exactly one zero in $(0, 1)$. Moreover, x satisfies $x'(t)^2 - x(t) \cdot x''(t) = 8\pi^2 > 0$. Together with this fact, Proposition 2.6 enlightens us to make the following assumption:

(Q3) $q \neq 0$ on $[0, 1]$, and $\bar{x}$ has only finite zeros in $(0, 1)$. Moreover, $\bar{x}'(t)^2 - \bar{x}(t) \cdot \bar{x}''(t) > 0$ on $[0, 1]$.

In addition, we list on the assumption of g which has been stated at the beginning of this paper.

(G) g is a continuous and T -periodic function with zero mean, not identically zero.

Secondly, we need to recall a lemma in [1] (see Lemma 1.2) because it is crucial to our arguments. Meanwhile, some prior statements are prepared.

X is an arbitrary Banach space. $L : X \rightarrow X$ is a Fredholm operator, the domain, kernel and range of L is denoted by $D(L)$, $N(L)$, and $R(L)$, respectively. Of course, $D(L) \subseteq X$. Then, there exists a closed linear subspace V and a linear subspace U of X , respectively, such that $X = N(L) \oplus V$ and $X = U \oplus R(L)$. Let Q be the projection onto $R(L)$, and the restriction of L to $D(L) \cap V$ has a continuous inverse $H : R(L) \rightarrow V$. Moreover, let F be a nonlinear map with domain $D(F) = X$ such that $D(L) \cap D(F) \neq \emptyset$.

Lemma 2.9 ([1]). *For L, V, Q, H, F mentioned above, assume that the map $HQF : X \rightarrow V$ is uniformly bounded and compact on bounded sets. Let*

$$\Sigma = \{(s, v) \in \mathbf{R} \times V \mid v + HQF(su + v) = 0\},$$

where $u \in N(L)$. Then, for any $\alpha > 0$, there exists a connected subset Σ_α of Σ such that $\text{proj}_{\mathbf{R}} \supseteq [-\alpha, \alpha]$.

Finally, let us consider to decompose a subspace space of $C[0, 1]$ and rewrite (1.1) correspondingly.

As we all know, the Banach space $C[0, 1]$ possesses the usual norm $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$. Set

$$X = \{x \in C[0, 1] \mid x(0) = 0 = (1)\}. \quad (2.3)$$

Then X is also a Banach space with the same norm of $C[0, 1]$. Moreover, set

$$W = \left\{ x \in X \mid \int_0^1 x(t) \cdot \bar{x}(t) dt = 0 \right\}, \quad (2.4)$$

which is a closed subspace of X . Then, we can decompose X as

$$X = \text{span}\{\bar{x}(\cdot)\} \oplus W. \quad (2.5)$$

Hence, for any $h \in X$, h has a (unique) decomposition

$$h(t) = r\bar{x}(t) + \tilde{h}(t), \quad r \in \mathbf{R}, \quad \tilde{h} \in W,$$

and (1.1) can be rewritten in the form

$$\begin{cases} (-1)^{p+1}x^{(2p)}(t) + q(t)x(t) + g(x(t)) = r\bar{x}(t) + \tilde{h}(t), & t \in (0, 1), \\ x(0) = 0 = x(1), \end{cases} \quad (2.6)$$

where $r \in \mathbf{R}$, $\tilde{h} \in W$.

In next section, we only investigate the solutions of (2.6).

3. Main Result and its Proof

Theorem 3.1. *If q and g satisfies (Q1) – (Q3) and (G), respectively, then for any given $p, q, g, \tilde{h}$ and each given natural number n , there is a positive constant ε_n (depending on $p, q, g, \tilde{h}$, and n) such that (2.6) has at least n solutions if $0 < |r| < \varepsilon_n$. Finally, if $r = 0$, then (2.6) has infinitely many solutions.*

Proof. We will take three steps to complete the proof of Theorem 3.1.

Step 1. For any given p, q, g , and $\tilde{h}$, there is the equivalent condition of the existence of solutions for (2.6).

Set $Y = \{x \in C^{(2p-2)}[0, 1] \mid x^{(2k-2)}(0) = 0 = x^{(2k-2)}(1), k = 1, 2, \dots, p\}$, with the norm $\|x\|_1 = \max_{0 \leq k \leq 2p-2} \|x^{(k)}(t)\|_\infty$ for any $x \in Y$ and $Z = C^{2p}[0, 1] \cap Y$.

Define the two operators

$$L : Z \rightarrow X, (Lx)(t) = (-1)^{p+2}x^{(2p)}(t) - q(t)x(t), \text{ for all } x \in Z, t \in [0, 1], \quad (3.1)$$

and

$$F : Y \rightarrow X, (Fx)(t) = r\bar{x}(t) + \tilde{h}(t) - g(x(t)), \text{ for all } x \in Y, t \in [0, 1]. \quad (3.2)$$

Clearly, (2.6) is equivalent to the operator equation

$$Lx + Fx = 0. \quad (3.3)$$

It is well-known that L is a Fredholm operator of index zero from (Q3). Moreover, (Q1) implies $N(L) = \text{span}\{\bar{x}(t)\}$ and Fredholm alternative theorem implies $\text{Im } L = W$. Hence, there exist a continuous projection $P : X \rightarrow X$ such that

$$R(P) = R(L), R(I - P) = N(L). \quad (3.4)$$

and (3.3) is equivalent to the alternative system

$$\tilde{x} + KPF(c\bar{x} + \tilde{x}) = 0 \quad (\text{auxiliary equation}), \quad (3.5)$$

$$(I - P)F(c\bar{x} + \tilde{x}) = 0 \quad (\text{bifurcation equation}), \quad (3.6)$$

where K is the inverse of $L : Z \cap W \rightarrow R(L)$, and any $x \in X$ is written in the form $x(t) = c\bar{x}(t) + \tilde{x}(t)$, $c \in \mathbf{R}$, $\tilde{x}(t) \in W$. (3.2) implies that

$$KPF(c\bar{x} + \tilde{x}) = KP(\tilde{h} + r\bar{x} - g(c\bar{x} + \tilde{x})).$$

Then, for any given $c \in \mathbf{R}$, there exists one solution $\tilde{x} \in W$ of (3.5) by applying the Schauder fixed point theorem. We denote by $\sum$ the solution set of (3.5), i.e.,

$$\sum = \{(c, \tilde{x}) \in \mathbf{R} \times W \mid \tilde{x} = KPF(c\bar{x} + \tilde{x})\}. \quad (3.7)$$

Thanks to (3.4), we can choose

$$(I - P)x(t) = \left(\int_0^1 x(t) \cdot \bar{x}(t) dt \right) \bar{x}(t), \text{ for all } x \in X.$$

Then, we have

$$r = \int_0^1 g(c\bar{x}(t) + \tilde{x}(t))\bar{x}(t)dt$$

from (3.6) and (Q2). Hence, we obtain an equivalent condition of the existence of solutions for (2.6):

For any given $p, q, g,$ and $\tilde{h}$, (2.6) has one solution if and only if r belongs to the range of the function $\Gamma : \Sigma \rightarrow \mathbf{R}$, defined by

$$\Gamma(c, \tilde{x}) = \int_0^1 g(c\bar{x}(t) + \tilde{x}(t))\bar{x}(t)dt. \quad (3.8)$$

Moreover, $\Gamma(\Sigma)$ is a bounded interval. Indeed, let $r_1, r_2 \in \Gamma(\Sigma)$ and $r_1 \leq r_2$. Then, for $h_1 = r_1\bar{x}(t) + \tilde{h}(t)$, $h_2 = r_2\bar{x}(t) + \tilde{h}(t)$, (3.7) shows that (2.6) has at least one solution $\hat{x}(t)$ and $\check{x}(t)$, respectively. Hence, for any $h = r\bar{x}(t) + \tilde{h}(t)$, $r_1 \leq r \leq r_2$, (2.6) has a lower solution $\hat{x}(t)$ and an upper solution $\check{x}(t)$ from (Q2). That is, (2.6) has at least one solution. Therefore, $r \in \Gamma(\Sigma)$. The arbitrariness of r deduce that $\Gamma(\Sigma)$ is an interval. In addition, (G) implies that $\Gamma(\Sigma)$ is also bounded.

Step 2. (2.6) has at least n solutions of if $|r| \leq \varepsilon_n$, where ε_n will be given later on.

Let us define the following two functions:

$$p_1 : \Sigma \rightarrow \mathbf{R}, \quad \text{by } p_1(c, \tilde{x}) = c, \quad \text{for all } (c, \tilde{x}) \in \Sigma,$$

and

$$p_2 : \Sigma \rightarrow \ker P, \quad \text{by } p_2(c, \tilde{x}) = \tilde{x}, \quad \text{for all } (c, \tilde{x}) \in \Sigma.$$

Combining with (G), (3.5) implies that there is a constant $M_0 > 0$, independent of $c \in \mathbf{R}$, such that

$$\|\tilde{x}\|_\infty \leq M_0, \|\tilde{x}'\|_\infty \leq M_0, \|\tilde{x}''\|_\infty \leq M_0, \quad \text{for all } \tilde{x} \in p_2(\sum). \quad (3.9)$$

It is easy to see that there exists a constant $c_0^- > 0$ (depending on p and M_0) such that for any $c \geq c_0^-$,

$$\left\| \frac{c\bar{x}(t) + \tilde{x}'(t)}{\|c\bar{x} + \tilde{x}\|_\infty} - \frac{\bar{x}'(t)}{\|\bar{x}\|_\infty} \right\|_\infty$$

is small. In addition, we know that $\bar{x}'$ has only finite zeros in $(0, 1)$ from (Q3). Hence, any zero t_1 of $c\bar{x}'(t) + \tilde{x}'(t)$ must be close to only one zero t_0 of $\bar{x}'(t)$ for any $c \geq c_0^-$ and any $\tilde{x} \in p_2(\sum)$. In other words, $c\bar{x}'(t) + \tilde{x}'(t)$ has only finite zeros in $(0, 1)$ for any $c \geq c_0^-$ and any $\tilde{x} \in p_2(\sum)$. Therefore, let G be a primitive of g , if $(c, \tilde{x}) \in \sum$ and $c \geq c_0^-$, then

$$\begin{aligned} \Gamma(c, \tilde{x}) &= \int_0^1 [-G(c\bar{x}(t) + \tilde{x}(t)) + G(\|c\bar{x} + \tilde{x}\|_\infty)] \\ &\quad \times \frac{c[\bar{x}'(t)^2 - \bar{x}(t) \cdot \bar{x}''(t)] + \bar{x}'(t) \cdot \tilde{x}'(t) - \bar{x}(t) \cdot \tilde{x}''(t)}{(c\bar{x}'(t) + \tilde{x}'(t))^2} dt. \end{aligned} \quad (3.10)$$

Set

$$M_1 = \max\{1, \|\bar{x}\|_\infty, \|\bar{x}'\|_\infty\} \quad \text{and} \quad M_2 = \min_{[0, 1]} (\bar{x}'(t)^2 - \bar{x}(t) \cdot \bar{x}''(t)). \quad (3.11)$$

Clearly, $M_1 > 0$. (Q3) implies $M_2 > 0$. Denote $c_0^+ = \frac{2M_1M_0+1}{M_2}$. (3.9) and

(3.11) imply that

$$c[\bar{x}'(t)^2 - \bar{x}(t) \cdot \bar{x}''(t)] + \bar{x}'(t) \cdot \tilde{x}'(t) - \bar{x}(t) \cdot \tilde{x}''(t) > 0, \quad (3.12)$$

for any $c \geq c_0^+$.

Now let us set

$$c_0 = \max\{c_0^-, c_0^+\}, \quad G(s_1) = \max_{[0, T]} G, \quad \text{and} \quad G(s_2) = \min_{[M_1 T, (M_1+1)T]} G.$$

Then, there exists some natural number n_1 such that $s_1 + n_1 T \geq c_0 M_1 + M_0$.

Hence, we can set

$$s_1^- = \frac{s_1 + n_1 T - M_0}{M_1}, \quad s_1^+ = \frac{s_1 + n_1 T + M_0}{M_1}.$$

For the spaces X, W and the operators L, P, F, K mentioned in (2.3), (2.4), (3.1), (3.4), and (3.5), respectively, we can verify that KPF is uniformly bounded and compact on bounded sets. Therefore, we can deduce from Lemma 2.9 that for L there exists a connected subset $\sum_1$ of $\sum$ such

that $p_1(\sum_1) \supseteq [-s_1^+, s_1^+] \supseteq [s_1^-, s_1^+]$ and $(s_1^-, \tilde{x}_{s_1^-}), (s_1^+, \tilde{x}_{s_1^+}) \in \sum_1$.

Since $\|s_1^- \bar{x} + \tilde{x}_{s_1^-}\|_\infty \leq s_1 + n_1 T$, $\|s_1^+ \bar{x} + \tilde{x}_{s_1^+}\|_\infty \geq s_1 + n_1 T$ and the norm function $\|\cdot\|_0$ is continuous, there exists $(c_1, \tilde{x}_{c_1}) \in \sum_1$ with $\|c_1 \bar{x}(t) + \tilde{x}_{c_1}(t)\|_0 = s_1 + n_1 T$ such that

$$\int_0^1 \frac{1 - G(c_1 \bar{x}(t) + \tilde{x}_{c_1}(t)) + G(\|c_1 \bar{x} + \tilde{x}_{c_1}\|_\infty)}{(c_1 \bar{x}'(t) + \tilde{x}_{c_1}'(t))^2} dt > 0. \quad (3.13)$$

(3.13) combining with (3.10) and (3.12) implies $\Gamma(c_1, \tilde{x}_{c_1}) > 0$.

Similarly, we also set

$$s_2^- = \frac{s_2 + n_1 T - M_0}{M_0}, \quad s_2^+ = \frac{s_2 + n_1 T + M_0}{M_1}.$$

Then, there exists another connected subset $\sum_2$ of $\sum$ such that

$p_1(\sum_2) \supseteq [-s_2^+, s_2^+] \supseteq [s_2^-, s_2^+]$ and $(s_2^-, \tilde{x}_{s_2^-}), (s_2^+, \tilde{x}_{s_2^+}) \in \sum_2$. Therefore,

there exists $(c_2, \tilde{x}_{c_2}) \in \sum_2$ with $\|c_2 \bar{x}(t) + \tilde{x}_{c_2}(t)\|_\infty = s_2 + n_1 T$ such that

$$\int_0^1 \frac{-G(c_2 \bar{x}(t) + \tilde{x}_{c_2}(t)) + G(\|c_2 \bar{x} + \tilde{x}_{c_2}\|_\infty)}{(c_2 \bar{x}'(t) + \tilde{x}'_{c_2}(t))^2} dt < 0. \quad (3.14)$$

Together with (3.10) and (3.12), (3.14) implies that $\Gamma(c_2, \tilde{x}_{c_2}) < 0$.

Now for the interval $[s_1^-, s_2^+]$, there exists another connected subset $\sum_{1,2}$ of $\sum$ such that $p_1(\sum_{1,2}) \supseteq [s_1^-, s_2^+]$ and $\Gamma(\sum_{1,2}) \supseteq [-\delta_1, \delta_1]$, where

$$\delta_1 = \min\{\Gamma(c_1, \tilde{x}_{c_1}), -\Gamma(c_2, \tilde{x}_{c_2})\}.$$

Similarly, we can denote

$$G(s_{2i-1}) = \max_{[(i-1)(M_1+1)T, (i-1)M_1+iT]} G, \quad i = 2, \dots, n,$$

and

$$G(s_{2i}) = \min_{[(i(M_1+1)-1)T, i(M_1+1)T]} G, \quad i = 2, \dots, n.$$

Then, there exists some natural numbers n_i such that $s_{2i-1} + n_i T \geq c_0 M_1 + M_0$. In addition, we can obtain the intervals $[s_{2i-1}^-, s_{2i}^+]$, where

$$s_{2i-1}^- = \frac{s_{2i-1} + n_i T - M_0}{M_1}, \quad s_{2i}^+ = \frac{s_{2i} + n_i T + M_0}{M_1}, \quad i = 2, \dots, n.$$

For each interval mentioned above, there is a connected subset $\sum_{2i-1, 2i}$ of $\sum$ such that $p_1(\sum_{2i-1, 2i}) \supseteq [-s_{2i-1}^+, s_{2i}^+] \supseteq [s_{2i-1}^-, s_{2i}^+]$ and $\Gamma(\sum_{2i-1, 2i}) \supseteq [-\delta_i, \delta_i]$, where

$$\delta_i = \min\{\Gamma(c_{2i-1}, \tilde{x}_{c_{2i-1}}), -\Gamma(c_{2i}, \tilde{x}_{c_{2i}})\}, \quad i = 2, \dots, n,$$

$(c_{2i-1}, \tilde{x}_{c_{2i-1}}) \in \sum_{2i-1}$, $(c_{2i}, \tilde{x}_{c_{2i}}) \in \sum_{2i}$, and $\sum_{2i-1}$, $\sum_{2i}$ are the connected subsets of $\sum$. Set $\varepsilon_n = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. Hence, if $|r| < \varepsilon_n$, then (2.6) has at least n solutions x of the form

$$x = c_i \bar{x} + \tilde{x} \quad \text{with} \quad c_i \in [s_{2i-1}^-, s_{2i}^+] \quad (i = 1, 2, \dots, n). \quad (3.15)$$

Step 3. If $r = 0$, then (2.6) has infinitely many solutions.

The structure of x in (3.15) implies that the conclusion mentioned above holds naturally.

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