Journal of Pure and Applied Mathematics: Advances and Applications
Volume 13, Number 2, 2015, Pages 97-113

NONTRIVIAL SOLUTIONS OF ASYMPTOTICALLY LINEAR SECOND ORDER INDEFINITE HAMILTONIAN SYSTEMS

YINGYING CHEN and JINHANG XU

Department of Mathematics
Nanjing Normal University
Nanjing, Jiangsu 210023
P. R. China
e-mail: chenyingying0620@163.com
xujinhang305@163.com

Abstract

Capietto, Dalbono and Portaluri proposed a class of strongly indefinite second order Hamiltonian systems with Dirichlet boundary conditions. We will investigate this system and obtain some new results.

1. Introduction and Main Results

Many famous mathematicians such as Ekeland [1]; Long [2, 3]; Mawhin and Willem [4]; and Rabinowitz [5] have investigated Hamiltonian systems. In [6, 7, 8, 9], some problems concerning Hamiltonian systems have also been investigated. In 2010, Capietto et al. [10] proved a multiplicity result for the class of strongly indefinite nonlinear second-order asymptotically linear systems with Dirichlet boundary conditions:

2010 Mathematics Subject Classification: 34A12, 34B15, 34L30, 49R50.
Keywords and phrases: strongly indefinite, Hamiltonian system, Dirichlet conditions.
Received May 11, 2015

$$J\ddot{x} + S(t,\ x(t))x(t) = 0, \tag{1.1}$$

$$x(0) = 0 = x(1), \tag{1.2}$$

where $J = \begin{pmatrix} I_{n-\nu} & 0 \\ 0 & -I_{\nu} \end{pmatrix}$, $S : [0, 1] \times \mathbf{R}^n \to \mathcal{L}_s(\mathbf{R}^n)$ is continuous. We first introduce an index theory for the linear second order indefinite Hamiltonian system (1.2) and

$$P\ddot{x} + B(t)x = 0, \tag{1.3}$$

where $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B \in L^{\infty}([0, 1],\ \mathcal{L}_s(\mathbf{R}^2))$ in Section 2. Then, in Section 3, we investigate nontrivial solutions of the asymptotically linear second-order indefinite Hamiltonian system (1.2) and

$$P\ddot{x} + V'(t,\ x) = 0, \tag{1.4}$$

where $V \in C^1([0, 1],\ \mathbf{R}^2)$.

In Section 4, we will recall some results from [11] concerning index theory for self-adjoint operator equations and multiple solutions for asymptotically linear operator equations, which are used in Sections 2-3. Our main results will be obtained by applying some associated results for operator equations from [11].

2. Index Theory for Second Order Linear Indefinite Hamiltonian Systems

In this chapter, we discuss the problem

$$P\ddot{x} + B(t)x = 0, \tag{2.1}$$

$$x(0) = 0 = x(1), \tag{2.2}$$

where $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B \in L^{\infty}([0, 1],\ \mathcal{L}_s(\mathbf{R}^2))$.

Define $X = L^2([0, 1], \mathbf{R}^2), (Ax)(t) = -P\ddot{x}(t)$ with $D(A) = \{x \in H^2([0, 1]; \mathbf{R}^2) | x(0) = 0 = x(1)\}$, then $A : D(A) \subset X \to X$.

We now prove A is self-adjoint. If so, $\sigma_r(A) = \emptyset$. To this end, it suffices to prove

(i) $(Ax, y) = (x, Ay), \forall x, y \in D(A)$, and

(ii) $(Ax, y) = (x, z), \forall x \in D(A) \Rightarrow y \in D(A)$, and $Ay = z$.

In fact $\forall x, y \in D(A)$,

$$\begin{aligned}
(Ax, y) &= \int_0^1 (-P\ddot{x}(t), y(t))dt = \int_0^1 -\ddot{x}_1(t)y_1(t)dt + \int_0^1 \ddot{x}_2(t)y_2(t)dt \\
&= -\dot{x}_1(t)y_1(t)|_0^1 + \int_0^1 \dot{x}_1(t)\dot{y}_1(t)dt + \dot{x}_2(t)y_2(t)|_0^1 - \int_0^1 \dot{x}_2(t)\dot{y}_2(t)dt \\
&= \int_0^1 \dot{x}_1(t)\dot{y}_1(t)dt - \int_0^1 \dot{x}_2(t)\dot{y}_2(t)dt \\
&= x_1(t)\dot{y}_1(t)|_0^1 - \int_0^1 x_1(t)\ddot{y}_1(t)dt - x_2(t)\dot{y}_2(t)|_0^1 + \int_0^1 x_2(t)\ddot{y}_2(t)dt \\
&= -\int_0^1 x_1(t)\ddot{y}_1(t)dt + \int_0^1 x_2(t)\ddot{y}_2(t)dt \\
&= \int_0^1 (x(t), -P\ddot{y}(t))dt \\
&= (x, Ay).
\end{aligned}$$

Thus (i) holds. To prove (ii), suppose

$$\int_0^1 (-P\ddot{x}(t), y(t))dt = \int_0^1 (x(t), z(t))dt, \quad \forall x \in D(A).$$

Because $z \in L^2([0, 1], \mathbf{R}^2)$, there exists

$$w \in H_0^2([0, 1]; \mathbf{R}^2) \equiv D(A),$$

such that $Aw = z$. In fact, there exists

$$x(t) = -t\int_0^1\int_0^s f(\tau)d\tau ds + \int_0^t\int_0^s f(\tau)d\tau ds = \int_0^1 G(t, s)f(s)ds,$$

with $G(t, s) = (t-1)s$ when $s \leq t$ and $G(t, s) = t(s-1)$ when $s > t$. $x(t)$ is a solution of $\ddot{x}(t) = f(t), t \in (0, 1), x(0) = 0 = x(1)$. Thus $w = -P\int_0^1 G(t, s)z(s)ds$, and $(Ax, y) = (x, Aw) = (Ax, w)$ so $(Ax, y - w) = 0$, $\forall x \in D(A)$, hence $y = w \in D(A)$.

Now, we want to prove $\sigma(A) = \sigma_d(A)$, and it suffices to prove

(i) $\sigma_p(A) = \{\pm k^2\pi^2\}$;

(ii) $\dim \ker(A - \lambda I_2) \leq 1$ for $\lambda = \pm k^2\pi^2$;

(iii) $\sigma_c(A) = \emptyset$.

At first $\lambda \in \sigma_p(A) \Leftrightarrow \exists x \in D(A) \setminus \{0\}$ such that $(A - \lambda I_2)x = 0$, equivalently, $-P\ddot{x} - \lambda x = 0, x(0) = 0 = x(1)$.

By substituting $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we obtain

$$\ddot{x}_1 + \lambda x_1 = 0, \quad \ddot{x}_2 - \lambda x_2 = 0,$$

$$x_1(0) = 0 = x_1(1), \quad x_2(0) = 0 = x_2(1).$$

So $\lambda \in \{\pm k^2\pi^2\}$ for $k \in N^*$ and (i) holds.

By definition, $\ker(A - \lambda I_2) = \begin{pmatrix} c \sin k\pi t \\ 0 \end{pmatrix}$ when $\lambda = k^2\pi^2$, and $\ker(A - \lambda I_2) = \begin{pmatrix} 0 \\ c \sin k\pi t \end{pmatrix}$ when $\lambda = -k^2\pi^2$, where c is a constant, otherwise $\ker(A - \lambda I_2) = \{0\}$.

(iii) That $\sigma_c(A) = \emptyset$ is equivalent to the following: given $\lambda \in \mathbf{R} \setminus \sigma_p(A)$, $R(A - \lambda I_2) = X$. This is equivalent to $\forall f \in X, -P\ddot{x} - \lambda x = f$, $x(0) = 0 = x(1)$ has a unique solution. So, we have three possibilities:

Case 1. $\lambda = 0$. In this case, $-P\ddot{x} = f(t)$, $t \in (0, 1)$, $x(0) = 0 = x(1)$ has a unique solution: $x = -P\int_0^1 G(t, s) f(s) ds$.

Case 2. $\lambda > 0$. Set $\lambda = u^2$, $u > 0$. Then, suppose the solution of $\ddot{x} + u^2 x = f(t)$ is

$$x(t) = c_1(t)\cos ut + c_2(t)\sin ut.$$

Because

$$\dot{x}(t) = -c_1(t)u \sin ut + c_2(t)u \cos ut + \dot{c}_1(t)\cos ut + \dot{c}_2(t)\sin ut.$$

By setting

$$\dot{c}_1(t)\cos ut + \dot{c}_2(t)\sin ut = 0,$$

and then

$$\ddot{x}(t) = -u^2 x(t) - \dot{c}_1(t)u \sin ut + \dot{c}_2(t)u \cos ut,$$

$x(0) = 0$ implies $c_1 = 0$, and because $x(1) = 0$,

$$0 = -\frac{1}{u}\int_0^1 f(\tau)\sin u\tau d\tau \cos u + \left(c_2 + \frac{1}{u}\int_0^1 f(\tau)\cos u\tau d\tau\right)\sin u.$$

Because $\lambda = u^2$, $\sin u \neq 0$, c_2 is determined uniquely, and hence $\ddot{x} + u^2 x = f(t)$, $x(0) = 0 = x(1)$ has a unique solution.

Case 3. $\lambda < 0$, let $\lambda = -u^2$, with $u > 0$.

Suppose the solution of $\ddot{x} - u^2 x = f(t)$ is

$$x(t) = c_1(t)e^{-ut} + c_2(t)e^{ut}.$$

Then

$$\dot{x}(t) = -uc_1(t)e^{-ut} + uc_2(t)e^{ut} + \dot{c}_1(t)e^{-ut} + \dot{c}_2(t)e^{ut}.$$

By setting

$$\dot{c}_1(t)e^{-ut} + \dot{c}_2(t)e^{ut} = 0.$$

$$\ddot{x}(t) = u^2 x(t) - u\dot{c}_1(t)u^{-ut} + u\dot{c}_2(t)u^{ut}.$$

Substituting into the equation $\ddot{x} - u^2 x = f(t)$, we obtain

$$-u\dot{c}_1(t)e^{-ut} + u\dot{c}_2(t)e^{ut} = f(t).$$

Thus

$$2\dot{c}_1(t)e^{-ut} = -\frac{1}{u}f(t), \quad 2\dot{c}_2(t)e^{ut} = \frac{1}{u}f(t),$$

and

$$c_1(t) = -\frac{1}{2u}\left(\int_0^t f(\tau)e^{u\tau}d\tau + c_1\right),$$

$$c_2(t) = \frac{1}{2u}\left(\int_0^t f(\tau)e^{-u\tau}d\tau + c_2\right).$$

Finally,

$$x(t) = -\frac{1}{2u}\left(\int_0^t f(\tau)e^{u\tau}d\tau + c_1\right)e^{-ut} + \frac{1}{2u}\left(\int_0^t f(\tau)e^{-u\tau}d\tau + c_2\right)e^{ut}.$$

Because $x(0) = 0 = x(1)$, c_1 and c_2 satisfy

$$c_1 - c_2 = 0, \quad \left(\int_0^1 f(\tau)e^{-u\tau}d\tau + c_2\right)e^u - \left(\int_0^1 f(\tau)e^{u\tau}d\tau + c_1\right)e^{-u} = 0.$$

Then c_1 and c_2 are determined uniquely and it means that

$$\ddot{x} - u^2 x = f(t), \quad x(0) = 0 = x(1)$$

has a unique solution.

Similar to Definition 4.1, we have the following definitions:

Definition 2.1. For any $\overline{B} \in L^2([0, 1], \mathcal{L}_s(\mathbf{R}^2))$, we define

$$\nu_P(\overline{B}) \equiv \dim \ker(A - B),$$

where $B \in \mathcal{L}_s(X)$ is defined by $(Bx)(t) = \overline{B}(t)x(t)$.

Definition 2.2. For any $\overline{B}_1, \overline{B}_2 \in L^2([0, 1], \mathcal{L}_s(\mathbf{R}^2))$ with $\overline{B}_1 < \overline{B}_2$, we define

$$I_P(\overline{B}_1, \overline{B}_2) \equiv \sum_{\lambda \in [0,1)} \nu_P((\overline{B}_1 + \lambda(\overline{B}_2 - \overline{B}_1))),$$

and for any $\overline{B}_1, \overline{B}_2 \in L^2([0, 1], \mathcal{L}_s(\mathbf{R}^2))$, we define

$$I_P(\overline{B}_1, \overline{B}_2) = I_P(\overline{B}_1, cI_2) - I_P(\overline{B}_2, cI_2),$$

for some constant c satisfying $cI_2 > \overline{B}_1$, $cI_2 > \overline{B}_2$.

Here for any $\overline{B}_1, \overline{B}_2 \in L^2([0, 1], \mathcal{L}_s(\mathbf{R}^2))$, we write $\overline{B}_1 \leq \overline{B}_2$ if $\overline{B}_1(t) \leq \overline{B}_2(t)$ for a.e. $t \in [0, 1]$; write $\overline{B}_1 < \overline{B}_2$ if $\overline{B}_1 \leq \overline{B}_2$ and $\overline{B}_1(t) < \overline{B}_2(t)$ on a subset of $[0, 1]$ with positive measure.

Definition 2.3. For any $\overline{B} \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^2))$, we define

$$i_P(\overline{B}) = I_P(0, B).$$

From Definitions 2.1-2.3, $\nu_P(cI_2) = 0$ when $c \in \mathbf{R} \setminus \{\pm k^2\pi^2\}(k \neq 0)$, $\upsilon_P(cI_2) = 1$ when $c \in \{\pm k^2\pi^2\}$; $i_P(cI_2) = k$ when $|c| \in (k^2\pi^2, (k+1)^2\pi^2]$, $i_P(cI_2) = 0$ when $c \in (-\pi^2, \pi^2)$.

Proposition 2.4. *For any* $\overline{B}_1, \overline{B}_2 \in L^2([0, 1], \mathcal{L}_s(\mathbf{R}^2))$. *If* $\overline{B}_1 \leq \overline{B}_2$, *then* $i_P(\overline{B}_1) \leq i_P(\overline{B}_2)$, $i_P(\overline{B}_1) + \nu_P(\overline{B}_1) \leq i_P(\overline{B}_2) + \nu_P(\overline{B}_2)$; *if* $\overline{B}_1 < \overline{B}_2$, *then* $i_P(\overline{B}_1) + \nu_P(\overline{B}_1) \leq i_P(\overline{B}_2)$.

This proposition comes from [11, Proposition 5.1.2 (iii)].

3. Nontrivial Solutions for Asymptotically Linear Second Order Indefinite Hamiltonian Systems

In this chapter, we investigate the following problem:

$$P\ddot{x} + V'(t, x) = 0, \tag{3.1}$$

$$x(0) = 0 = x(1), \tag{3.2}$$

where $V \in C^1([0, 1] \times \mathbf{R}^{2n})$.

From Theorems 4.3-4.5, we have the following theorems:

Theorem 3.1. *Assume that* $V \in C^1([0, 1] \times \mathbf{R}^2)$ *satisfies*

(V_1) *there exist* $\overline{B}_1, \overline{B}_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^2))$ *with* $i_P(\overline{B}_1) = i_P(\overline{B}_2)$, $\nu_P(\overline{B}_2) = 0$ *such that* $V'(t, x) = \overline{B}(t, x)x + h(t, x)$, *where* $\overline{B}_1(t) \leq \overline{B}(t, x) \leq \overline{B}_2(t)$, $\forall (t, x) \in C^1([0, 1] \times \mathbf{R}^{2n})$ *and* $h(t, x)$ *bounded*, *then* (3.1)-(3.2) *has one solution*; *if we assume*

(V_2) *there exist a symmetric* $n \times n$ *matrix* $C(t, x)$, *which is continuous with respect to* $(t, x) \in [0, 1] \times \mathbf{R}^n$ *and* $C_1, C_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^2))$ *with* $i_P(C_1) = i_P(C_2)$, $\nu_P(C_2) = 0$ *such that* $C_1(t) \leq C(t, x) \leq C_2(t)$, $\forall (t, x) \in [0, 1] \times \mathbf{R}^2$ *and* $V'(t, x) = C(t, x)x + o(|x|)$ *as* $|x| \to 0$.

Then (3.1)-(3.2) *has a nontrivial solution if* $i_P(\overline{B}_1) - i_P(C_1)$ *is odd.*

Theorem 3.2. *Assume that*

(V_3) $V \in C^2([0, 1] \times \mathbf{R}^2)$ *and* $V'(t, 0) \equiv 0$;

(V_4) *there exist* $\overline{B}_1, \overline{B}_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^2))$ *satisfying* $i_P(\overline{B}_1) = i_P(\overline{B}_2)$, $\nu_P(\overline{B}_2) = 0$ *and*

$$\overline{B}_1(t) \leq V''(t, x) \leq \overline{B}_2(t),$$

$\forall (t, x)$ *with* $|x| \geq r > 0$;

(V_5) *with* $\overline{B}_0 \equiv V''(t, 0)$, *we have*

$$i(B_1) \notin [i_P(\overline{B}_0), i_P(\overline{B}_0) + \nu_P(\overline{B}_0)].$$

Then (3.1)-(3.2) *has one nontrivial solution. Under the further assumption*,

(V_6) $\nu_P(\overline{B}_0) = 0$ *and* $|i_P(\overline{B}_1) - i_P(\overline{B}_0)| \geq 2$, (3.1)-(3.2) *has two nontrivial solutions.*

Proof of Theorem 3.1. Define $X = L^2([0, 1], \mathbf{R}^{2n})$, $(Ax)(t) = -P\ddot{x}(t)$, then $D(A) = \{x \in H^2([0, 1], \mathbf{R}^2) | x(0) = 0 = x(1)\} \subset X$. As we have proved in Section 2, A is an unbounded self-adjoint operator with $\sigma(A) = \sigma_d(A)$ is unbounded from both above and below and $Z \equiv D(|A|^{\frac{1}{2}}) = H_0^1([0, 1], \mathbf{R}^2)$ as we have proved in [12, Proposition 1.17]. Define

$$\Phi(x) = \int_0^1 V(t, x(t))dt, \quad \forall x \in X. \tag{3.3}$$

By assumption (V_1), $\Phi \in C^1(X)$ and

$$\Phi'(x)y = (\nabla\Phi(x), y) = \int_0^1 V'(t, x(t)) \cdot y(t)dt, \quad \forall x \in X. \tag{3.4}$$

Set

$$(B(x)y)(t) = \overline{B}(t,\ x(t))y(t),\ C(x)(t) = h(t,\ x(t)),\ (B_i x)(t) = \overline{B}_i(t)x(t),$$

$$i = 1,\ 2 \tag{3.5}$$

in view of Theorem 4.3, there exists one solution for (3.1)-(3.2); if we further assume (V_2), Φ defined as in (3.3) satisfies (Φ_2) of Theorem 4.3, then (3.1)-(3.2) has at least one nontrivial solution $i_P(\overline{B}_1) - i_P(C_1)$ is odd.

Proof of Theorem 3.2. The following trick comes from [11]. Note that

$$V'(t,\ x) = \int_0^1 V''(t,\ \theta x)d\theta x + V'(t,\ 0).$$

Define

$$B(t,\ x) = \int_0^1 V''(t,\ \theta x)d\theta, \quad |x| \geq r / \delta,$$

and otherwise

$$B(t,\ x) = B_1(t).$$

When $\|x\| \geq r / \delta$

$$\int_0^1 V''(t,\ \theta x)d\theta = \int_\delta^1 + \int_0^\delta V''(t,\ \theta x)d\theta = I_1 + I_2,$$

then, in view of (V_3)

$$(1 - \delta)B_1(t) \leq I_1 \leq B_2(t)(1 - \delta).$$

There exists $M > 0$ such that

$$-M \leq V''(t,\ x) \leq M,$$

for all $(t, x) \in [0, 1] \times \mathbf{R}^n$. Thus

$$-M\delta \leq I_2 \leq M\delta,$$

and if $B_2(t) \leq M$,

$$I_1 + I_2 \leq B_2(t) + M\delta - B_2(t)\delta \leq B_2(t) + 2\delta M;$$

and at the same time $B_1(t) \leq B_2(t) \leq M$, then

$$I_1 + I_2 \geq B_1(t) - \delta B_1(t) - \delta M \geq B_1(t) - 2\delta M.$$

Because $i(B_1) = i(B_2)$ and $\nu(B_1) = \nu(B_2) = 0$, there exists $\epsilon > 0$ such that

$$i(B_1 - \epsilon) = i(B_1) = i(B_2) = i(B_2 + \epsilon),$$

and

$$\nu(B_1 - \epsilon) = 0 = \nu(B_2 + \epsilon).$$

Choose $M > 0$ such that

$$-M \leq B_1(t) \leq B_2(t) \leq M,$$

and $\delta > 0$ such that

$$2\delta M < \epsilon.$$

Then

$$B_1(t) - \epsilon \leq B(t, x) \leq B_2(t) + \epsilon,$$

and

$$h(t, x) = V'(t, x) - B(t, x)x$$

is bounded. Theorem 4.4 applies and the results hold.

Corollary 3.3. *Assume that*

$$((k-1)^2\pi^2 + \epsilon)I_n \leq V''(t, x) \leq (k^2\pi^2 - \epsilon)I_n, \quad \forall (t, x),$$

where $\epsilon > 0$ *is small and* $k \in \mathbf{N}$. *Then* (3.1)-(3.2) *has at least one solution.*

Proof. Let $B_1(t) := (k-1)^2\pi^2 I_n$, $B_2(t) := (k^2\pi^2 - \frac{\epsilon}{2})I_n$, then $N(t, x) := V(t, x) - \frac{1}{2}(B_1(t)x, x)$ is convex with respect to x, since $N''(t, x) \geq \epsilon I_{2n}$, and

$$V(t, x) = \int_0^1 ds(\int_0^1 V''(t, \tau sx)xsd\tau, x) + (V'(t, \theta), x) + V(t, \theta)$$

$$\leq \frac{1}{2}(k^2\pi^2 - \epsilon)|x|^2 + (V'(t, \theta), x) + V(t, \theta)$$

$$\leq \frac{1}{2}b|x|^2 + c,$$

where $b := k^2\pi^2 - \frac{\epsilon}{2}$, $c > 0$ is a constant. Since $i_P(bI_n) = i_P((k-1)^2\pi^2 I_n)$ $+ \nu_P((k-1)^2\pi^2 I_n)$, (3.1)-(3.2) has a solution in view of Theorem 3.2.

In the end of this chapter as in Dong [11], we have the following result:

Theorem 3.4. *Assume that V satisfies*

(V_7) $V \in C^2([0, 1] \times \mathbf{R}^{2n})$ *and there exist* $B_1, B_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^{2n}))$ *with* $i_P(B_1) = i_P(B_2)$, $\nu_P(B_2) = 0$ *such that*

$$B_1(t) \leq V''(t, x) \leq B_2(t), \tag{3.6}$$

$\forall (t, x) \in [0, 1] \times \mathbf{R}^{2n}$ *with* $|x| \geq r > 0$.

(V_8) $V'(t, 0) \equiv 0$ *and* $\nu_P(B_0) = 0$, *where we set* $B_0(t) \equiv V''(t, 0)$.

(V_9) $V(t, -x) = V(t, x)$ *for all* $(t, x) \in [0, 1] \times \mathbf{R}^n$,

then (3.1)-(3.2) *has* $|i_P(B_0) - i_P(B_1)|$ *distinct pairs of solutions.*

Proof. From (3.4) and (V_8),

$$\Phi'(x)y = \int_0^1 V'(t, x)ydt = \int_0^1 \int_0^1 V''(t, \theta x)xyd\theta dt, \tag{3.7}$$

thus (Φ_1) is satisfied in view of (V_7).

And $(\Phi''(x)y, z) = \int_0^1 (V''(t, x)y, z)dt$, with $V''(t, 0) = B_0(t)$ and $\nu_P(B_0) = 0$, (Φ_4) is satisfied, thus Theorem 4.5 yields $|i_P(B_0) - i_P(B_1)|$ distinct pairs of solutions for (3.1)-(3.2).

4. Appendix

In this section, we will recall some results from [11] concerning index theory for self-adjoint operator equations and multiple solutions of asymptotically linear operator equations, which are used in Sections 2-3.

Let X be a real separable infinite dimensional Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be an unbounded linear self-adjoint operator with domain $D(A)$ satisfying $\sigma(A) = \sigma_d(A)$ and which is unbounded from both above and below. In [11], the authors establish an index theory for the equation

$$Ax - Bx = 0, \tag{4.1}$$

for any $B \in \mathcal{L}_s(X)$.

Definition 4.1. (i) For any $B \in \mathcal{L}_s(X)$, we define

$$\nu_A(B) = \dim \ker(A - B), \tag{4.2}$$

$\nu_A(B)$ is called the nullity of B.

(ii) For any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, we define

$$I_A(B_1, B_2) = \sum_{\lambda \in [0,1)} \nu_A((1 - \lambda)B_1 + \lambda B_2); \tag{4.3}$$

and for any $B_1, B_2 \in \mathcal{L}_s(X)$, we define

$$I_A(B_1, B_2) = I_A(B_1, kI) - I_A(B_2, kI), \tag{4.4}$$

where $I : X \to X$ is the identity map and $kI > B_1, kI > B_2$ for some real number $k > 0$.

(iii) For any $B \in \mathcal{L}_s(X)$, we define

$$i_A(B) = i_A(B_0) + i_A(B_0, B), \tag{4.5}$$

where $B_0 \in \mathcal{L}_s(X)$ is fixed and $i_A(B_0)$ is a prescribed integer.

Here for any $B_1, B_2 \in \mathcal{L}_s(X)$, we write $B_1 \leq B_2$ with respect to X_1 (a linear subspace of X) if and only if $(B_1x, x) \leq (B_2x, x)$ for any $x \in X_1$; we write $B_1 < B_2$ with respect to X_1 if and only if $(B_1x, x) < (B_2x, x)$ for any $x \in X_1 \setminus \{0\}$. If $X_1 = X$, we just write $B_1 \leq B_2$ or $B_1 < B_2$. As in [11], we call $I_A(B_1, B_2)$ the relative Morse index between B_1 and B_2, we call $i_A(B)$ the index of B and $i_A(B_0)$ is called an initial index. Generally, the initial index can be any prescribed integer and the index $i_A(B)$ also depends on B_0 and the initial index. However, for a concrete operator A in applications, we will choose special B_0 and $i_A(B_0)$ so that $i_A(B)$ looks like natural as much as possible.

The index and nullity established above have the following monotone property.

Proposition 4.2. *For any* $B_1, B_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^{2n}))$, *if* $B_1 \leq B_2$, *then* $i_A(B_1) \leq i_A(B_2)$ *and* $i_A(B_1) + \nu_A(B_1) \leq i_A(B_2) + \nu_A(B_2)$; *if* $B_1 < B_2$, *then* $i_A(B_1) + \nu_A(B_1) \leq i_A(B_2)$.

Consider the following operator equation:

$$Ax - \nabla\Phi(x) = 0, \tag{4.6}$$

where $\nabla\Phi : Z \equiv D(|A|^{\frac{1}{2}}) \to X$ satisfying $(\nabla\Phi(x), y) = \Phi'(x)y, \forall x, y \in Z$.

Theorem 4.3 [11, Theorem 8.4.1]**.** *Assume that* $\Phi \in C^1(Z, \mathbf{R})$ *satisfies*

(Φ_1) *There exist* $B : X \to \mathcal{L}_s(X), B_1, B_2 \in \mathcal{L}_s(X)$ *with* $i_A(B_1) = i_A(B_2)$, $\nu_A(B_2) = 0$ *such that* $B_1 \leq B(x) \leq B_2$ *and* $\nabla\Phi(x) - B(x)x = C(x)$, $\|C(x)\| \leq M$ *for all* $x \in Z$ *and some* $M > 0$, *then* (4.6) *has at least one solution*, *if we further assume that*

(Φ_2) *there exist* $B_0 : X \to \mathcal{L}_s(X)$ *and* $B_{01}, B_{02} \in \mathcal{L}_s(X)$ *with* $i_A(B_{01}) = i_A(B_{02}), \nu_A(B_{02}) = 0$ *and some* $r > 0$ *such that* $B_{01} \leq B_0(x) \leq B_{02}$, $\nabla\Phi(x) = B_0(x)$ *for all* $x \in D(A)$ *with* $\|x\| \leq r$.

Then (4.6) *has one nontrivial solution provided* $i_A(B_1) - i_A(B_{01})$ *is odd.*

Theorem 4.4. *Assume that*

(i) $\Phi \in C^2(Z)$ *with* $Z := D(|A|^{\frac{1}{2}})$, $\Phi'(0) = 0$, $\Phi''(x)$ *exists and is bounded for* $x \in X$;

(ii) *there exist* $B_1, B_2 \in \mathcal{L}_s(X)$ *with* $i_A(B_1) = i_A(B_2)$, $\nu_A(B_2) = 0$ *and* $B : X \to \mathcal{L}_s(X)$, $C : X \to X$ *such that*

$$\Phi'(x) = B(x)x + C(x) \text{ for any } x,$$

$$B_1 \leq B(x) \leq B_2,\ C(x) \text{ is bounded}; \tag{4.7}$$

(iii) *with* $B_0 := \Phi''(0)$, *we have*

$$i_A(B_1) \notin [i_A(B_0), i_A(B_0) + \nu_A(B_0)].$$

Then (4.2) *has a nontrivial solution* $x = x_0$.

Under the further assumption that

(iv) $\nu_A(B_0) = 0$ *and* $|i_A(B_1) - i_A(B_0)| \geq \nu_A(\Phi''(x_0))$, (4.2) *has two nontrivial solutions.*

Theorem 4.5 [11, Theorem 8.5.1]. *Assume that* $\Phi \in C^1(Z, \mathbf{R}^2)$ *satisfies* (Φ_1) *and*

(Φ_3) $\Phi(-x) = \Phi(x),\ \forall x \in Z,$

(Φ_4) $\Phi \in C^2(Z)$ *and there exist* $\widetilde{B} : Z \to \mathcal{L}_s(X)$ *and* $M > 0$ *with* $|\widetilde{B}(x)| \le M$ *such that*

$$\Phi''(x)(u, v) = (\widetilde{B}(x)u, v), \quad \forall x, u, v \in Z, \tag{4.8}$$

and let $B_0 \equiv \widetilde{B}(0)$ *with* $\nu_A(B_0) = 0$.

Then (4.6) *has* $|i_A(B_0) - i_A(B_1)|$ *distinct pairs of nontrivial solutions.*

References

[1] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Berlin, Springer, 1990.

[2] Y. Long, Index Theory and it's Applications in Hamiltonian Systems (in Chinese), Academic Press, 1993.

[3] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Mathematics, No. 207, Birkhauser, Basel, 2002.

[4] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1998.

[5] P. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, C. B. M. 65 (1986).

[6] A. Du, Morse Index Theory and Existence of Asymptotically Linear $2p$-order Hamiltonian Systems, Nanjing Normal University, 2008.

[7] I. Ekeland, N. Ghoussoub and H. Terani, Multiple solutions for a classical problem in the calculus of varations, J. Differential Equations 131 (1996), 299-343.

[8] Z. Wang, Multiple solutions for indefinite functionals and applications to asymptotically linear problem, Acta Math. Sinica (N. S.) 5 (1989), 101-113.

[9] J. Yue, Index Theory and Existence of Asymptotically Linear Second Order Ordinary Differential Systems, Nanjing Normal University, 2008.

[10] A. Capietto, F. Dalbono and A. Portaluri, A multiplicity result for a class of strongly indefinite asymptotically linear second order systems, Nonlinear Anal. 72 (2010), 2874-2890.

[11] Y. Dong, Index Theory for Hamiltonian Systems and Multiple Solution Problems, Science Press, Beijing, 2015.

[12] Y. Chen, Y. Dong and Y. Shan, Existence of solutions for sub-linear or super-linear operator equations, Science China Mathematics, Online preprint.

■