A LOCAL STRONG SOLUTION OF THE NAVIER-STOKES PROBLEM IN $L_2(\Omega)$

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Abstract

In this paper, we prove that the Navier-Stokes initial value problem (1) has a unique smooth local strong solution $u(t, x)$ and $p(t, x)$ if the following condition are satisfied:

- (1) $f \in DL_2(\Omega)$ and f is Hölder continuous about t on $(0, \infty)$;
- (2) The initial value $u_0 \in D((-\Delta)^{\frac{1}{2}})$ is smooth, divergence-free vector field.

The Navier-Stokes initial value problem can be written in its classical form as

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \Delta u - \nabla p - (u \bullet \nabla)u + f, \quad x \in \Omega, \quad t \in (0, \infty), \\ \nabla \bullet u \equiv \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \quad t \in (0, \infty), \\ u|_{t=0} = u_0, \quad x \in \Omega, \end{array} \right. \quad (1)$$

where Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$ of class C^3 , $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity field, $u_0 = u_0(x)$ is the initial velocity, $p = p(t, x)$ is the pressure, $f = f(t, x)$ is the external force,

$$\Delta u = \left(\sum_{i=1}^3 \frac{\partial^2 u_1}{\partial x_i^2}, \sum_{i=1}^3 \frac{\partial^2 u_2}{\partial x_i^2}, \sum_{i=1}^3 \frac{\partial^2 u_3}{\partial x_i^2} \right).$$

In these four equations u, p are unknown and f, u_0 are given. The boundary condition imposed on the velocity at Ω is homogeneous.

The existence, uniqueness, and regularity properties of solutions for the Navier-Stokes problems are extensively studied. There is an extensive literature on the solvability of the initial value problem for Navier-Stokes equations.

Let $L_2(\Omega)$ be the Banach space of real vector functions in $L^2(\Omega)$.

That is,

$$L_2(\Omega) = \{h : \Omega \rightarrow R^3, h = (h_1, h_2, h_3), h_i \in L^2(\Omega) (i = 1, 2, 3)\}.$$

For $u = (u_1, u_2, u_3) \in L_2(\Omega)$, we define the norm

$$\|u\|_{L_2(\Omega)} = (\sum_{i=1}^3 \|u_i\|_{L^2(\Omega)}^2)^{\frac{1}{2}},$$

then $L_2(\Omega)$ is a Banach space. The set of all real vector functions u such that $\operatorname{div} u = 0$ and $u \in C_0^\infty(\Omega)$ is denoted by $DL_2(\Omega)$. Let $DL_2(\Omega)$ be the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L_2(\Omega)$. If $u \in C^\infty(\Omega)$, then

$u \in DL_2(\Omega)$ if and only if $\operatorname{div} u = 0$ in Ω and $u_n = 0$ on $\partial\Omega$ (p. 270 in [3]). (2)

In this paper, we always consider the spaces of vector value functions on Ω .

We have

$$DL_2(\Omega) \subset L_2(\Omega) = W^{0,2}(\Omega), \quad \|\bullet\|_{DL_2(\Omega)} = \|\bullet\|_{L_2(\Omega)},$$

$$L_2(\Omega) = DL_2(\Omega) \oplus DL_2(\Omega)^\perp.$$

From [2] and [3], we see that $DL_2(\Omega)^\perp = \{\nabla h; h \in W^{1,2}(\Omega)\}$. Let P be the orthogonal projection from $L_2(\Omega)$ onto $DL_2(\Omega)$. $A = -P\Delta$ is called the Stokes operator. By applying P to the first equation of (1.1) and taking account of the other equations, we are let the following abstract initial value problem, Pr. II

$$\begin{cases} \frac{du}{dt} = P\Delta u + Fu + Pf, t \in (0, \infty), \\ u|_{t=0} = u_0, x \in \Omega, \end{cases} \quad (3)$$

where $Fu = -P(u \bullet \nabla)u$.

We consider Equation (3) in integral form Pr. III

$$u(t) = e^{tP\Delta}u_0 + \int_0^t e^{(t-s)P\Delta}Fu(s)ds. \quad (4)$$

In Theorem 1.6 of [3], Fujita and Kato proved that if u is a solution of (3) then u is of the class C^∞ as $L_2(\Omega)$ -valued functions. They gave some sufficient conditions for (3) having a solution. In Theorem 3.4 of [2], Giga and Miyakawa proved that the solution of (4) belongs to $(C^\infty(\overline{\Omega} \times (0, T]))^n$. They gave some sufficient conditions for (3) and (4) having a solution. In [8], Otelbaev proved that if all $\frac{\partial u}{\partial t}, \Delta u, (u \bullet \nabla)u, \operatorname{div} p \in L_2((0, T) \times \Omega)$, then (1) have the unique local solution. But he did not prove that this solution is smooth. In [10], Shakhmurov discussed nonlocal Navier-Stokes problems in abstract function space $DL_2(\Omega)$. He gave some sufficient conditions for (4) having a solution. In this paper, we will prove that the Navier-Stokes initial value problem (1) has the unique local strong solution $u(t, x) \in (C^\infty([0, t_1] \times \Omega))^3$, $p(t, x) \in C^\infty([0, t_1] \times \Omega)$ in the state space $DL_2(\Omega)$ by using the theory of semigroup of operators.

For $u = (u_1, u_2, u_3) \in L_2(\Omega)$ we define $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ and $\nabla u = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3} \right)$. Since the operator $-\nabla = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is strongly elliptic of order 2. Theorem 7.3.6 in [11] implies that Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^2(\Omega)$ with $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Hence Δ is also the infinitesimal generator of an analytic semigroup of contraction on $L_2(\Omega)$ with $D(\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega)$, where $H_2(\Omega)$ and $H_{1,0}(\Omega)$ are the Sobolev spaces of vector value in $H^2(\Omega)$ and $H_0^1(\Omega)$, respectively. We will prove that Δ is also the infinitesimal generator of an analytic semigroup of contraction on $DL_2(\Omega)$.

An operator A is called preserving divergence-free on a vector value functions space X if A maps every $u \in X$ with $\operatorname{div} u = 0$ to an Au with $\operatorname{div} Au = 0$.

Lemma 1. *For every $u \in L_2(\Omega)$, $\operatorname{div} u = 0$ if and only if $\operatorname{div}(\lambda I - \Delta)u = 0$ for*

$$\lambda \in \Sigma_{\vartheta} = \{\lambda : \vartheta - \pi < \arg \lambda < \pi - \vartheta, |\lambda| \geq r\},$$

where $0 < \vartheta < \pi/2$.

Proof. Let $u = (u_1, u_2, u_3) \in L_2(\Omega)$. Then

$$\begin{aligned} \Delta u &= \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}, \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}, \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right), \\ \operatorname{div}(\Delta u) &= \frac{\partial^3 u_1}{\partial x_1^3} + \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_1}{\partial x_1 \partial x_3^2} + \frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} + \frac{\partial^3 u_2}{\partial x_2^3} \\ &\quad + \frac{\partial^3 u_2}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u_3}{\partial x_3 \partial x_1^2} + \frac{\partial^3 u_3}{\partial x_3 \partial x_2^2} + \frac{\partial^3 u_3}{\partial x_3^3} \\ &= \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + \frac{\partial^2}{\partial x_2^2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \\ &\quad + \frac{\partial^2}{\partial x_3^2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \\ &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = \Delta(\operatorname{div} u). \end{aligned}$$

So we have

$$\operatorname{div}[(\lambda I - \Delta)u] = (\lambda I - \Delta)(\operatorname{div} u). \quad (5)$$

From (5), it is clear that $\operatorname{div} u = 0$ implies that $\operatorname{div}(\lambda I - \Delta)u = 0$.

Since $-\Delta$ is a strongly elliptic operator of order 2 on Ω . From Theorem 7.3.2 in [11], it follows that there exist constant $C > 0$, $R \geq 0$ and $0 < \vartheta < \pi/2$ such that

$$\|u\|_{L_2(\Omega)} \leq \frac{C}{|\lambda|} \|(\lambda I - \Delta)u\|_{L_2(\Omega)}, \quad (6)$$

for $u \in D(\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega) \subset L_2(\Omega)$ and

$$\lambda \in \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg \lambda < \pi - \vartheta, |\lambda| \geq r\}.$$

From (6), it follows that for every $\lambda \in \Sigma_\vartheta$ the operator $\lambda I - \Delta$ is injective from $D(\Delta)$ into $L_2(\Omega)$. From (5), it follows that $\operatorname{div}(\lambda I - \Delta)u = 0$ implies that $\operatorname{div} u = 0$. $\square$

Lemma 2 (1.5.12 in [5]). *Let $\{T(t) : t \geq 0\}$ be a C_0 -semigroup on a Banach space X . If Y is a closed subspace of X such that $T(t)Y \subset Y$ for all $t \geq 0$, i.e., if Y is $T(t)_{t \geq 0}$ -invariant, then the restrictions*

$$T(t)|_Y := T(t)|_Y$$

form a C_0 -semigroup $\{T(t)|_Y : t \geq 0\}$, called the subspace semigroup, on the Banach space Y .

Lemma 3 (Proposition 2.2.3 in [5]). *Let $(A, D(A))$ be the generator of a C_0 -semigroup $\{T(t) : t \geq 0\}$ on a Banach space X and assume that the restricted semigroup (subspace semigroup) $\{T(t)|_Y : t \geq 0\}$ is a C_0 -semigroup on some $(T(t))_{t \geq 0}$ -invariant Banach space $Y \rightarrow X$. Then the generator of $\{T(t)|_Y : t \geq 0\}$ is the part $(A|_Y, D(A|_Y))$ of A in Y .*

Lemma 4. *The operator Δ with $D(\Delta) \subset DL_2(\Omega)$ is the infinitesimal generator of an analytic semigroup of contractions on $DL_2(\Omega)$.*

Proof. From Theorem 7.3.6 in [11] Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^2(\Omega)$. Then Δ is also the infinitesimal generator of an analytic semigroup of contractions on $L_2(\Omega)$. Let $\{T(t)|t \geq 0\}$ be the restriction of the analytic semigroup generated by Δ on $L_2(\Omega)$ to the real axis, $\{T(t)|t \geq 0\}$ is a C_0 semigroup of contractions by Theorem 7.2.5 and Theorem 3.1.1 in [11]. We have already noted that $DL_2(\Omega)$ is a closed subspace of $L_2(\Omega)$ and is also a Hilbert space. We want to show that $DL_2(\Omega)$ is $T(t)_{t \geq 0}$ -invariant. For every $u \in L_2(\Omega)$ with $\operatorname{div} u = 0$ and $\lambda \in \rho(\Delta) \cap \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg \lambda < \pi - \vartheta, |\lambda| \geq r\}$ we have $(\lambda I - \Delta)[R(\lambda : \Delta)u] = u$, where Σ_ϑ is the same as in the proof of Lemma 1. From Lemma 1 it follows that $\operatorname{div} R(\lambda : \Delta)u = 0$. That is to say that $R(\lambda : \Delta)$ is preserving divergence-free for $\lambda \in \rho(\Delta) \cap \Sigma_\vartheta$. From Theorem 2.5.2(c) in [11] it follows that $\rho(\Delta) \supset R^+$, and so $\rho(\Delta) \cap \Sigma_\vartheta \supset \{\lambda : \lambda \geq r\}$. Hence $R(\lambda : \Delta)$ is preserving divergence free for every $\lambda \geq r$. Let $u \in DL_2(\Omega)$ then there exists a sequence u_n such that $\lim_{n \rightarrow \infty} u_n = u$ and $\operatorname{div} u_n = 0$ for $n = 1, 2, \dots$. Since $R(\lambda : \Delta)$ is bounded and so is continuous. Hence $\lim_{n \rightarrow \infty} R(\lambda : \Delta)u_n = R(\lambda : \Delta)u$ and $\operatorname{div} R(\lambda : \Delta)u_n = 0$ for every $\lambda \geq r$. Therefore $R(\lambda : \Delta)u \in DL_2(\Omega)$ for every $\lambda \geq r$. It follows that $DL_2(\Omega)$ is $R(\lambda : \Delta)$ -invariant for every $\lambda \geq r$. Now, the Theorem 4.5.1 in [11] implies that $DL_2(\Omega)$ is $T(t)_{t \geq 0}$ -invariant. From Lemmas 2 and 3, it follows that $\Delta|_{DL_2(\Omega)}$ with $D(\Delta|_{DL_2(\Omega)}) = D(\Delta) \cap DL_2(\Omega)$ is the infinitesimal generator of the C_0 semigroup $\{T(t)|_{DL_2(\Omega)} : t \geq 0\}$ of contractions on $DL_2(\Omega)$.

We will prove that $\{T(t)|_{DL_2(\Omega)} : t \geq 0\}$ can also be extended to an analytic semigroup on $DL_2(\Omega)$. Suppose that $\lambda \in \rho(\Delta) \cap \Sigma_\vartheta$, i.e., there exists $R(\lambda : \Delta)$ from $L_2(\Omega)$ into $D(\Delta)$. Then for any $u \in DL_2(\Omega) \subset L_2(\Omega)$, $R(\lambda : \Delta)u \in DL_2(\Omega)$. We have

$$(\lambda I - \Delta)R(\lambda : \Delta)u = u \quad \text{and} \quad R(\lambda : \Delta)(\lambda I - \Delta)u = u. \quad (7)$$

Since $R(\lambda : \Delta)$ is preserving divergence-free we have

$$R(\lambda : \Delta)u \in DL_2(\Omega) \quad \text{and} \quad (\lambda I - \Delta)u \in DL_2(\Omega).$$

Thus the formula (7) becomes

$$(\lambda I - \Delta|_{DL_2(\Omega)})R(\lambda : \Delta)u = u \quad \text{and} \quad R(\lambda : \Delta)(\lambda I - \Delta|_{DL_2(\Omega)})u = u.$$

Hence $(\lambda I - \Delta|_{DL_2(\Omega)})R(\lambda : \Delta) = I$ and $R(\lambda : \Delta)(\lambda I - \Delta|_{DL_2(\Omega)}) = I$. We get

$$(\lambda I - \Delta|_{DL_2(\Omega)})^{-1} = R(\lambda : \Delta)|_{DL_2(\Omega)}. \quad (8)$$

From the formula (8) and Theorem 2.5.2(c) in [11], we have

$$\begin{aligned} \rho(\Delta|_{DL_2(\Omega)}) &\supset \rho(\Delta) \cap \Sigma_\vartheta \supset \Sigma = (\{\lambda : |\arg \lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}) \cap \Sigma_\vartheta \\ &= \Sigma_\vartheta = \{\lambda : |\arg \lambda|_{<} \vartheta_1, |\lambda| \geq r\}, \end{aligned}$$

where $0 < \delta < \frac{\pi}{2}$ and $\vartheta_1 = \min\{\frac{\pi}{2} + \delta, \pi - \vartheta\}$. Thus, for $\lambda \in \Sigma_{\vartheta_1}$,

$\lambda I - \Delta|_{DL_2(\Omega)}$ is invertible. From Theorem 2.5.2(c) in [11], we have

$$\begin{aligned} \|R(\lambda : \Delta|_{DL_2(\Omega)})\| &= \sup_{\substack{u \in DL_2(\Omega) \\ \|u\|_{DL_2(\Omega)}=1}} \|R(\lambda : \Delta|_{DL_2(\Omega)})u\| \\ &\leq \sup_{\substack{u \in L_2(\Omega) \\ \|u\|_{L_2(\Omega)}=1}} \|R(\lambda : \Delta)u\| = \|R(\lambda, \Delta)\|_{L_2(\Omega)} \leq \frac{M}{|\lambda|}. \end{aligned} \quad (9)$$

Now Theorem 2.5.2(c) in [11] implies that $\{T(t)|_{DL_2(\Omega)} : \lambda \geq 0\}$ can also be extended to an analytic semigroup on $DL_2(\Omega)$. Therefore $\Delta|_{DL_2(\Omega)}$ is a infinitesimal generator of an analytic semigroup of contraction on $DL_2(\Omega)$. $\square$

Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space X . From the results of Subsection 2.6 in [11], we can define the fraction powers A^α for $0 \leq \alpha \leq 1$ and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ dense in X . $D(A^\alpha)$ equipped with the norm $\|x\|_\alpha = \|A^\alpha x\|$ is a Banach space denoted by X_α . It is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the embedding of X_β into X_α is continuous. If $-A = \Delta$ and $\gamma > 1/2$, then $X_\gamma \subset X_{1/2}$ and $D((-\Delta)^\gamma) \subset D((-\Delta)^{1/2})$, so the norms $\|\bullet\|_{DL_2(\Omega)_\gamma}$ and $\|\bullet\|_{DL_2(\Omega)}$ are equivalent (see p. 291 in [6]), i.e., there exists $L_0 > 0$ such that for any $u \in D((-\Delta)^\gamma) \cap \| \bullet \|_{DL_2(\Omega)_\gamma} \leq L_0 \| \bullet \|_{DL_2(\Omega)}$

$$\|(-\Delta)^\gamma u\|_{DL_2(\Omega)} = \|u\|_{DL_2(\Omega)_\gamma} \leq L_0 \|u\|_{DL_2(\Omega)^{1/2}}. \quad (10)$$

For $u \in D(\Delta)$, we have

$$\|\nabla u\|_{DL_2(\Omega)} = \|-\nabla u\|_{DL_2(\Omega)} = \|(-\Delta)^{1/2} u\|_{DL_2(\Omega)} = \|u\|_{DL_2(\Omega)^{1/2}}. \quad (11)$$

In [2], Giga and Miyakawa proved the following result:

Lemma 5 (Lemma 2.2 in [2]). *Let $0 \leq \delta < 1/2$ and $n(1 - r^{-1})/2$. Then*

$$\|A^{-\delta} P(u, \nabla) v\|_{0,r} \leq M \|A^\vartheta u\|_{0,r} \|A^\omega v\|_{0,r},$$

with some constant $M = M(\delta, \vartheta, \omega, r)$, provided $\delta + \vartheta + \omega \geq n/2r + 1/2$, $\vartheta > 0$, $\omega > 0$, $\omega + \delta > 1/2$.

From the Lemma 5 and the formula (10) we see that if take $n = 3$, $r = 2$, $\delta = 0$, $\vartheta = 3/4$ and $\omega = 3/4$, then

$$\begin{aligned} \|(u \bullet \nabla)v\|_{DL_2(\Omega)} &= \|(u \bullet \nabla)v\|_{L_2(\Omega)} = \|(u \bullet \nabla)v\|_{W^{0,2}(\Omega)} \\ &\leq M \left\| (-\Delta)^{\vartheta} u \right\|_{DL_2(\Omega)} \left\| (-\Delta)^{\omega} v \right\|_{DL_2(\Omega)} \\ &\leq ML_0^2 \|u\|_{DL_2(\Omega)_{1/2}} \|v\|_{DL_2(\Omega)_{1/2}}, \end{aligned}$$

with some constant M for any $u, v \in DL_2(\Omega)$. Hence we have

Lemma 6. *Suppose that $u, v \in DL_2(\Omega)$ are velocity fields and $(u \bullet \nabla)v \in DL_2(\Omega)$, then*

$$\|(u \bullet \nabla)v\|_{DL_2(\Omega)} \leq ML_0^2 \|u\|_{DL_2(\Omega)_{1/2}} \|v\|_{DL_2(\Omega)_{1/2}}.$$

Assumption (F). Let $X = DL_2(\Omega)$ and U be an open subset in $R^+ \times X_\alpha$ ($0 < \alpha < 1$). The function $f : U \rightarrow X$ satisfies the Assumption (F) if for every $(t, u) \in U$ there is a neighbourhood $V \subset U$ and constants $L \geq 0$, $0 < \vartheta \leq 1$ such that for all $(t_i, u_i) \in V$ ($i = 1, 2$)

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq L(|t_1 - t_2|^{\vartheta} + \|u_1 - u_2\|_\alpha).$$

Lemma 7 (Theorem 6.3.1 in [11]). *Let $-A$ be the generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ and assume that $0 \in \rho(-A)$. If, $0 < \alpha < 1$ and f satisfies the Assumption (F) then for every initial date $(t_0, u_0) \in U$ the initial value problem*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t \in (t_0, \infty), \\ u(t_0) = u_0 \end{cases} \quad (12)$$

has a unique local solution $u \in C([t_0, t_1] : X) \cap C^1((t_0, t_1) : X)$, where $t_1 = t_1(t_0, u_0) > t_0$.

Now we study the Navier-Stokes initial value problem (1).

A function u which is differentiable almost everywhere on $[0, T]$ such that $u' \in L^1[0, T : DL_2(\Omega)]$ is called a strong solution of the initial value problem (1) if $u(0) = u_0$ and u satisfies (1) a.e. on $[0, T]$.

Let $C^\mu([0, \infty); DL_2(\Omega)_{1/2})$ denote the space of all Hölder continuous functions on $[0, T]$ with exponent μ and with values in a Banach space $DL_2(\Omega)_{1/2}$.

Theorem. *The initial value problem (1) has a unique smooth local strong solution if the initial value $u_0 \in D((-\Delta)^{\frac{1}{2}})$ is smooth, divergence-free vector field and f is Hölder continuous about t on $(0, \infty)$ with exponent β_1 , i.e., there exist constant C_1 and $0 < \beta_1 < 1$ such that*

$$\|f(s, x) - f(t, x)\| \leq C_1 |s - t|^{\beta_1}, \quad \text{for } s, t \in (0, \infty) \quad \text{and} \quad x \in \Omega. \quad (13)$$

Proof. We will find that by incorporating the divergence-free condition, we can remove the pressure term from our equation (see p. 271³ in [3], p. 234₆ and p. 239₉ in [9]). So first we can rewrite (1) into a abstract initial value problem on $DL_2(\Omega)$

$$\begin{cases} \frac{du}{dt} = \Delta u + F(t, u(t)), & t \in (0, \infty), \\ u|_{t=0} = u_0, & x \in \Omega, \end{cases} \quad (14)$$

where $F(t, u(t)) = -(u \bullet \nabla)u + f$ is a abstract function. From Lemma 4, Δ is the generator of an analytic semigroup $T(t)$ of contraction on $DL_2(\Omega)$ and $\|T(t)\| \leq 1$. From Theorem 2.5.2(c) in [11] $0 \in \rho(\Delta)$.

Let U be the subset of $R^+ \times DL_2(\Omega)_{1/2}$ such that $(t_1, u_1) \in U$ iff $u_1 = u(t_1)$ for some $u(t) \in C^{\mu}([0, \infty); DL_2(\Omega)_{1/2})$ and $(u_1 \bullet \nabla)u_1 \in DL_p(\Omega)$. Let $u_k(t, x) = (k, k, k)(t \in [0, \infty), x \in \Omega, k \in R)$. Then $(t, u_k) \in U$ for $t \in [0, \infty)$ and all $k \in R$. For any $u \in D((-\Delta)^{\frac{1}{2}})$ let $u(t) \equiv u$ ($t \in [0, \infty)$). Then $u(t) \in C^{\mu}([0, \infty); DL_2(\Omega)_{1/2})$. Hence U is not empty.

Let U_1 be the open kernel of U . From Lemma 8 $F(t, u(t)) = -(u \bullet \nabla)u + f$ is a function: $U_1 \rightarrow DL_2(\Omega)$. For any $(t_1, u_1), (t_2, u_2) \in U_1$, we have $(u_1 \bullet \nabla)u_1, (u_2 \bullet \nabla)u_2 \in DL_2(\Omega)$ and

$$\begin{aligned}
& \|(u_1 \bullet \nabla)u_1 - (u_2 \bullet \nabla)u_2\|_{DL_2(\Omega)} \\
&= \|(u_1 \bullet \nabla)u_1 - (u_1 \bullet \nabla)u_2\|_{DL_2(\Omega)} \\
&\quad + \|(u_1 \bullet \nabla)u_2 - (u_2 \bullet \nabla)u_2\|_{DL_2(\Omega)} \\
&= \|(u_1 \bullet \nabla)(u_1 - u_2)\|_{DL_2(\Omega)} + \|[(u_1 - u_2) \bullet \nabla]u_2\|_{DL_2(\Omega)} \\
&\leq ML_0^2(\|u_1\|_{DL_2(\Omega)_{1/2}}\|(u_1 - u_2)\|_{DL_2(\Omega)_{1/2}} \\
&\quad + \|u_1 - u_2\|_{DL_2(\Omega)_{1/2}}\|u_2\|_{DL_2(\Omega)_{1/2}}) \\
&= ML_0^2\left(\|u_1\|_{DL_2(\Omega)_{1/2}} + \|u_2\|_{DL_2(\Omega)_{1/2}}\right)\|u_1 - u_2\|_{DL_2(\Omega)_{1/2}}. \quad (15)
\end{aligned}$$

We used Lemma 6 in the above third step. Therefore $(u \bullet \nabla)u$ is local Lipschitz continuous. Since $u(t)$ is Hölder continuous about t on $[0, \infty)$ in $DL_2(\Omega)_{1/2}$, so there is a constant C_2 and $0 < \beta_2 > 1$ such that

$$\|u(s, x) - u(t, x)\|_{DL_2(\Omega)_{1/2}} \leq C_2|s - t|^{\beta_2} \quad \text{for } s, t \in [0, \infty). \quad (16)$$

Hence

$$\begin{aligned}
& \| (u(t_1) \bullet \nabla) u(t_1) - (u(t_2) \bullet \nabla) u(t_2) \|_{DL_2(\Omega)} \\
& \leq \| (u(t_1) \bullet \nabla) u(t_1) - (u(t_1) \bullet \nabla) u(t_2) \|_{DL_2(\Omega)} \\
& \quad + \| (u(t_1) \bullet \nabla) u(t_2) - (u(t_2) \bullet \nabla) u(t_2) \|_{DL_2(\Omega)} \\
& = \| (u(t_1) \bullet \nabla) (u(t_1) - u(t_2)) \|_{DL_2(\Omega)} \\
& \quad + \| [(u(t_1) - u(t_2)) \bullet \nabla] u(t_2) \|_{DL_2(\Omega)} \\
& \leq ML_0^2 (\|u(t_1)\|_{DL_2(\Omega)_{1/2}} \|u(t_1) - u(t_2)\|_{DL_2(\Omega)_{1/2}} \\
& \quad + \|u(t_1) - u(t_2)\|_{DL_2(\Omega)_{1/2}} \|u(t_2)\|_{DL_2(\Omega)_{1/2}}) \\
& = ML_0^2 (\|u(t_1)\|_{DL_2(\Omega)_{1/2}} + \|u(t_2)\|_{DL_2(\Omega)_{1/2}}) \|u(t_1) - u(t_2)\|_{DL_2(\Omega)_{1/2}} \\
& \leq ML_0^2 C_2 \left(\|u(t_1)\|_{DL_2(\Omega)_{1/2}} + \|u(t_2)\|_{DL_2(\Omega)_{1/2}} \right) \cdot |t_1 - t_2|^{\beta_2}. \tag{17}
\end{aligned}$$

We used the Lemma 6 in the above third step, and the formula (16) in fifth step.

If $u_0 \in D((- \Delta)^{\frac{1}{2}})$, then $(0, u_0) \in U_1$. Set

$$V = B_\varepsilon(0, u_0) = \left\{ (t, u) \in U : |t - 0| \leq \varepsilon, \|u - u_0\|_{DL_2(\Omega)_{1/2}} \leq \varepsilon \right\}.$$

Then for $(t, u) \in V$,

$$\begin{aligned}
\|u\|_{DL_2(\Omega)_{1/2}} & = \|u - u_0 + u_0\|_{DL_2(\Omega)_{1/2}} \\
& \leq \|u - u_0\|_{DL_2(\Omega)_{1/2}} + \|u_0\|_{DL_2(\Omega)_{1/2}} \\
& \leq \varepsilon + \|u_0\|_{DL_2(\Omega)_{1/2}}.
\end{aligned}$$

Let $L = \varepsilon + \|u_0\|_{DL_2(\Omega)_{1/2}}$, $L_1 = 2ML_0^2L$, $L_2 = 2ML_0^2LC_2$, and $L_3 = \text{Max}(C_1, L_2, L_1)$, $\vartheta = \text{Max}(\beta_1, \beta_2)$ then from (13), (15) and (17) it follows that for all $(t_i, u_i) \in V$

$$\begin{aligned}
& \|F(t_1, u_1(t_1)) - F(t_2, u_2(t_2))\|_{DL_2(\Omega)} \\
& \leq \|F(t_1, u_1(t_1)) - F(t_1, u_2(t_1))\|_{DL_2(\Omega)} \\
& \quad + \|F(t_1, u_2(t_1)) - F(t_2, u_2(t_2))\|_{DL_2(\Omega)} \\
& = \|(u_1(t_1) \bullet \nabla)u_1(t_1) - (u_2(t_1) \bullet \nabla)u_2(t_1)\|_{DL_2(\Omega)} \\
& \quad + \|(u_2(t_1) \bullet \nabla)u_2(t_1) - (u_2(t_2) \bullet \nabla)u_2(t_2)\|_{DL_2(\Omega)} \\
& \quad + \|f(t_1, x) - f(t_2, x)\|_{DL_2(\Omega)} \\
& \leq 2ML_0^2L\|u_1 - u_2\|_{DL_2(\Omega)_{1/2}} + 2ML_0^2LC_2|t_1 - t_2|^{\beta_2} + C_1|t_1 - t_2|^{\beta_1} \\
& \leq L_1\|u_1 - u_2\|_{DL_2(\Omega)_{1/2}} + L_2|t_1 - t_2|^{\beta_2} + C_1|t_1 - t_2|^{\beta_1} \\
& \leq L_3(|t_1 - t_2|^{\vartheta} + \|u_1 - u_2\|_{DL_2(\Omega)_{1/2}}).
\end{aligned}$$

Hence $F(t, u(t))$ satisfies the Assumption (F), then by Lemma 7 for every initial data $(0, u_0) \in U_1$ the initial value problem (14) has a unique local solution

$$u(t) \in C([0, t_1] : DL_2(\Omega)) \cap C^1((0, t_1] : DL_2(\Omega)), \quad (18)$$

where $t_1 = t_1(u_0)$.

We consider Equation (3) in integral form

$$u(t) = e^{tP\Delta}u_0 + \int_0^t e^{(t-s)P\Delta}Fu(s)ds. \quad (19)$$

In a similar induction way as Theorem 3.9 in [2] or as Theorem 5.1 in [10] we can prove that the solution $u(t, x) \in \left(C^\infty((0, t_1] \times \overline{\Omega})\right)^3$ (see Appendix). We can also prove directly that $u(t, x)$ is smooth. In fact, the solution (18) of (14) is also the solution of (19). The Theorem 3.4 in [2] mean that as long as the solution of (19) exists, this solution is smooth. From Theorem 3.4 in [2] we have the solution $u(t, x) \in \left(C^\infty((0, t_1] \times \overline{\Omega})\right)^3$. Substituting $u(t, x)$ into (1) we get the solution $p(t, x)$. We also have $p(t, x) \in C^\infty((0, t_1] \times \overline{\Omega})$. It follows from the formula (2) and $u \in DL_2(\Omega)$ that the solution $u(t, x)$ is divergence-free. Changing the value of u on $\partial\Omega$ to zero we get a unique smooth local strong solution $u = u(t, x)$, $p(t, x)$ of the Navier-Stokes initial value problem (1). $\square$

The above Theorem extents Theorem 1.2 in [3] and Theorem 2.5 in [2]. In another paper, we will prove that if we take $f(t, x)$ to be identically zero and assume that any initial value $u_0 \in D((-\Delta)^{\frac{1}{2}}) \cap C^\infty([0, \infty) \times R^3)$ satisfies $|\partial_x^\alpha u_0(x)| \leq C_{\alpha k} (1 + |x|)^{-k}$ on R^3 for any α and k , then the Navier-Stokes initial value problem has a global smooth solution with bounded energy.

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Appendix

Lemma 1 (Lemma 3.1 in [2]). *Let $u \in D(-\Delta)$ and $-\Delta u \in (W^{m,r}(\Omega))^n$ ($1 < r < \infty$) for some integer $m \geq 0$, then $u \in (W^{m+2,r}(\Omega))^n$ and satisfies*

$$\|u\|_{m+2,r} \leq C_{mr} \|-\Delta u\|_{m,r},$$

with a constant $C_{mr} > 0$ independent of u and $-\Delta u$.

Let $C^\mu([0, T]; X)$ denote the space of Hölder continuous functions on $[0, T]$ with exponent μ and with values in a Banach space X . Similarly, let $C^\mu((0, T]; X)$ denote the space of functions which are Hölder continuous on every subinterval $[\epsilon, T]$ of $(0, T]$, with exponent μ .

Lemma 2 (Lemma 3.2 in [2]). *Let $f(t) \in C^\mu([0, T]; DL_2(\Omega))$, for some $0 < \mu < 1$. Then the function*

$$v(t) = \int_0^t e^{-(t-s)(-\Delta)} f(s) ds \in C^\mu((0, T]; D(-\Delta)) \cap C^{1+\nu}((0, T]; DL_2(\Omega)),$$

for every ν such that $0 < \nu < \mu$.

Let P_r be the continuous projection from $L_2(\Omega)$ to $DL_2(\Omega)$.

Lemma 3 (Lemma 3.3 in [2]). (i) *For $m > 3/r$, there exists a constant $C_{m,r} > 0$ such that*

$$\|P_r(u \bullet \nabla)v\|_{m,r} \leq C_{m,r} \|u\|_{m,r} \|v\|_{m+1,r},$$

for every $u \in (W^{m,r}(\Omega))^3$, $v \in (W^{m+1,r}(\Omega))^3$ ($1 < r < \infty$).

(ii) *When $r > 3$, we have*

$$\|P_r(u \bullet \nabla)v\|_{0,r} \leq C_r \|u\|_{1,r} \|v\|_{1,r},$$

for $u, v \in (W^{1,r}(\Omega))^3$.

We will say that $u(t)$ has property (P_m) ($m \geq 1$) if

$$u^{(m)} \in C^\mu\left((0, T]; D((-\Delta)^{\frac{1}{2}})\right),$$

$$u^{(j)} \in C^\mu\left((0, T]; \left(W^{m+1-j, 2}(\Omega)\right)^n\right), 1 \leq j \leq m-1,$$

$$u \in C^\mu\left((0, T]; \left(W^{m+2, 2}(\Omega)\right)^n\right),$$

for all μ , $0 < \mu < \frac{1}{2}$. Here $u^{(j)} = d^j u / dt^j$.

Let the solution in the above Theorem be

$$u(t, x) = T(t)u_0 + \int_0^t T(t-s)F(s, (-\Delta)^{-\frac{1}{2}}y_0(s))ds.$$

Lemma 2 and Lemma 3(i) now imply.

Lemma 4 (Lemma 3.6 in [2]). $u \in C^\mu((0, T]; D(-\Delta))$ and $u' = du/dt \in C^\mu((0, T]; DL_2(\Omega))$ for all μ , $0 < \mu < \frac{1}{2}$. Moreover $F(s, (-\Delta)^{-\frac{1}{2}}y_0(s)) \in C^\mu((0, T]; (W^{1, 2}(\Omega))^3)$.

Lemma 5 (Lemma 3.7 in [2]). We have $u' = \partial u / \partial x \in C^\mu\left((0, T]; D\left((-\Delta)^{\frac{1}{2}}\right)\right)$ for all μ , $0 < \mu < \frac{1}{2}$.

The proof is similar to Lemma 3.7 in [2].

Since $D((-\Delta)^{\frac{1}{2}}) \subset (W^{1, r}(\Omega))^3$, Lemma 1, Lemma 4 and the identity $u = (-\Delta)^{-1}(Fu - u')$ show that

$$u \in C^\mu\left((0, T]; (W^{3, 2}(\Omega))^3\right). \quad (20)$$

Lemma 5 and the above formula (20) show that $u(t)$ has property $(P)_1$.

Lemma 6 (Lemma 3.8 in [2]). $(P)_m$ implies $(P)_{m+1}$.

The proof is the same to Lemma 3.8 in [2].

Therefore we can prove the following theorem in a similar way as Theorem 3.9 in [2] or as Theorem 5.1 in [10].

Theorem. *The solution in the above Theorem is smooth.*