A REMARK ON HARDY-TYPE INEQUALITY IN SPHERES

SONGTING YIN and YU REN

Department of Mathematics and Computer Science

Tongling University

Tongling, 244000 Anhui

P. R. China

e-mail: yst419@163.com

ry86901@163.com

Abstract

We obtain a Hardy type inequality with the best constant on the sphere and provide a simple and short proof. The result complements some inequalities in recent literatures.

1. Introduction

The classical Hardy inequality states that, for $n \geq 3$ and all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} dx.$$

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The constant $\frac{(n-2)^2}{4}$ is optimal and not attained in the Sobolev space $W^{1,2}(\mathbb{R}^n)$.

In [12], Kome and Özaydin obtained the improved Hardy inequalities in the Poincaré conformal disc model

$$\int_{\mathbb{B}^n} |\nabla f|^2 dV \geq \frac{(n-2)^2}{4} \int_{\mathbb{B}^n} \frac{f^2}{r^2} dV,$$

where $f \in C_0^\infty(\mathbb{B}^n)$ and $r = \log \frac{1+|x|}{1-|x|}$ is the geodesic distance.

Furthermore, the constant $\frac{(n-2)^2}{4}$ is sharp.

There has been a lot of research concerning Hardy inequality in the Euclidean spaces and Riemannian manifolds due to its application to singular problems. We see [2], [3], [10], [11], [14] for the Euclid case and [4], [7], [8], [9], [12], [16] for the Riemannian case, and so on.

However, there are not many literatures discussing Hardy inequality on the sphere up to now. See [1], [5], [15], [17] and the references therein. Recently, Xiao ([15]) studied this issue and derived the following inequality:

$$C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geq \frac{(n-2)^2}{4} \left(\int_{\mathbb{S}^n} \frac{f^2}{d(p, x)^2} dV + \int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(p, x))^2} dV \right).$$

Another Hardy-type inequality was obtained by the first author as follows ([17]):

$$\frac{n-2}{2} \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geq \frac{(n-2)^2}{4} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p, x)} dV.$$

Here $d(p, x)$ is the geodesic distance from p to x on $\mathbb{S}^n$, and the constant

$\frac{(n-2)^2}{4}$ is sharp. To see L^p -Hardy inequality on the sphere, we refer to [13].

In this short note, we will give a new Hardy-type inequality on the sphere with the best constant.

2. The Main Result and its Proof

Theorem 2.1. *Let $(\mathbb{S}^n, g)$ ($n \geq 3$) be the n -sphere with sectional curvature 1. Then for any function $f \in C^\infty(\mathbb{S}^n)$, it holds*

$$\frac{n-2}{2} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p, x)} dV \leq \left(\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p, x)} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\nabla f|^2 dV \right)^{\frac{1}{2}} + \frac{1}{2} \int_{\mathbb{S}^n} f^2 dV,$$

where p is a fixed point in $\mathbb{S}^n$, and the constant $\frac{n-2}{2}$ is sharp.

Remark 2.2. Since on the sphere the Laplacian of the distance function is $\Delta d(p, x) = (n-1) \cot d(p, x)$, we let the inequality involve the term $\frac{f^2}{\tan^2 d(p, x)}$. So, it is a bit different in form from that in Euclidean

spaces and some other type of Hardy inequalities. In Theorem 2.1, the second term in the right side of the inequality cannot be removed because it will lead to a contradiction if f is a nonzero constant. To prove the result, we follow the method and idea from [6].

Proof. Let f be a smooth function on $\mathbb{S}^n$ and $r(x) = d(p, x)$ denotes the distance function from the fixed point $p \in \mathbb{S}^n$. Then

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla r}{\tan r} \right) &= \frac{\Delta r}{\tan r} + \nabla r \cdot (\tan^{-1} r) \\ &= \frac{n-1}{\tan^2 r} - \frac{\sec^2 r}{\tan^2 r} \\ &= \frac{n-2}{\tan^2 r} - 1, \end{aligned}$$

which together with divergence theorem yields

$$\begin{aligned}
\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV &= \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 \operatorname{div} \left(\frac{\nabla r}{\tan r} \right) dV + \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 dV \\
&= -\frac{1}{n-2} \int_{\mathbb{S}^n} \frac{\nabla r(f^2)}{\tan r} dV + \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 dV \\
&= -\frac{2}{n-2} \int_{\mathbb{S}^n} \frac{f \langle \nabla r, \nabla f \rangle}{\tan r} dV + \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 dV \\
&\leq \frac{2}{n-2} \int_{\mathbb{S}^n} \frac{f |\nabla f|}{|\tan r|} dV + \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 dV \\
&\leq \frac{2}{n-2} \left(\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\nabla f|^2 dV \right)^{\frac{1}{2}} + \frac{1}{n-2} \int_{\mathbb{S}^n} f^2 dV.
\end{aligned}$$

It follows that

$$\frac{n-2}{2} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV \leq \left(\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\nabla f|^2 dV \right)^{\frac{1}{2}} + \frac{1}{2} \int_{\mathbb{S}^n} f^2 dV.$$

In what follows, we show the constant $\frac{(n-2)}{2}$ is sharp.

Let

$$f_\delta(x) = \frac{1}{|\tan r|^\delta}, \text{ where } \delta = \frac{n-2}{2}.$$

Then $|\nabla f_\delta| = \frac{\delta \sec^2 r}{|\tan r|^{\delta+1}}$, and thus

$$\int_{\mathbb{S}^n} \frac{f_\delta^2}{\tan^2 r} dV = \operatorname{Vol}(\mathbb{S}^{n-1}) \lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr,$$

$$\int_{\mathbb{S}^n} |\nabla f_\delta|^2 dV = \text{Vol}(\mathbb{S}^{n-1}) \lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{\delta^2 \sec^4 r (\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr,$$

$$\int_{\mathbb{S}^n} f_\delta^2 dV = \text{Vol}(\mathbb{S}^{n-1}) \lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta}} dr.$$

Since $f_\delta(x) \in C^\infty(\mathbb{S}^n \setminus \{p, r^{-1}(\pi)\})$, and can be approximated by a family of smooth functions on the sphere $\mathbb{S}^n$, we compute

$$\begin{aligned} C &= \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{\left(\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\nabla f|^2 dV \right)^{\frac{1}{2}} + \frac{1}{2} \int_{\mathbb{S}^n} f^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r} dV} \\ &\leq \frac{\left(\int_{\mathbb{S}^n} \frac{f_\delta^2}{\tan^2 r} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\nabla f_\delta|^2 dV \right)^{\frac{1}{2}} + \frac{1}{2} \int_{\mathbb{S}^n} f_\delta^2 dV}{\int_{\mathbb{S}^n} \frac{f_\delta^2}{\tan^2 r} dV} \\ &= \frac{\left(\lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr \right)^{\frac{1}{2}} \left(\lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{\delta^2 \sec^4 r (\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr \right)^{\frac{1}{2}} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta}} dr}{\lim_{\epsilon \rightarrow 0} \int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr} \end{aligned}$$

$$=: I + II,$$

where, by L'Hospital rule,

$$I = \left(\lim_{\epsilon \rightarrow 0} \frac{\int_\epsilon^{\pi-\epsilon} \frac{\delta^2 \sec^4 r (\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr}{\int_\epsilon^{\pi-\epsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr} \right)^{\frac{1}{2}}$$

$$\begin{aligned}
&= \left(\lim_{\varepsilon \rightarrow 0} \frac{\frac{\delta^2 \sec^4 \varepsilon (\sin \varepsilon)^{n-1}}{|\tan \varepsilon|^{2\delta+2}}}{\frac{(\sin \varepsilon)^{n-1}}{|\tan \varepsilon|^{2\delta+2}}} \right)^{\frac{1}{2}} \\
&= \delta, \\
\\
II &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \frac{\int_{\varepsilon}^{\pi-\varepsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta}} dr}{\int_{\varepsilon}^{\pi-\varepsilon} \frac{(\sin r)^{n-1}}{|\tan r|^{2\delta+2}} dr} = 0.
\end{aligned}$$

This implies that

$$C \leq \delta = \frac{n-2}{2}.$$

Thus the constant $\frac{n-2}{2}$ is sharp and the proof is complete. $\square$

In [18], the first author derived a Rellich type inequality on the sphere as follows:

$$C(n) \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(p, x)} dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{S}^n} \frac{f^2}{\sin^4 d(p, x)} dV,$$

where $C(n) = \frac{n(n-4)(n^2-2n-4)}{8}$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp. One can also use the method above to obtain a similar result.

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