ON A STRENGTHENED FORM OF A GENERAL HARDY-TYPE INEQUALITY

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Abstract

In this paper, we employed the concept of superquadratic functions to obtain a class of Hardy-type inequalities involving a more general Hardy operator. Our results are new and in some cases refine and improve similar results in the literature.

1. Introduction

In a note [1] published in 1920, Hardy announced the following famous classical integral inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad (1.1)$$

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so-called Hardy's inequality, where $p > 1$ and $f \in L_p(0, \infty)$ is a nonnegative function. He later proved the results in [2] and therein showed that the constant $\left(\frac{p}{p-1}\right)^p$ is sharp, i.e., the smallest possible.

Due to the significance and usefulness of (1.1), a lot of attention has been given to its applications, methods of proofs, variants and extensions in many directions. For instance, the readers may consult [4, 6, 7, 8, 12] and the references cited therein.

In 2009, Krulić et al. [9] studied a more general class of Hardy-type inequalities involving the general Hardy integral operator

$$\mathcal{A}_k f(x) := \frac{1}{K(x)} \int_{\Omega} k(x, y) f(y) d\mu_2(y), \quad x \in \Lambda, \quad (1.2)$$

using convexity argument:

Theorem 1.1. *Let $0 < p \leq q < \infty$ and let $(\Lambda, \sum_1, \mu_1)$, $(\Omega, \sum_2, \mu_2)$ be measure spaces with positive σ -finite measures and let $u : \Lambda \rightarrow \mathbb{R}$ and $k : \Lambda \times \Omega \rightarrow \mathbb{R}$ be nonnegative measurable functions. Further, suppose that $K : \Lambda \rightarrow \mathbb{R}$ is defined by*

$$K(x) = \int_{\Omega} k(x, y) d\mu_2(y), \quad x \in \Lambda,$$

and

$$v(y) = \left(\int_{\Lambda} u(x) \left(\frac{k(x, y)}{K(x)} \right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{p}{q}} < \infty, \quad y \in \Omega.$$

If $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is a non-negative superquadratic function, then the inequality

$$\left(\int_{\Lambda} u(x) \varphi^{\frac{q}{p}}(\mathcal{A}_k f(x)) d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int_{\Omega} v(y) \varphi(f(y)) d\mu_2(y) \right)^{\frac{1}{p}}, \quad (1.3)$$

holds for all non-negative measurable functions $f : \Omega \rightarrow \mathbb{R}$ and for $\mathcal{A}_k f : \Lambda \rightarrow \mathbb{R}$ defined by (1.2).

In 2010, Abramovich et al. [10] employed superquadracity argument instead of convexity argument to obtain, under the same settings of Theorem 1.1 above, the following refinement of inequality (1.3) for the special case $p = q$:

$$\begin{aligned} & \int_{\Lambda} u(x) \varphi(\mathcal{A}_k f(x)) d\mu_1(x) \\ & + \int_{\Omega} \int_{\Lambda} u(x) \frac{k(x, y)}{K(x)} \varphi(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y) \\ & \leq \left(\int_{\Omega} v(y) \varphi(f(y)) d\mu_2(y) \right). \end{aligned} \quad (1.4)$$

For more information on superquadratic functions and their applications, we refer interested readers to [3, 5, 13] and the references cited therein.

In 2013, Čižmešija and Krnić [11] extended the inequality (1.4) as

$$\begin{aligned} & \int_{\Lambda} u(x) \varphi^r(\mathcal{A}_k f(x)) d\mu_1(x) \\ & + r \int_{\Omega} \int_{\Lambda} u(x) \frac{k(x, y)}{K(x)} \varphi^{r-1}(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y) \\ & \leq \left(\int_{\Omega} v(y) \varphi(f(y)) d\mu_2(y) \right)^r, \end{aligned} \quad (1.5)$$

where $r \geq 1$ and

$$v(y) = \left(\int_{\Lambda} u(x) \left(\frac{k(x, y)}{K(x)} \right)^r d\mu_1(x) \right)^{1/r} < \infty, \quad y \in \Omega.$$

In this paper, we derive a further refinement of the general Hardy-type inequality (1.3) for all parameters p and q satisfying $0 < p \leq q < \infty$. In particular, we prove some results that show that it is possible to add, yet another refinement term to the left hand side of inequality (1.5) when $r \geq 2$.

This paper is organized as follows: In Section 2, we gave definition and some results needed to prove our main results. In Section 3, our main results which in some cases yield some new refined Hardy-type inequalities, provide generalizations of the inequalities (1.3) and (1.5) (see Theorem 3.1).

2. Some Basic Results on Superquadraticity

In this section, we present some basic definitions and results on superquadratic and subquadratic functions needed in the proof of our main results in the next section.

Definition 2.1 ([3]). A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is said to be superquadratic provided for each $x \geq 0$, there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) - C_x(y - x) \geq 0, \quad (2.1)$$

for all $y \in [0, \infty)$. φ is subquadratic if $-\varphi$ is superquadratic.

For example, the function

$$\phi(x) = x^p, \quad x \in (0, \infty)$$

is superquadratic for any $p \geq 1$. We refer to [3] for more examples of superquadratic functions.

Lemma 2.2 ([3]). *Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a superquadratic function with C_x as in Definition 2.1. Then*

$$(1) \quad \phi(0) \leq 0.$$

(2) *If $\phi(0) = \phi'(0) = 0$, then $C_x = \phi'(x)$ whenever ϕ is differentiable at $x > 0$.*

$$(3) \quad \text{If } \phi \geq 0, \text{ then } \phi \text{ is convex and } \phi(0) = \phi'(0) = 0.$$

Next is the refined Jensen's inequality for superquadratic and subquadratic functions.

Theorem 2.3 ([3]). *Let (Ω, Σ, μ) be a probability measure space. Then the inequality*

$$\varphi\left(\int_{\Omega} f(x)d\mu(x)\right) \leq \int_{\Omega} \left[\varphi(f(x)) - \varphi\left(\left|f(x) - \int_{\Omega} f(y)d\mu(y)\right|\right)\right] d\mu(x), \quad (2.2)$$

holds for all probability measures μ and all non-negative μ -integrable functions f if and only if $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is superquadratic. Moreover (2.1) holds in the reversed direction if and only if φ is subquadratic.

Proposition 2.4 ([13]). *Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be differentiable function such that $\phi(0) = \phi'(0)$. Then*

$$\phi(y) - \phi(1) - \phi'(1)(y-1) - \phi(|y-1|) \begin{cases} \geq 0 & \text{if } \phi \text{ is superquadratic,} \\ \leq 0 & \text{if } \phi \text{ is subquadratic,} \end{cases} \quad (2.3)$$

holds for all $y \geq 0$. If $\phi(x) = x^p$, then equality in (2.3) holds for all y if and only if $p = 2$.

A direct consequence of Proposition 2.4 or Theorem 2.3 yields the following refinement of the well-known Bernoulli's inequality which plays a central role in the proof of our main results.

Lemma 2.5 ([13]). *Let $h > 0$ then,*

$$h^p - p(h-1) - 1 - |h-1|^p \begin{cases} \geq 0 & \text{if } p \geq 0, \\ \leq 0 & \text{if } 1 < p \leq 2. \end{cases}$$

Equality holds for all $h > 0$ if and only if $p = 2$ and also when $p \neq 2$ if and only if $h = 1$.

3. Main Results

Our result is presented as follows:

Theorem 3.1. *Let $(\Lambda, \sum_1, \mu_1)$, $(\Omega, \sum_2, \mu_2)$ be measure spaces with positive σ -finite measures and let $\zeta : \Lambda \rightarrow \mathbb{R}$ and $k : \Lambda \times \Omega \rightarrow \mathbb{R}$ be nonnegative measurable functions. Further, suppose that $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is a non-negative superquadratic function, $K : \Lambda \rightarrow \mathbb{R}$ is defined by*

$$K(x) = \int_{\Omega} k(x, y) d\mu_2(y), \quad x \in \Lambda,$$

and

$$\eta(y) = \left(\int_{\Lambda} \zeta(x) \left(\frac{k(x, y)}{K(x)} \right)^r d\mu_1(x) \right)^{1/r} < \infty, \quad y \in \Omega,$$

where $r \geq 1$.

(1) *If $r \geq 2$, then the inequality*

$$\begin{aligned} & \int_{\Lambda} \zeta(x) \varphi^r(\mathcal{A}_k f(x)) d\mu_1(x) \\ & + r \int_{\Omega} \int_{\Lambda} \zeta(x) \frac{k(x, y)}{K(x)} \varphi^{r-1}(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y) \\ & + \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_2(y) \right)^r d\mu_1(x) \\ & \leq \left(\int_{\Omega} \eta(y) \varphi(f(y)) d\mu_2(y) \right)^r, \end{aligned} \tag{3.1}$$

holds for all non-negative measurable functions $f : \Omega \rightarrow \mathbb{R}$ and for $\mathcal{A}_k f : \Lambda \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_k f(x) := \frac{1}{K(x)} \int_{\Omega} k(x, y) f(y) d\mu_2(y), \quad x \in \Lambda.$$

(2) *If $1 < r \leq 2$, then (3.1) holds in the reverse direction.*

Proof. (1) Applying the inequality (2.2) to the superquadratic function φ , we find that

$$\begin{aligned}
& \varphi \left(\frac{1}{K(x)} \int_{\Omega} k(x, y) f(y) d\mu_2(y) \right) \\
& + \frac{1}{K(x)} \int_{\Omega} k(x, y) \varphi \left(\left| f(y) - \frac{1}{K(x)} \int_{\Omega} k(x, y) f(y) d\mu_2(y) \right| \right) d\mu_2(y) \\
& \leq \frac{1}{K(x)} \int_{\Omega} k(x, y) \varphi(f(y)) d\mu_2(y),
\end{aligned} \tag{3.2}$$

holds for each fixed $x \in \Lambda$.

Now, it follows from Lemma 2.5, after raising inequality (3.2) to power r , that

$$\begin{aligned}
& \varphi^r \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) \\
& + r \varphi^{r-1} \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) \int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \\
& + \left(\int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \right)^r \\
& \leq \left(\varphi \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) + \int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \right)^r \\
& \leq \left(\int_{\Omega} \varphi(f(y)) d^* \mu_2(y) \right)^r,
\end{aligned} \tag{3.3}$$

where $d^* \mu_2(y) = \frac{k(x, y)}{K(x)} d\mu_2(y)$; yielding

$$\begin{aligned}
& \varphi^r \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) \\
& + r \varphi^{r-1} \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) \int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \\
& + \left(\int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \right)^r \\
& \leq \left(\int_{\Omega} \varphi(f(y)) d^* \mu_2(y) \right)^r. \tag{3.4}
\end{aligned}$$

By multiplying inequality (3.4) by $\zeta(x)$ and integrating the resulting inequality with respect to $d\mu_1(x)$ over Λ , we find that

$$\begin{aligned}
& \int_{\Lambda} \zeta(x) \varphi^r \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) d\mu_1(x) + r \int_{\Lambda} \zeta(x) \varphi^{r-1} \left(\int_{\Omega} f(y) d^* \mu_2(y) \right) \\
& \times \left(\int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \right) d\mu_1(x) \\
& + \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \varphi \left(\left| f(y) - \int_{\Omega} f(y) d^* \mu_2(y) \right| \right) d^* \mu_2(y) \right)^r d\mu_1(x) \\
& \leq \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \varphi(f(y)) d^* \mu_2(y) \right)^r d\mu_1(x). \tag{3.5}
\end{aligned}$$

Employing the Minkowski integral inequality (see [14]), we get that

$$\begin{aligned}
& \left(\int_{\Lambda} \zeta(x) \left(\int_{\Omega} \varphi(f(y)) d^* \mu_2(y) \right)^r d\mu_1(x) \right)^{1/r} \\
&= \left(\int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(f(y)) d\mu_2(y) \right)^r d\mu_1(x) \right)^{1/r} \\
&\leq \int_{\Omega} \varphi(f(y)) \left(\int_{\Lambda} \zeta(x) \left(\frac{k(x, y)}{K(x)} \right)^r d\mu_1(x) \right)^{1/r} d\mu_2(y), \tag{3.6}
\end{aligned}$$

where from

$$\begin{aligned}
& \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \varphi(f(y)) d^* \mu_2(y) \right)^r d\mu_1(x) \\
&= \left(\int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(f(y)) d\mu_2(y) \right)^r d\mu_1(x) \right)^{1/r} \\
&\leq \left(\int_{\Omega} \varphi(f(y)) \left(\int_{\Lambda} \zeta(x) \left(\frac{k(x, y)}{K(x)} \right)^r d\mu_1(x) \right)^{1/r} d\mu_2(y) \right)^r. \tag{3.7}
\end{aligned}$$

Finally, combine inequalities (3.5) and (3.7) and apply the Fubini's theorem on the resulting inequality. Then, we obtain, after taking into account, the following definitions:

$$\begin{aligned}
\eta(y) &:= \left(\int_{\Lambda} \zeta(x) \left(\frac{k(x, y)}{K(x)} \right)^r d\mu_1(x) \right)^{1/r}, \\
d^* \mu_2(y) &:= \frac{k(x, y)}{K(x)} d\mu_2(y), \\
\mathcal{A}_k f(x) &:= \frac{1}{K(x)} \int_{\Omega} k(x, y) f(y) d\mu_2(y), \quad x \in \Lambda, \tag{3.8}
\end{aligned}$$

that

$$\begin{aligned}
& \int_{\Lambda} \zeta(x) \varphi^r(\mathcal{A}_k f(x)) d\mu_1(x) \\
& + r \int_{\Omega} \int_{\Lambda} \zeta(x) \frac{k(x, y)}{K(x)} \varphi^{r-1}(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y) \\
& + \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_2(y) \right)^r d\mu_1(x) \\
& \leq \left(\int_{\Omega} \eta(y) \varphi(f(y)) d\mu_2(y) \right)^r. \tag{3.9}
\end{aligned}$$

(2) The proof of (2) is similar to the proof of (1) above except that the signs of the inequality is reversed.

□

Remark 3.2. Note, first, that the third term on the left-hand side of inequality (3.1) is non-negative, since φ is a non-negative superquadratic function. Now, let $r = \frac{q}{p}$. Then, Theorem 3.1 shows that the following inequality:

$$\begin{aligned}
& \int_{\Lambda} \zeta(x) \varphi^{\frac{q}{p}}(\mathcal{A}_k f(x)) d\mu_1(x) \\
& + r \int_{\Omega} \int_{\Lambda} \zeta(x) \frac{k(x, y)}{K(x)} \varphi^{\frac{q}{p}-1}(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y) \\
& + \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_2(y) \right)^{\frac{q}{p}} d\mu_1(x) \\
& \leq \left(\int_{\Omega} \eta(y) \varphi(f(y)) d\mu_2(y) \right)^{\frac{q}{p}}, \tag{3.10}
\end{aligned}$$

holds for any $\frac{q}{p} \geq 2$. The above inequality holds in the reverse direction

when $1 < \frac{q}{p} \leq 2$.

Example 3.3. Put

$$\begin{aligned}
 I_1 &= \int_{\Lambda} \zeta(x) \varphi^{\frac{q}{p}}(\mathcal{A}_k f(x)) d\mu_1(x) \\
 &\quad + r \int_{\Omega} \int_{\Lambda} \zeta(x) \frac{k(x, y)}{K(x)} \varphi^{\frac{q}{p}-1}(\mathcal{A}_k f(x)) \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_1(x) d\mu_2(y), \\
 I_2 &= \int_{\Lambda} \zeta(x) \left(\int_{\Omega} \frac{k(x, y)}{K(x)} \varphi(|f(y) - \mathcal{A}_k f(x)|) d\mu_2(y) \right)^r d\mu_1(x), \\
 \text{and } I_3 &= \left(\int_{\Omega} \eta(y) \varphi(f(y)) d\mu_2(y) \right)^{\frac{q}{p}}; \tag{3.11}
 \end{aligned}$$

in (3.10), then it shows that the generalized Hardy-type inequality (1.5)

$$I_1 \leq I_3,$$

can be refined to

$$I_1 + I_2 \leq I_3,$$

for $p \geq 2$ while for $1 < p \leq 2$ we have the following two-sided estimate:

$$I_1 \leq I_3 \leq I_1 + I_3.$$

Hence, our Theorem 3.1 gives a refinement of (1.5) and, of course, (1.3).

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