AN ACH METHOD FOR SOLVING THE PERIODICITY PROBLEM OF A CLASS OF DIFFERENTIAL EQUATIONS

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Abstract

In this paper, using the idea of the aggregate function method, an ACH method is proposed to solve the periodicity problem of a class of differential equations. Under suitable conditions, we give a constructive proof the existence of the periodic solutions. Following the solution path, we can find the periodic solution via the reduced predictor-corrector algorithm by the ACH method.

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1. Introduction

As is well known, the periodicity problem plays a central role in the qualitative theory of differential equations for its significance in the physical sciences. Hence finding periodic solutions of ordinary differential equations is naturally an attracting topic. This paper deals with the problems of finding periodic solutions for vector ordinary differential equations of the form:

$$x' = f(t, x) = f(t + T, x), \quad (1.1)$$

where T is a fixed positive number. In [1], we used a homotopy interior point method to find the periodic solutions of (1.1). In [2], we applied proper perturbations to the equality constraints and hence enlarge the scope of choice of initial points. This point can improve the computational efficiency of the algorithm. However, the algorithms in [1, 2] become inefficient when the number of constraint functions increases largely. To overcome the difficulties mentioned above, we combine the aggregate function method [3, 4] with the homotopy method [5-9] to develop an ACH method to solve problem (1.1). Compared with the algorithms in [1, 2], the ACH method can reduce the dimensions of the linear systems greatly, so this method can deal with problem (1.1) when the number of constraint functions increases largely. Under suitable conditions, the existence of a smooth path from a given interior point of the feasible region to the periodic solution point of problem (1.1) will be proven. Following this solution path, we can give an implementable globally convergent algorithm. Using the ACH method, we are able to solve problem (1.1) and thus improve the computational efficiency of the algorithm in [1, 2] greatly.

2. Main Results

In this section, let $u_i(x)$, $i = 1, \dots, m$ be C^3 functions, then let $X = \{x \in R^n : u_i(x) \leq 0, i = 1, \dots, m\}$, $X^0 = \{x \in R^n : u_i(x) < 0, i = 1, \dots, m\}$ and $\partial X = X \setminus X^0$ be the boundary set of X . In addition, denote the nonnegative and positive orthants R^1 by R_+^1 and R_{++}^1 , respectively. For all $x \in \partial X$, denote the active index set by $I(x) = \{i \in \{1, \dots, m\} : u_i(x) = 0\}$.

By introducing C^2 functions $\alpha_i(x) \in R^n$, $i = 1, \dots, m$, we need the following conditions which are often used in practice:

(C₁) X^0 is nonempty and X is bounded.

(C₂) For any $x \in \partial X$, if

$$\sum_{i \in I(x)} \alpha_i(x) \beta_i = 0, \beta_i \geq 0,$$

then $\beta_i = 0$, $i \in I(x)$.

(C₃) There exists a closed subset $\hat{X} \subset X^0$ with nonempty interior $\hat{X}^0$, such that for any $x \in \partial X$, we obtain

$$\{x + \sum_{i \in I(x)} \beta_i \alpha_i(x) : \beta_i \geq 0 \text{ for } i \in I(x) \text{ and } \sum_{i \in I(x)} \beta_i > 0\} \cap \hat{X} = \emptyset.$$

In the following, we are devoted to obtaining the global convergence of the ACH method under conditions (C₁) – (C₃).

Lemma 2.1. *Let $\bar{u}(x) = \max_{1 \leq i \leq m} \{u_i(x)\}$, $X' = \{x \in R^n : \bar{u}(x) \leq 0\}$, then $X' = X$.*

The non-differentiability of $\bar{u}(x)$ prevents us to give a effective method to solve the periodicity problem. To overcome this difficulty, we use the technique of the aggregate function, and hence consider *the*

aggregate form of the inequality constraint functions $u_i(x)$, $i = 1, \dots, m$ as follows:

$$u(\theta\lambda, x) = \theta\lambda \ln \sum_{i=1}^m \exp(u_i(x) / \theta\lambda),$$

where $\theta \in (0, 1]$.

Let $X(\theta\lambda) = \{x \in R^n : u(\theta\lambda, x) \leq 0\}$, $X^0(\theta\lambda) = \{x \in R^n : u(\theta\lambda, x) < 0\}$ and $\partial X(\theta\lambda) = X(\theta\lambda) \setminus X^0(\theta\lambda)$. The following Lemmas 2.2-2.6 are easily to deduce.

Lemma 2.2. *For a given $\theta \in (0, 1]$, the following two conclusions hold:*

- (1) *For any $\lambda \in (0, 1]$, $\bar{u}(x) \leq u(\theta\lambda, x) \leq \bar{u}(x) + \theta\lambda \ln m$.*
- (2) *For $0 < \lambda_1 \leq \lambda_2 \leq 1$, $u(\theta\lambda_1, x) \leq u(\theta\lambda_2, x)$, in addition, when $\lambda \rightarrow 0$, $u(\theta\lambda, x) \rightarrow \bar{u}(x)$.*

Lemma 2.3. *If $u_i(x)$, $i = 1, \dots, m$ is C^3 , then $u(\theta\lambda, x)$ is also C^3 , and the gradient of $u(\theta\lambda, x)$ with respect to x is*

$$\nabla_x u(\theta\lambda, x) = \sum_{i=1}^m y_i(\theta\lambda, x) \nabla u_i(x), \quad (2.1)$$

where

$$y_i(\theta\lambda, x) = \exp(u_i(x) / \theta\lambda) / \sum_{i=1}^m \exp(u_i(x) / \theta\lambda). \quad (2.2)$$

Lemma 2.4. $y_i(\theta\lambda, x)$, $i = 1, \dots, m$ in (2.2) satisfies that

- (1) $0 \leq y_i(\theta\lambda, x)$ and $\sum_{i=1}^m y_i(\theta\lambda, x) = 1$.
- (2) $y_i(\theta\lambda, x) \rightarrow 0$ as $\lambda \rightarrow 0^+$ and $X^0 \ni x \rightarrow \bar{x} \in \partial X$ for $i \notin I(\bar{x})$.

Lemma 2.5. *Under condition (C_1) , we obtain:*

(1) *For any $\theta \in (0, 1]$, and $\lambda \in (0, 1]$, $X(\theta\lambda) \subset X$.*

(2) *For any closed subset $N \subset X^0$, there exists a $\theta \in (0, 1]$, such that $N \subset X^0(\theta)$.*

Set $\alpha(\theta\lambda, x) = \sum_{i=1}^m y_i(\theta\lambda, x)\alpha_i(x)$, then the following lemma holds:

Lemma 2.6. *Under conditions $(C_1) - (C_3)$, then the following two conclusions hold:*

(1) *There exists a $\theta \in (0, 1]$, such that for any $\lambda \in (0, 1]$ and $x \in \partial X(\theta\lambda)$, if*

$$\alpha(\theta\lambda, x)\beta = 0, \beta \geq 0,$$

then $\beta = 0$.

(2) *For any closed subset $N \subset \hat{X}$, there exists a $\theta \in (0, 1]$, such that for any $\lambda \in (0, 1]$,*

$$\{x + y\alpha(\theta\lambda, x) : y > 0\} \cap N = \emptyset, \quad (2.3)$$

where $x \in \partial X(\theta\lambda)$.

Proof. (1) If not, then there exist sequences $\{\lambda_k\}_{k=1}^\infty, \{x^{(k)}\}_{k=1}^\infty$, such that $\lambda_k \rightarrow 0, x^{(k)} \in \partial X(\theta\lambda_k)$ and

$$\alpha(\theta\lambda_k, x^{(k)}) = \sum_{i=1}^m y_i(\theta\lambda_k, x^{(k)})\alpha_i(x^{(k)}) = 0. \quad (2.4)$$

By Lemmas 2.4 and 2.5, we conclude that

$$0 \leq y_i(\theta\lambda_k, x^{(k)}), \quad \sum_{i=1}^m y_i(\theta\lambda_k, x^{(k)}) = 1, \quad (2.5)$$

and $X(\theta\lambda_k)$ is bounded. Hence there exist subsequences $\{\lambda_{k_i}\}_{i=1}^\infty$, $\{x^{(k_i)}\}_{i=1}^\infty$, and $\{y_i(\theta\lambda_{k_i}, x^{(k_i)})\}_{i=1}^\infty$ such that $\lambda_{k_i} \rightarrow 0$, $x^{(k_i)} \rightarrow x^* \in \partial X$, and $y_i(\theta\lambda_{k_i}, x^{(k_i)}) \rightarrow y_i(0, x^*)$. Moreover, by Lemma 2.4, $y_i(0, x^*) = 0$ for $i \notin I(x^*)$. Therefore when $k_i \rightarrow \infty$, from (2.4) and (2.5), we obtain

$$\alpha(\theta\lambda_k, x^{(k)}) \rightarrow \sum_{i \in I(x^*)} y_i(0, x^*) \alpha_i(x^*) = 0,$$

and

$$y_i(0, x^*) \geq 0, \quad \sum_{i \in I(x^*)} y_i(0, x^*) = 1,$$

which contradicts condition (C_2) .

(2) If not, then there exist sequences $\{\lambda_k\}_{k=1}^\infty$, $\{\hat{x}^{(k)}\}_{k=1}^\infty$, $\{\tilde{x}^{(k)}\}_{k=1}^\infty$, and $\{\tilde{z}^{(k)}\}_{k=1}^\infty$ such that $\lambda_k \rightarrow 0$, $\hat{x}^{(k)} \in N$, $\tilde{x}^{(k)} \in \partial\Omega(\theta\mu_k)$, $\tilde{z}^{(k)} > 0$, and

$$\hat{x}^{(k)} = \tilde{x}^{(k)} + \alpha(\theta\lambda_k, \tilde{x}^{(k)})\tilde{z}^{(k)} = \tilde{x}^{(k)} + \sum_{i=1}^m \alpha_i(\tilde{x}^{(k)}) y_i(\theta\lambda_k, \tilde{x}^{(k)}) \tilde{z}^{(k)}. \quad (2.6)$$

By Lemmas 2.4 and 2.5, we conclude that

$$0 \leq y_i(\theta\lambda_k, \tilde{x}^{(k)}), \quad \sum_{i=1}^m y_i(\theta\lambda_k, \tilde{x}^{(k)}) = 1, \quad (2.7)$$

and $X(\theta\lambda_k)$ is bounded. Hence there exist subsequences $\{\lambda_{k_i}\}_{i=1}^\infty$, $\{\hat{x}^{(k_i)}\}_{i=1}^\infty$, $\{\tilde{x}^{(k_i)}\}_{i=1}^\infty$, and $\{y_i(\theta\lambda_{k_i}, \tilde{x}^{(k_i)})\}_{i=1}^\infty$ such that $\lambda_{k_i} \rightarrow 0$, $\hat{x}^{(k_i)} \rightarrow \hat{x} \in N \subset \hat{X}$, $\tilde{x}^{(k_i)} \rightarrow \tilde{x} \in \partial X$, and $y_i(\theta\lambda_{k_i}, \tilde{x}^{(k_i)}) \rightarrow y_i(0, \tilde{x})$. Moreover,

by Lemma 2.4, we have $y_i(0, \tilde{x}) = 0$ for $i \notin I(\tilde{x})$. So from (2.7), we obtain

$$0 \leq y_i(0, \tilde{x}), \quad \sum_{i \in I(\tilde{x})} y_i(0, \tilde{x}) = 1.$$

By (2.6) and condition (C₂), as $k_i \rightarrow \infty$, we have $\tilde{z}^{(k_i)} \rightarrow \tilde{z}$ and $0 \leq \tilde{z} < \infty$. Then taking limits in (2.6), we have

$$\hat{x} = \tilde{x} + \sum_{i \in B(\tilde{x})} y_i(0, \tilde{x}) \tilde{z} \alpha_i(\tilde{x}),$$

which contradicts condition (C₃). $\square$

For finding the periodic solution of the differential equations, we construct the following aggregate homotopy:

$$H(P, P^{(0)}, \lambda) = \begin{pmatrix} (1 - \lambda)(x - x(T, x) + \alpha(\theta\lambda, x)y) + \lambda(x - x^{(0)}) \\ yu(\theta\lambda, x) - \lambda y^{(0)}u(\theta, x^{(0)}) \end{pmatrix} = 0, \quad (2.8)$$

where $P = (x, y) \in R^n \times R^1$ and $P^{(0)} = (x^{(0)}, y^{(0)}) \in \hat{X}^0 \times R_{++}^1$.

Given a point $P^{(0)}$, denote $H(P, P^{(0)}, \lambda)$ by $H_{P^{(0)}}(P, \lambda)$. Then let

$$H_{P^{(0)}}^{-1}(0) = \{(P, \lambda) \in X(\theta\lambda) \times R_+^1 \times (0, 1] : H_{P^{(0)}}(P, \lambda) = 0\},$$

be the zero-point set of $H_{P^{(0)}}$.

The following theorem is our main result, which presents a theoretic base of the ACH method.

Theorem 2.1. *Let H be defined as in (2.8). In addition, let conditions (C₁) – (C₃) hold, let $u_i(x)$, $i = 1, \dots, m$ be C^3 functions and let $\alpha_i(x)$, $i = 1, \dots, m$ be C^2 mappings. Then the Equation (1.1) has a T -periodic*

solution in X . Moreover, for almost every (x_0, y_0) there exists a C^1 path $(P(s), \lambda(s))$ of dimension 1 such that

$$H(P^{(0)}, P(s), \lambda(s)) = 0, \quad (P(0), \lambda(0)) = (P^{(0)}, 1), \quad (2.2)$$

and $\lambda(s)$ leads to 0; while $P(s)$ leads to a point (x^*, y^*) ; in particular, $x(t, x^*)$ is a T -periodic solution of (1.1).

Proof. Let the Jacobi matrix of $H(P, P^{(0)}, \lambda)$ be $DH(P, P^{(0)}, \lambda)$, we obtain

$$DH(P, P^{(0)}, \lambda) = \left(\frac{\partial H(P, P^{(0)}, \lambda)}{\partial P}, \frac{\partial H(P, P^{(0)}, \lambda)}{\partial P^{(0)}}, \frac{\partial H(P, P^{(0)}, \lambda)}{\partial \lambda} \right).$$

$\forall (P, \lambda) \in R^{n+1} \times (0, 1]$, the matrix is

$$\frac{\partial H(P, P^{(0)}, \lambda)}{\partial P^{(0)}} = \begin{pmatrix} -\lambda I & 0 \\ -\lambda y^{(0)} \nabla u(\theta, x^{(0)}) - \lambda u(\theta, x^{(0)}) \end{pmatrix}.$$

$DH(P, P^{(0)}, \lambda)$ is of full row rank because $\partial H(P, P^{(0)}, \lambda) / \partial P^{(0)}$ is nonsingular, so 0 is a regular value of $H(P, P^{(0)}, \lambda)$. Then for almost all $P^{(0)}$, 0 is a regular value of $H_{P^{(0)}}$. So by the fact that $H(P, P^{(0)}, 1) = 0$, we conclude that there exists a C^1 curve, which is denoted by $\Gamma_{P^{(0)}}$, such that

$$H(P(s), P^{(0)}, \lambda(s)) = 0, \quad (P(0), \lambda(0)) = (P^{(0)}, 1).$$

Because

$$\frac{\partial H_{P^{(0)}}(P^{(0)}, 1)}{\partial P} = \begin{pmatrix} I & 0 \\ y^{(0)} \nabla u(\theta, x^{(0)}) u(\theta, x^{(0)}) \end{pmatrix}$$

is nonsingular, therefore by using the classification theorem of one-dimensional smooth manifold, $\Gamma_{P^{(0)}}$ is diffeomorphic to the unit interval $(0, 1]$.

Let (P^*, λ^*) be a limit point of $\Gamma_{P^{(0)}}$. When $\lambda^* = 1$, then from (2.8), we obtain

$$\begin{aligned} x - x^{(0)} &= 0, \\ yu(\theta, x) - y^{(0)}u(\theta, x^{(0)}) &= 0. \end{aligned}$$

So $H(P, P^{(0)}, 1) = 0$ has a unique solution $P = P^{(0)}$. Then the fact that $(P^*, \lambda^*) = (x^*, y^*, \lambda^*) \in X^0(1) \times R_{++}^1 \times \{1\}$ does not occur.

Now we are devoting to proving the fact that $(P^*, \lambda^*) = (x^*, y^*, \lambda^*) \in \partial(X(\lambda^*) \times R_+^1) \times (0, 1]$ does not occur. If the conclusion does not hold, there exists a sequence of points $(x^{(k)}, y^{(k)}, \lambda_k) \in \Gamma_{P^{(0)}}$ such that $x^{(k)} \rightarrow x^*$, $\|y^{(k)}\| \rightarrow \infty$, and $\lambda_k \rightarrow \lambda^*$ as $k \rightarrow \infty$. From (2.8), we obtain

$$(1 - \lambda_k)\alpha(\theta\lambda_k, x^{(k)})y^{(k)} + x^{(k)} - x^{(0)} = (1 - \lambda_k)(x(T, x^{(k)}) - x^{(0)}). \quad (2.9)$$

As $k \rightarrow \infty$, the following equation holds:

$$\left(\lim_{k \rightarrow \infty} (1 - \lambda_k)y^{(k)} \right) \alpha(\theta, x^*) + x^* = x^{(0)}. \quad (2.10)$$

Condition (C₂) and (2.10) yield that

$$\lim_{k \rightarrow \infty} (1 - \lambda_k)y^{(k)} = \beta^*, \quad (2.11)$$

where $\beta^* \geq 0$. So (2.10) and (2.11) yield that

$$x^* + \alpha(\theta, x^*)\beta^* = x^{(0)}, \quad (2.12)$$

which contradicts Lemma 2.6.

If $\lambda^* < 1$, then (2.9) becomes

$$\begin{aligned} & (1 - \lambda_k) \alpha(\theta \lambda_k, x^{(k)}) y^{(k)} \\ &= (1 - \lambda_k) (x^{(k)} - x(T, x^{(k)})) - \lambda_k (x^{(k)} - x^{(0)}). \end{aligned} \quad (2.13)$$

When $k \rightarrow \infty$, it is easy to show that the right-hand side of (2.13) is bounded and the left-hand side of (2.13) is infinite,. This results in a contradiction.

As a result, the conclusion that $(P^*, \lambda^*) = (x^*, y^*, \lambda^*) \in X \times R_+^m \times \{0\}$ holds. In the following, it is easy to show that $x(t, x^*)$ is a T -periodic solution of (1.1). $\square$

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