p -RIP CONDITION FOR SIGNAL RECOVERY VIA l_p MINIMIZATION

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Abstract

We are concerned with the recovery of both sparse and partially sparse signals via l_p minimization. In sparse recovery, the concept of null space property via l_p -quasinorm (p -NSP) and the concept of p -restricted isometry property (p -RIP) for $0 < p \leq 1$ are very important, while the relation between them has not been studied. First, we will investigate the relation between the p -RIP and the p -NSP. Using the p -RIP, we get a sufficient condition for sparse signal

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recovery from noisy measurements. Then, we generalize these properties to the partially sparse recovery and extend the result to partially sparse recovery based on the partially p -RIP.

1. Introduction

We consider the signal recovery problem in which one part of the signal is known to be sparse while the other part is possibly dense. It is the so called partially sparse recovery (PSSR) problem [1, 2]. It is clear that the PSSR contains the SSR (sparse signal recovery) as a special case. As the PSSR widely appears in signal and image processing such as the magnetic resonance imaging (MRI) reconstruction, it has recently received much attention [1, 2, 19, 24].

Suppose that the signal $x \in R^N$ is split into two parts $x = (x_1, x_2)$, where $x_1 \in R^{N-r}$ is sparse and $x_2 \in R^r$ is possibly dense. The PSSR is stated as the following minimization problem:

$$\min \|x_1\|_0 \quad \text{s.t.} \quad A_1 x_1 + A_2 x_2 = y, \quad (1.1)$$

where $\|x_1\|_0$ stands for the number of nonzero elements in x_1 , $A_1 \in R^{k \times (N-r)}$ and $A_2 \in R^{k \times r}$. This problem is of course NP-hard because its special case where $r = 0$ is NP-hard. Like the SSR, a possible way to deal with problem (1.1) is to solve its convex approximation problem [1]

$$\min \|x_1\|_1 \quad \text{s.t.} \quad A_1 x_1 + A_2 x_2 = y, \quad (1.2)$$

where $\|x\|_1$ stands for the l_1 -norm of vector x . It is a natural generalization of l_1 -minimization for problem SSR corresponding to the case $r = 0$.

We say that matrix A satisfies the restricted isometry property (RIP) of order s , with the restricted isometry constant (RIC) δ_s , if δ_s is the smallest number, such that the inequalities

$$(1 - \delta_s) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s) \|x\|_2^2,$$

hold for all s -sparse vector x . We say that a vector x is s -sparse if at most s components of x are nonzero, i.e., $\|x\|_0 \leq s$. There have been well-known sufficient conditions that guarantee SSR (exactly or approximately) via l_1 minimization, e.g., $\delta_{2s} \leq \sqrt{2} - 1$ [3] and $\delta_{2s} + \delta_{3s} < 1$ [10]. Recently, Vaswani and Lu [24] proposed a modified compressive sensing (abbreviated as modified-CS) for signal recovery with partially known support (which is actually equivalent to the PSSR). Sufficient conditions for exact recovery were given there. These conditions were shown to be much weaker than the two sufficient conditions in [10] and [3]. Jacques [19] extended the results of Vaswani and Lu [24] to the case of compressible signals and noisy measurements. We notice that these sufficient conditions are obtained under the assumption that the sensing matrix A satisfies the RIP. They are some what weaker than the existing RIP conditions for SSR problem.

Recently, Bandeira et al. [1] introduced the concept of the so-called partial restricted isometry property (partial RIP) and partial null space property (PNSP) for the partial sparse recovery. It was proved that the PNSP provided a sufficient and necessary condition for exact recovery of problem (1.2), while the partial RIP is sufficient for partial sparse recovery with noisy measurements [1]. The results obtained by Bandeira et al. [1] extended the classical results in [3] to the partial sparse recovery in noisy case.

We are interested in the following l_p minimization problem for PSSR:

$$\min \|x_1\|_p^p \quad \text{s.t.} \quad A_1 x_1 + A_2 x_2 = y. \quad (1.3)$$

It is a natural generalization of the l_p minimization problem for SSR. A great number of numerical experiments have shown that fewer measurements were required for exact recovery of SSR, if we replace the l_1 -norm by l_p -norm ($0 < p < 1$) (see [4] for details). The study for SSR

problem via l_p minimization has taken good progress (see, e.g., [4, 8, 16, 21-23]). On the other hand, however, related study about PSSR via l_p minimization problem is relatively fewer. Recently, a sufficient condition for exact recovery of PSSR via l_p minimization was established in [26], by generalizing the p -RIP to PSSR problem.

In this paper, we further study the PSSR via l_p minimization problem (1.3). We first extend a result in [22] to the compressible signal recovery problem where the measurements contain noise. We then generalize the concept of p -RIP to partial p -RIP, and extend the related results in [22] to the problem in which the signals are not partially sparse, but partially compressible. The condition we obtained here is shown to be weaker than that in [26].

The remainder of the paper is organized as follows. Section 2 is dedicated to the SSR problem via l_p minimization. In this section, we will derive a relation between the concept of the p -NSP and the concept of p -RIP. We will also give a p -RIP condition for SSR. It is a generalization of the related result obtained in [22] in the noisy case. Section 3 is concerned with the PSSR problem. In the section, we will generalize the p -RIP to partial p -RIP and establish a partial p -RIP condition for PSSR.

Throughout the paper, we will use the following notations. We use $[N]$ to denote the set of integers $\{1, 2, \dots, N\}$, and $[N]^s$ to denote the set of all subsets of $[N]$ with cardinality $s \leq N$. For matrix A , we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ its null space and range, respectively. For a vector $v \in R^N$ and an index set $S \in [N]^s$, we use $v_S \in R^N$ to denote the vector whose elements are defined by $(v_S)_i = v_i, \forall i \in S$ and $(v_S)_i = 0, \forall i \notin S$. $|v|$ would be a vector with the i -th element $|v_i|$. We denote by S^c the complement of S with respect to $\{1, 2, \dots, N\}$. In many places, we will use the following function:

$$\sigma_t(\bar{x})_p = \inf_{\|z\|_0 \leq t} \|\bar{x} - z\|_p.$$

It is the best t -term approximation error of a vector $\bar{x} \in R^N$ in l_p -norm.

2. Sparse Signal Recovery via l_p Minimization

In this section, we are going to derive a sufficient condition for recovering compressible signals in noisy case via l_p minimization. We will investigate the relationship between the p -RIP and the p -NSP. The concept of the p -NSP was introduced by Foucart and Rauhut in [15] and has been shown to be a sufficient and necessary condition for exact SSR via l_p minimization problem. The concept of the p -RIP was introduced by Chartrand and Staneva in [8]. The success of sparse recovery was then established under a p -RIP condition for l_1 minimization problem.

2.1. A relation between p -NSP and p -RIP

The main question addressed in compressed sensing is under what conditions on the sensing matrix A can a sparse vector $\bar{x} \in R^N$ be recovered by solving the following problem:

$$\min_{x \in R^N} \|x\|_1 \quad \text{s.t.} \quad Ax = y, \quad (2.1)$$

where $y := A\bar{x}$. The null space property (NSP) is a well known characterization of such matrices, and has been extensively studied (see, e.g., [6, 12-14, 25, 27, 28]). The definition of the NSP may be slightly different in the literature. Here we adopt the definition in [6].

Definition 2.1 (NSP). Matrix $A \in R^{k \times N}$ is said to satisfy the Null Space Property (NSP) of order s if for every $\eta \in \mathcal{N}(A) \setminus \{0\}$ and every $S \in [N]^s$, one has

$$\|\eta_S\|_1 < \frac{1}{2} \|\eta\|_1.$$

It is well known that the NSP is a necessary and sufficient condition for the recovery of an s -sparse vector $\bar{x}$ via l_1 minimization.

Foucart and Rauhut [15] extended the above NSP to the p -NSP concerning the l_p -minimization problem

$$\min_{x \in R^N} \|x\|_p^p \quad \text{s.t.} \quad Ax = y. \quad (2.2)$$

Definition 2.2 (p -NSP). A matrix $A \in R^{k \times N}$ is said to satisfy the p -NSP of order s if the inequality

$$\|\eta_S\|_p < \|\eta_{S^c}\|_p,$$

holds for all index sets S with $|S| \leq s$ and all $\eta \in \mathcal{N}(A) \setminus \{0\}$.

The theorem below shows that the p -NSP is a sufficient and necessary condition for SSR via l_p minimization problem. Details can be found in [15].

Theorem 2.3. *Every s sparse vector $\bar{x}$ is the unique solution of problem (2.2) for $y := A\bar{x}$ if and only if matrix A satisfies the p -NSP of order s .*

The following concept of p -RIP was from [8].

Definition 2.4 (p -RIP). Given a matrix $A \in R^{k \times N}$. Given positive integer s and $0 < p \leq 1$. We say that matrix A satisfies the restricted p -isometry property (p -RIP) with the constant $\delta_s \in (0, 1)$, if δ_s is the smallest number such that

$$(1 - \delta_s) \|x\|_2^p \leq \|Ax\|_p^p \leq (1 + \delta_s) \|x\|_2^p,$$

for all s -sparse vector x .

In [18], Fornasier and Rauhut have shown that the NSP implied by the RIP. The following theorem shows that p -NSP can also be implied by p -RIP.

The following theorem established a relationship between the p -NSP and the p -RIP.

Theorem 2.5. *Assume that $A \in R^{k \times N}$ satisfies the p -RIP of the order $s = t + h$ with constant $\delta_s \in (0, 1)$. Then A satisfies the p -NSP of order t if $\gamma \triangleq \frac{1 + \delta_h}{1 - \delta_s} \left(\frac{t}{h} \right)^{1-p/2} < 1$.*

Proof. Let $\eta \in \mathcal{N}(A) \setminus \{0\}$, $T \in [N]^t$. Define $T_0 = T$ and T_1 be the set of indices of h largest absolute value components of η_{T^c} , T_2 the next h largest, and so on. Assume that the last index set is T_J . Then the size of T_J is not larger than h . With this rearrangement of η , we obtain

$$|\eta_l| \leq |\eta_{l'}|, \text{ for all } l' \in T_{k'}, l \in T_k, 1 \leq k' < k \leq J. \quad (2.3)$$

Let $T_{01} = T_0 \cup T_1$. Note that $A\eta = 0$ implies $A\eta_{T_{01}} = -\sum_{j=2}^J A\eta_{T_j}$. It follows from the Hölder inequality and the p -RIP that

$$\begin{aligned} \|\eta_T\|_p^p &= \|\eta_{T_0}\|_p^p \leq t^{1-\frac{p}{2}} \|\eta_{T_0}\|_2^p \leq t^{1-\frac{p}{2}} \|\eta_{T_{01}}\|_2^p \\ &\leq t^{1-\frac{p}{2}} \frac{1}{1-\delta_s} \|A\eta_{T_{01}}\|_p^p \leq t^{1-\frac{p}{2}} \frac{1}{1-\delta_s} \sum_{i=2}^J \|A\eta_{T_i}\|_p^p \\ &\leq t^{1-\frac{p}{2}} \frac{1+\delta_h}{1-\delta_s} \sum_{i=2}^J \|\eta_{T_i}\|_2^p. \end{aligned} \quad (2.4)$$

By (2.3), we derive that for $l \in T_{i+1} (i = 1, 2, \dots, J-1)$,

$$h|\eta_l|^p \leq \|\eta_{T_i}\|_p^p.$$

Taking the p -th root and then squaring both sides of the inequality, we get

$$h^{\frac{2}{p}} |\eta_l|^2 \leq \|\eta_{T_i}\|_p^2.$$

Making summation in both sides of the inequality over $l \in T_{i+1}$, we obtain

$$\|\eta_{T_{i+1}}\|_2^2 \leq h^{1-\frac{2}{p}} \|\eta_{T_i}\|_p^2.$$

This implies

$$\sum_{j=2}^J \|\eta_{T_j}\|_2^p \leq h^{\frac{p}{2}-1} \sum_{j=1}^J \|\eta_{T_j}\|_p^p = h^{\frac{p}{2}-1} \|\eta_{T^c}\|_p^p.$$

Combining this with (2.4), we get

$$\|\eta_T\|_p^p \leq \frac{1 + \delta_h}{1 - \delta_s} \left(\frac{t}{h}\right)^{1-\frac{p}{2}} \|\eta_{T^c}\|_p^p.$$

Since $\gamma = \frac{1 + \delta_h}{1 - \delta_s} \left(\frac{t}{h}\right)^{1-\frac{p}{2}} < 1$, we get the desired result. The proof is complete. $\square$

2.2. A p -RIP condition for compressible signal recovery in the presence of noise

In [4], Chartrand has shown that for sparse signal, the p -RIP condition:

$$\delta_{aK} + b\delta_{(a+1)K} < b - 1, \quad b > 1, \quad a = b^{\frac{2}{2-p}}, \quad \text{and} \quad K = \|\bar{x}\|_0,$$

can guarantee exact recovery from l_p minimization problem (2.2). This p -RIP condition is a generalization of Theorem 2.1 in [5]. Numerical experiments have shown that the p -RIP condition is met by fewer measurements for smaller p . In most realistic applications, the true signal vector $\bar{x}$ is not sparse but rather compressible, meaning that most components are very small but not necessarily zero. In this case, Shen and Li [22] generalized the result in [4] to the following theorem:

Theorem 2.6. *Let t and h be positive integers and $0 < p < 1$.*

Suppose that matrix A satisfies $\gamma := \frac{1 + \delta_h}{1 - \delta_{h+t}} (t/h)^{1-p/2} < 1$. Then the

solution of problem (2.2) x^ obeys*

$$\|\bar{x} - x^*\|_p^p \leq \frac{2 + 2\gamma}{1 - \gamma} \sigma_t(\bar{x})_p^p,$$

and

$$\|\bar{x} - x^*\|_2^p \leq C_{2,p} \frac{\sigma_t(\bar{x})_p^p}{t^{1-p/2}},$$

for all $\bar{x} \in R^N$, where $C_{2,p} = \frac{2\gamma(2 + \delta_h - \delta_{h+t})}{(1 + \delta_h)(1 - \gamma)}$.

In practice, the observed measurement vector y may contain noise. If the signal $\bar{x}$ is compressible and the measurement vector contains noise, then the minimization problem becomes

$$\min \|x\|_p^p \quad \text{s.t.} \quad \|Ax - y\|_2 \leq \epsilon. \quad (2.5)$$

We generalize Theorem 2.6 to this case and get the following result:

Theorem 2.7. *Let t and h be positive integers and $0 < p \leq 1$.*

Suppose that matrix A satisfies the p -RIP of order $s = t + h$ with the

p -RIP condition: $\gamma := \frac{1 + \delta_h}{1 - \delta_s} \left(\frac{t}{h}\right)^{1-p/2} < 1$. Then the solution x^ to*

problem (2.5) obeys

$$\|\bar{x} - x^*\|_2^p \leq C_1 \epsilon^p + C_2 \sigma_t(\bar{x})_p^p, \quad (2.6)$$

and

$$\|\bar{x} - x^*\|_p^p \leq C_3 \epsilon^p + C_4 \sigma_t(\bar{x})_p^p, \quad (2.7)$$

where $\bar{x}$ is the origin signal,

$$C_1 = \frac{2(1 + (t/h)^{1-\frac{p}{2}})N^{1-\frac{p}{2}}}{1 - \delta_s - (t/h)^{1-\frac{p}{2}}(1 + \delta_h)}, C_2 = 2h^{\frac{p}{2}-1} + \frac{2(1 + (t/h)^{1-\frac{p}{2}})(1 + \delta_h)h^{\frac{p}{2}-1}}{1 - \delta_s - (t/h)^{1-\frac{p}{2}}(1 + \delta_h)},$$

$$C_3 = \frac{4(tN)^{1-\frac{p}{2}}}{1 - \delta_s - (t/h)^{1-\frac{p}{2}}(1 + \delta_h)}, C_4 = 2 + \frac{4(t/h)^{1-\frac{p}{2}}(1 + \delta_h)}{1 - \delta_s - (t/h)^{1-\frac{p}{2}}(1 + \delta_h)}.$$

Proof. Denote $v = x^* - \bar{x}$. Let T_0 denote the set of indexes of t largest absolute value components of v , T_1 the next h largest of $v_{T_0^c}$, T_2 the next h largest of $v_{T_{01}^c}$, and so on. Here $T_{01} = T_0 \cup T_1$. We have

$$\begin{aligned} \|\bar{x}\|_p^p &\geq \|x^*\|_p^p = \|\bar{x}_{T_0} + v_{T_0} + \bar{x}_{T_0^c} + v_{T_0^c}\|_p^p \\ &= \|\bar{x}_{T_0} + v_{T_0}\|_p^p + \|\bar{x}_{T_0^c} + v_{T_0^c}\|_p^p \\ &\geq \|\bar{x}_{T_0}\|_p^p - \|v_{T_0}\|_p^p + \|v_{T_0^c}\|_p^p - \|\bar{x}_{T_0^c}\|_p^p, \end{aligned}$$

which implies

$$\|v_{T_0}\|_p^p + 2\|\bar{x}_{T_0^c}\|_p^p \geq \|v_{T_0^c}\|_p^p. \quad (2.8)$$

Notice that

$$\|\bar{x} - x^*\|_2^p = \|v\|_2^p = \|v_{T_{01}} + v_{T_{01}^c}\|_2^p \leq \|v_{T_{01}}\|_2^p + \|v_{T_{01}^c}\|_2^p. \quad (2.9)$$

We are going to estimate the two terms $\|v_{T_{01}}\|_2^p$ and $\|v_{T_{01}^c}\|_2^p$ at the right-hand side of (2.9). We first derive an upper bound for $\|v_{T_{01}^c}\|_2^p$. For $j > 0$ and $u \in T_{j+1}$, we denote $|(v_{T_{j+1}})_u|$ the u -th element in $|v_{T_{j+1}}|$. We have

$$h|(v_{T_{j+1}})_u|^p \leq \|v_{T_j}\|_p^p.$$

Taking the p -th root, squaring, and making summation over $u \in T_{j+1}$ in both sides of the last inequality, we obtain

$$\|v_{T_{j+1}}\|_2^2 \leq h^{1-2/p} \|v_{T_j}\|_p^2.$$

It then follows that

$$\sum_{j>0} \|v_{T_{j+1}}\|_2^p \leq h^{p/2-1} \sum_{j>0} \|v_{T_j}\|_p^p.$$

That is to say

$$\|v_{T_{01}}^c\|_2^p \leq \sum_{j>0} \|v_{T_{j+1}}\|_2^p \leq h^{p/2-1} \sum_{j>0} \|v_{T_j}\|_p^p = h^{\frac{p}{2}-1} \|v_{T_0}^c\|_p^p.$$

Using (2.8), we get

$$\|v_{T_{01}}^c\|_2^p \leq h^{\frac{p}{2}-1} (\|v_{T_0}\|_p^p + 2\|\bar{x}_{T_0}^c\|_p^p). \quad (2.10)$$

By Hölder inequality, we have $\|v_{T_0}\|_p^p \leq t^{1-\frac{p}{2}} \|v_{T_0}\|_2^p$, with together with the fact $\|v_{T_0}\|_2^p \leq \|v_{T_{01}}\|_2^p$ and (2.10) implies

$$\|v_{T_{01}}^c\|_2^p \leq h^{\frac{p}{2}-1} (t^{1-\frac{p}{2}} \|v_{T_{01}}\|_2^p + 2\|\bar{x}_{T_0}^c\|_p^p). \quad (2.11)$$

The last inequality gives an upper bound for $\|v_{T_{01}}^c\|_2^p$. We turn to the estimate of an upper bound for $\|v_{T_{01}}\|_2^p$. Since

$$\|Av\|_p^p \geq \|Av_{T_{01}}\|_p^p - \sum_{j>1} \|Av_{T_j}\|_p^p,$$

by the p -RIP and (2.11), we obtain

$$\begin{aligned}
\|Av\|_p^p &\geq (1 - \delta_s) \|v_{T_{01}}\|_2^p - (1 + \delta_h) \sum_{j>1} \|v_{T_j}\|_2^p \\
&\geq (1 - \delta_s) \|v_{T_{01}}\|_2^p - (1 + \delta_h) h^{\frac{p}{2}-1} (t^{1-\frac{p}{2}} \|v_{T_{01}}\|_2^p + 2 \|\bar{x}_{T_0^c}\|_p^p) \\
&= [1 - \delta_s - h^{\frac{p}{2}-1} t^{1-\frac{p}{2}} (1 + \delta_h)] \|v_{T_{01}}\|_2^p - 2(1 + \delta_h) h^{\frac{p}{2}-1} \|\bar{x}_{T_0^c}\|_p^p.
\end{aligned} \tag{2.12}$$

Taking into account that both $\bar{x}$ and x^* satisfy the constraint $\|Ax - y\|_2 \leq \epsilon$, we get

$$\|Av\|_p^p \leq \|A\bar{x} - y\|_p^p + \|Ax^* - y\|_p^p \leq 2N^{1-\frac{p}{2}} \epsilon^p. \tag{2.13}$$

Combining (2.12) with (2.13), we can get

$$\|v_{T_{01}}\|_2^p \leq \frac{2N^{1-\frac{p}{2}} \epsilon^p + 2(1 + \delta_h) h^{\frac{p}{2}-1} \|\bar{x}_{T_0^c}\|_p^p}{1 - \delta_s - h^{\frac{p}{2}-1} t^{1-\frac{p}{2}} (1 + \delta_h)}. \tag{2.14}$$

Finally, according to (2.9), (2.11), and (2.14), we obtain

$$\begin{aligned}
\|v\|_2^p &\leq (1 + (\frac{t}{h})^{1-\frac{p}{2}}) \|v_{T_{01}}\|_2^p + 2h^{\frac{p}{2}-1} \|\bar{x}_{T_0^c}\|_p^p \\
&\leq \frac{2(1 + (\frac{t}{h})^{1-\frac{p}{2}}) N^{1-\frac{p}{2}}}{1 - \delta_s - (t/h)^{1-\frac{p}{2}} (1 + \delta_h)} \epsilon^p \\
&\quad + (2h^{\frac{p}{2}-1} + \frac{2(1 + (t/h)^{1-\frac{p}{2}}) (1 + \delta_h) h^{\frac{p}{2}-1}}{1 - \delta_s - (t/h)^{1-\frac{p}{2}} (1 + \delta_h)}) \|\bar{x}_{T_0^c}\|_p^p.
\end{aligned}$$

This together with the fact $\|\bar{x}_{T_0^c}\|_p^p = \sigma_t(\bar{x})_p^p$ yields (2.6).

Similarly, by the use of (2.8), the Hölder inequality and (2.14), we can prove (2.7). $\square$

3. A p -RIP Condition for PSSR via l_p Minimization

In this section, we extend the concept of p -NSP and p -RIP to the case of PSSR. We will also show that the p -NSP for PSSR is also a necessary and sufficient condition for the PSSR via l_p minimization problem. The results obtained in this section extends the results in the last section. We first extend the concept of p -NSP to the case of PSSR.

Definition 3.1 (p -NSP for PSSR). We say that $A = (A_1, A_2)$ satisfies the p -NSP of order $s - r$ for PSSR of size $N - r$ with $r \leq s$ if A_2 is of full column rank ($\mathcal{N}(A_2) = \{0\}$) and for every $v_1 \in R^{N-r} \setminus \{0\}$ such that $A_1 v_1 \in \mathcal{R}(A_2)$ and every $S \in [N - r]^{(s-r)}$, it holds that

$$\|(v_1)_S\|_p < \|(v_1)_{S^c}\|_p. \quad (3.1)$$

It is easy to see that if $r = 0$, the p -NSP for PSSR reduce to the p -NSP for SSR in Definition 2.2. In this case, the p -NSP is a necessary and sufficient condition for $\bar{x}$ to be the unique solution of problem (2.2) with $y = A\bar{x}$. The following theorem extends this result to the case of PSSR.

Theorem 3.2. *The matrix $A = (A_1, A_2)$ satisfies the p -NSP of order $s - r$ for PSSR of size $N - r$ if and only if for every $\bar{x} = (\bar{x}_1, \bar{x}_2)$ such that $\bar{x}_1 \in R^{N-r}$ is $(s - r)$ -sparse and $\bar{x}_2 \in R^r$, problem (1.3) with $y = A\bar{x}$ has a unique solution and it is given by $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$.*

Proof. We prove the result using a similar way to that in [1] with appropriate modifications.

Firstly, suppose that for any vector $(\bar{x}_1, \bar{x}_2) \in R^N$, where $\bar{x}_1$ is an $(s-r)$ -sparse vector and $\bar{x}_2 \in R^r$, the minimizer (x_1, x_2) of $\|x_1\|_p^p$ subject to $A_1x_1 + A_2x_2 = A\bar{x}$ satisfies $x_1 = \bar{x}_1$. For any $v_1 \neq 0$ such that $A_1v_1 \in \mathcal{R}(A_2)$, consider the minimization problem

$$\begin{aligned} \min \quad & \|x_1\|_p^p, \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 = A_1(v_1)_S + A_2v_2, \end{aligned}$$

where $v_2 \in R^r$ and $\in [N-r]^{(s-r)}$. By assumption, the minimizer (x_1, x_2) satisfies $x_1 = (v_1)_S$. Since $A_1v_1 \in \mathcal{R}(A_2)$, there exists u_2 such that $A_1(-(v_1)_{S^c}) + A_2u_2 = A_1(v_1)_S + A_2v_2$. Due to the fact $-(v_1)_{S^c} \neq (v_1)_S$, $(-(v_1)_{S^c}, u_2)$ will not be the solution of the problem. Consequently, $\|(v_1)_{S^c}\|_p^p > \|(v_1)_S\|_p^p$, and (3.1) holds.

Conversely, suppose that matrix A satisfies the p -NSP of order $s-r$ for PSSR of size $N-r$. For a given vector $(\bar{x}_1, \bar{x}_2) \in R^N$, where $\bar{x}_1$ is $(s-r)$ -sparse and $\bar{x}_2 \in R^r$, and a vector $(u_1, u_2) \in R^N$ with $u_1 \neq \bar{x}_1$ and satisfying $A_1u_1 + A_2u_2 = A_1\bar{x}_1 + A_2\bar{x}_2$, consider $(v_1, v_2) = ((\bar{x}_1 - u_1), (\bar{x}_2 - u_2)) \in \mathcal{N}(A)$. It is clear that $A_1v_1 \in \mathcal{R}(A_2)$ and $v_1 \neq 0$. Thus, setting S to be the support of $\bar{x}$, one has

$$\begin{aligned} \|\bar{x}_1\|_p^p &\leq \|\bar{x}_1 - (u_1)_S\|_p^p + \|(u_1)_S\|_p^p = \|(v_1)_S\|_p^p + \|(u_1)_S\|_p^p \\ &< \|(v_1)_{S^c}\|_p^p + \|(u_1)_S\|_p^p = \|-(u_1)_{S^c}\|_p^p + \|(u_1)_S\|_p^p \\ &= \|u_1\|_p^p, \end{aligned}$$

where the strict inequality follows from (3.1). This shows that every solution (x_1, x_2) to the problem (1.3) with $y = A\bar{x}$ satisfies $x_1 = \bar{x}_1$. On

the other hand, it is obvious that $x_2 = \bar{x}_2$ is uniquely determined by solving $A_2 x_2 = y - A_1 \bar{x}_1$ if and only if A_2 is full column rank. The proof is complete. $\square$

In what follows, we extend the concept of the p -RIP to the case of PSSR. Let $A = (A_1, A_2)$ with A_2 being full column rank. Denote by $\mathcal{P}$ the orthogonal projection operator from R^N onto $\mathcal{R}(A_2)^\perp$, namely,

$$\mathcal{P} = I - A_2(A_2^T A_2)^{-1} A_2^T. \quad (3.2)$$

Definition 3.3 (p -RIP for PSSR). We say that $\delta_{s-r}^r > 0$ is the partial p -RIC of order $s - r$ of the matrix $A = (A_1, A_2) \in R^{k \times N}$ for recovery of size $N - r$ with $r \leq s$, if A_2 is full column rank and δ_{s-r}^r is the p -RIP constant of order $s - r$ of the matrix $\mathcal{P}A_1$, where $\mathcal{P}$ is given by (3.2). That is, δ_{s-r}^r is the smallest number, such that the inequalities

$$(1 - \delta_{s-r}^r) \|x\|_2^p \leq \|\mathcal{P}A_1 x\|_p^p \leq (1 + \delta_{s-r}^r) \|x\|_2^p,$$

are satisfied for all $(s - r)$ -sparse vectors x .

Remark 3.4. It is easy to see that when $r = 0$, the partial p -RIP reduces to the p -RIP.

Again, we consider the PSSR where the true signal $\bar{x}$ is not sparse but rather partially compressible (i.e., $\bar{x}_1$ is compressible), and the measurement vector y contains noise. This results in the following minimization problem:

$$\begin{cases} \min \|x_1\|_p^p, \\ \text{s.t. } \|Ax - y\|_2 \leq \epsilon. \end{cases} \quad (3.3)$$

We have the theoretical result below

Theorem 3.5. Let $A = (A_1, A_2) \in R^{k \times N}$ with $A_1 \in R^{k \times (N-r)}$ and $A_2 \in R^{k \times r}$ satisfy the partial p -RIP of order $s - r$ with the partial p -RIC δ_{s-r}^r . For any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in R^N$, we let the noisy measurements $y = A\bar{x} + \xi$ satisfy $\|\xi\|_2 \leq \epsilon$. Let $x^* = (x_1^*, x_2^*)$ be a solution of problem (3.3). If $s - r = t + h$ and $\gamma := \frac{1 + \delta_{h-r}^r}{1 - \delta_{s-r}^r} \left(\frac{t}{h}\right)^{1-p/2} < 1$, then

$$\|\bar{x}_1 - x_1^*\|_2^p \leq C_1 \epsilon^p + C_2 \sigma_t(\bar{x}_1)_p^p, \quad (3.4)$$

and

$$\|\bar{x}_2 - x_2^*\|_2^p \leq D_2((2\epsilon)^p + D_1 \|\bar{x}_1 - x_1^*\|_2^p), \quad (3.5)$$

where

$$D_1 = \|A_1\|_2^p = \max_{x_1 \neq 0} \frac{\|A_1 x_1\|_2^p}{\|x_1\|_2^p} \quad \text{and} \quad D_2 = \|A_2^\dagger\|^p = \frac{1}{\min_{x_2 \neq 0} \frac{\|A_2 x_2\|_2^p}{\|x_2\|_2^p}}.$$

Proof. It follows from Definition 3.3 that matrix $\mathcal{P}A_1$ satisfies the condition in Theorem 2.7. Since $\mathcal{P}$ is a projection matrix, it holds that

$$\|\mathcal{P}A_1 \bar{x}_1 - \mathcal{P}y\|_2 = \|\mathcal{P}A\bar{x} - \mathcal{P}y\|_2 \leq \|A\bar{x} - y\| \leq \epsilon,$$

and a solution x'_1 of

$$\min_{x_1 \in R^{N-r}} \|x_1\|_p^p, \quad \text{s.t. } \|\mathcal{P}A_1 x_1 - \mathcal{P}y\|_2 \leq \epsilon, \quad (3.6)$$

satisfies

$$\|\bar{x}_1 - x'_1\|_2^p \leq C_1 \epsilon^p + C_2 \sigma_t(\bar{x}_1)_p^p. \quad (3.7)$$

We first show that problems (3.3) and (3.6) are equivalent, which will complete the proof of (3.4). We show this equivalency by showing that both problems have the same feasible region.

Let (x_1^*, x_2^*) be a feasible point of (3.3). We have

$$\|\mathcal{P}A_1x_1^* - \mathcal{P}y\|_2 = \|\mathcal{P}(A_1x_1^* + A_2x_2^* - y)\|_2 \leq \|A_1x_1^* + A_2x_2^* - y\|_2 \leq \epsilon.$$

It implies that x_1^* is a feasible point of (3.6). Now let x_1' be a feasible point of (3.6). Since $I - \mathcal{P}$ projects (orthogonally) onto the column space of A_2 , there must be an x_2' such that $A_2x_2' = (I - \mathcal{P})(y - A_1x_1')$, and hence $\|A_1x_1' + A_2x_2' - y\|_2 = \|\mathcal{P}A_1x_1' - \mathcal{P}y\|_2 \leq \epsilon$. It shows that (x_1', x_2') is a feasible point of (3.3).

The above arguments have shown that the sets of all feasible points of problems (3.3) and (3.6) coincide. Consequently, both problems are equivalent. The inequality (3.4) then follows from (3.7) directly.

We turn to the proof of (3.5). Since both $\bar{x}$ and x^* satisfy the measurements constraints $\|Ax - y\|_2 \leq \epsilon$, we have $\|Ax^* - A\bar{x}\|_2^p \leq 2\epsilon^p$, and thus

$$\|A_2(x_2^* - \bar{x}_2)\|_2^p \leq 2\epsilon^p + \|A_1(x_1^* - \bar{x}_1)\|_2^p. \quad (3.8)$$

By the definitions of D_1 and C_2 , we have $\|x_2^* - \bar{x}_2\|_2^p \leq D_2(2\epsilon^p + D_1\|x_1^* - \bar{x}_1\|_2^p)$. The result (3.5) then follows from (3.4) and (3.8). $\square$

Remark 3.6. Let $k = s - r$. If $h = at$ and $a > 1$, then the partial p -RIP condition implies $\delta_k < \frac{a^{1-p/2} - 1}{a^{1-p/2} + 1}$. Since $\frac{a^{1-p/2} - 1}{a^{1-p/2} + 1}$ is decreasing in p and $0 < p \leq 1$, we obviously have $\delta_k < \frac{\sqrt{a} - 1}{\sqrt{a} + 1}$, which is weaker than $\delta_{(a+1)k} < \frac{\sqrt{a} - 1}{\sqrt{a} + 1}$ in [26]. Consequently, the sufficient condition obtained in the paper is weaker than that in [26].

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