k PART PARTITIONS OF INTEGERS

ZDZISLAW TRUKSZYN and RYSZARD PALKA

Department of Electrical Machines and Drives

West Pomeranian University of Technology

Szczecin Al. Piastow 17

70-310 Szczecin

Poland

e-mail: rpalka@zut.edu.pl

Abstract

This paper presents formulas (together with their proofs) determining 3 and 4 part partitions of any integer. These formulas were derived using the properties of the floor function and Bernoulli formulas for various powers of finite sums of the floor function series. This made it possible to obtain above formulas in a much simpler way than most traditional methods.

1. Introduction

Partitions of natural numbers $p(n)$, understood as the number of sets of positive integers, whose sum is equal to n , have been the subject of interest of mathematicians for hundreds of years [1], [2], [4], [5], [7]. This problem was studied, among others, by Leonhard Euler, and the most significant result of these studies was the presentation, and after several years the proving, of the recursive formula for $p(n)$, using the pentagonal numbers

2020 Mathematics Subject Classification: Primary 11P81; Secondary 11P84.

Keywords and phrases: k part partitions, floor function.

Received March 12, 2021

$$\begin{aligned}
p(n) &= \sum_{m \geq 1} (-1)^{m-1} p[n - m(3m \pm 1)/2] \\
&= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) \dots \quad (1)
\end{aligned}$$

The asymptotic formula for $p(n)$ was given and proved by Hardy and Ramanujan, and then refined by Rademacher [4], [5]

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \quad n \rightarrow \infty, \quad (2)$$

and

$$p(n) \approx \frac{1}{\pi\sqrt{2}} \sum_{m=1}^{\infty} A_m(n) \sqrt{m} \frac{d}{dn} \left\{ \frac{1}{\sqrt{n-1/24}} \sinh \left[\frac{\pi}{m} \sqrt{\frac{2}{3}} (n-1/24) \right] \right\}. \quad (3)$$

The function $p(n, k)$ is defined as the set of k part partitions of number n , that is, the number of all strings with natural numbers:

$$x_1 + x_2 + \dots + x_k = n \text{ wherein } x_1 \geq x_2 \geq \dots \geq x_k. \quad (4)$$

In this paper, the study of partitions of natural numbers was carried out using the floor function [3], [6], [8]. The floor function is defined as the largest integer k not greater than x . The floor function will be denoted here as $\text{floor}(x) = [x]$ – for this reason, all subsequent non-floor function related formulas will use other types of parentheses. Some properties of the floor function, which will be used here to study partitions of natural numbers, have already been derived and proved in [9].

For 2 part partitions, we have

$$p(n, 2) = \left[\frac{n}{2} \right]. \quad (5)$$

Formula (5) is self-evident and requires no proof [7]. Formulas describing 3 and 4 part partitions are presented i.e., in [1], [2]:

$$p(n, 3) = \frac{1}{72} \left\{ 6n^2 - 7 - 9(-1)^n + 16 \cos\left(\frac{2}{3} \pi n\right) \right\}, \quad (6)$$

and

$$p(n, 4) = \frac{3(n+1)\{2n^2 + 4n - 13 + 9(-1)^n\}}{864} + \frac{-96 \cos(\frac{2}{3}\pi n) + 108(-1)^{\frac{n}{2} \bmod(n+1, 2)} + 32\sqrt{3} \sin(\frac{2}{3}\pi n)}{864}. \quad (7)$$

The formulas derived in the following chapters are based on calculating the partition $p(n, k)$ as the sum of the series of k part partitions using the floor function and the identity that allows to represent each natural number $n \geq k$ as $n = k\left[\frac{n}{k}\right] + m_k$, where $0 \leq m_k \leq k - 1$. The dependence $p(n, k) = p(n - 1, k - 1) + p(n - k, k)$ and Bernoulli's formulas for various powers of finite sums of series of natural numbers of the floor function will also be used [7], [9].

2. 3 Part Partitions of any Integer

Theorem 2.1. *The 3 part partition of any integer n can be calculated from the formula:*

$$p(n, 3) = \frac{1}{2} \left(\left[\frac{n}{3} \right]^2 + \left[\frac{n}{3} \right] \right) + \left[\frac{\left[\frac{n}{3} \right] + m_3}{2} \right] \left[\frac{\left[\frac{n}{3} \right] + m_3 - 1}{2} \right]. \quad (8)$$

The truth of formula (8) is proved by the mathematical induction method by examining the expressions $p(n, 3)$ for $l = 3n, 3n + 1, 3n + 2$, including all natural numbers $n \geq 3$.

Proof.

(a) $l = 3n$

$$\begin{aligned}
 p(3n, 3) &= \frac{1}{2} \left(\left[\frac{3n}{3} \right]^2 + \left[\frac{3n}{3} \right] \right) + \left[\frac{\left[\frac{3n}{3} \right]}{2} \right] \left[\frac{\left[\frac{3n}{3} \right] - 1}{2} \right] \\
 &= \frac{1}{2} (n^2 + n) + \left[\frac{n}{2} \right] \left[\frac{n-1}{2} \right].
 \end{aligned} \tag{9}$$

For $n = 1$, we have $p(3, 3) = (1^2 + 1)/2 = 1$. Assuming the correctness of the formula (9) for $p(3n, 3)$, we will examine the relationship $p(3(n+1), 3)$:

$$\begin{aligned}
 p(3(n+1), 3) &= \frac{1}{2} \left(\left[\frac{3(n+1)}{3} \right]^2 + \left[\frac{3(n+1)}{3} \right] \right) + \left[\frac{n+1}{2} \right] \left[\frac{n-1+1}{2} \right] \\
 &= \frac{1}{2} (n^2 + 3n + 2) + \left[\frac{n+1}{2} \right] \left[\frac{n}{2} \right].
 \end{aligned} \tag{10}$$

On the other hand, $p(3n+3, 3) = p(3n, 3) + p(3n+2, 2)$, thus

$$\begin{aligned}
 p(3(n+1), 3) &= \frac{1}{2} (n^2 + n) + \left[\frac{n}{2} \right] \left[\frac{n-1}{2} \right] + (n+1) + \left[\frac{n}{2} \right] \\
 &= \frac{1}{2} (n^2 + 3n + 2) + \left[\frac{n+1}{2} \right] \left[\frac{n}{2} \right].
 \end{aligned} \tag{11}$$

Both expressions are identical, which ends the proof.

(b) $l = 3n + 1$

$$\begin{aligned}
 p(3n+1, 3) &= \frac{1}{2} \left(\left[\frac{3n+1}{3} \right]^2 + \left[\frac{3n+1}{3} \right] \right) + \left[\frac{n+1}{2} \right] \left[\frac{n-1+1}{2} \right] \\
 &= \frac{1}{2} (n^2 + n) + \left[\frac{n+1}{2} \right] \left[\frac{n}{2} \right].
 \end{aligned} \tag{12}$$

For $n = 1$, we have $p(3 \cdot 1 + 1, 3) = \frac{1}{2}(1^2 + 1) + \left\lceil \frac{1+1}{2} \right\rceil \left\lfloor \frac{1}{2} \right\rfloor = 1$ and

$$\begin{aligned} p(3(n+1) + 1, 3) &= \frac{1}{2} \left(\left\lceil \frac{3(n+1)+1}{3} \right\rceil^2 + \left\lceil \frac{3(n+1)+1}{3} \right\rceil \right) + \left\lceil \frac{n+2}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor \\ &= \frac{1}{2} (n^2 + 3n + 2) + \left\lceil \frac{n+2}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor, \end{aligned} \quad (13)$$

at the same time

$$\begin{aligned} p(3(n+1) + 1, 3) &= p(3n + 1, 3) + p(3n + 3, 2) \\ &= \frac{1}{2} (n^2 + n) + \left\lceil \frac{n+1}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{3n+3}{2} \right\rceil \\ &= \frac{1}{2} (n^2 + 3n + 2) + \left\lceil \frac{n+2}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned} \quad (14)$$

The equality of expressions (13) and (14) proves the correctness of the formula (8).

(c) $l = 3n + 2$

$$\begin{aligned} p(3n + 2, 3) &= \frac{1}{2} \left(\left\lceil \frac{3n+2}{3} \right\rceil^2 + \left\lceil \frac{3n+2}{3} \right\rceil \right) + \left\lceil \frac{n+2}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor \\ &= \frac{1}{2} (n^2 + n) + \left\lceil \frac{n+2}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned} \quad (15)$$

For $n = 1$, we have $p(3 \cdot 1 + 2, 3) = \frac{1}{2}(1^2 + 1) + \left\lceil \frac{1+2}{2} \right\rceil \left\lfloor \frac{1+1}{2} \right\rfloor = 2$ and

$$\begin{aligned} p(3(n+1) + 2, 3) &= \frac{1}{2} \left(\left\lceil \frac{3(n+1)+2}{3} \right\rceil^2 + \left\lceil \frac{3(n+1)+2}{3} \right\rceil \right) \\ &\quad + \left\lceil \frac{n+2+1}{2} \right\rceil \left\lfloor \frac{n+1+1}{2} \right\rfloor = \frac{1}{2} (n^2 + 3n + 2) + \left\lceil \frac{n+3}{2} \right\rceil \left\lfloor \frac{n+2}{2} \right\rfloor, \end{aligned} \quad (16)$$

and, analogous to the previous

$$\begin{aligned}
 p(3(n+1) + 2, 3) &= p(3n + 2, 3) + p(3n + 4, 2) \\
 &= \frac{1}{2}(n^2 + n) + \left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{3n+4}{2} \right\rfloor \\
 &= \frac{1}{2}(n^2 + 3n + 2) + \left\lfloor \frac{n+3}{2} \right\rfloor \left\lfloor \frac{n+2}{2} \right\rfloor. \tag{17}
 \end{aligned}$$

The identity of both expressions closes the proof. The proven three cases of expressions, derived from the general formula, cover every natural number, thus the general formula is true.

3. 4 Part Partitions of any Integer

Theorem 3.1. *The 4 part partition of any integer n can be calculated from the formula:*

$$\begin{aligned}
 p(n, 4) &= \frac{1}{9} \left\{ 4 \left\lfloor \frac{n}{4} \right\rfloor^3 + 3(m_4 + 1) \left\lfloor \frac{n}{4} \right\rfloor^2 + 3m_4 \left\lfloor \frac{m_4}{2} \right\rfloor \left\lfloor \frac{n}{4} \right\rfloor \right\} \\
 &\quad + \frac{1}{9} \left\{ \left\lfloor \frac{m_3 + 1}{2} \right\rfloor (3 - m_3) - \left\lfloor \frac{m_4 + 1}{2} \right\rfloor (3 - m_4) \right\}. \tag{18}
 \end{aligned}$$

In the proof of this formula, we will consider various combinations of $m_3 = 0, 1, 2$ and $m_4 = 0, 1, 2, 3$. There are 12 such combinations, therefore it is necessary to examine 12 expressions with n , which are special cases of the general formula. These expressions are shown in the following Table 1.

Table 1.

	m_3, m_4	
1	1, 0	$p(12n - 8, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 8}{4} \right]^3 + 3 \left[\frac{12n - 8}{4} \right]^2 + 2 \right)$ $= 12n^3 - 21n^2 + 12n - 2$
2	2, 1	$p(12n - 7, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 7}{4} \right]^3 + 6 \left[\frac{12n - 7}{4} \right]^2 - 1 \right)$ $= 12n^3 - 18n^2 + 8n - 1$
3	0, 2	$p(12n - 6, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 6}{4} \right]^3 + 9 \left[\frac{12n - 6}{4} \right]^2 + 6 \left[\frac{12n - 6}{4} \right] - 1 \right)$ $= 12n^3 - 15n^2 + 6n - 1$
4	1, 3	$p(12n - 5, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 4}{4} \right]^3 + 3 \left[\frac{12n - 4}{4} \right] + 1 \right)$ $= 12n^3 - 12n^2 + 3n$
5	2, 0	$p(12n - 4, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 4}{4} \right]^3 + 3 \left[\frac{12n - 4}{4} \right]^2 + 1 \right)$ $= 12n^3 - 9n^2 + 2n$
6	0, 1	$p(12n - 3, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 3}{4} \right]^3 + 6 \left[\frac{12n - 3}{4} \right]^2 - 2 \right) = 12n^3 - 6n^2$
7	1, 2	$p(12n - 2, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 2}{4} \right]^3 + 9 \left[\frac{12n - 2}{4} \right]^2 + 6 \left[\frac{12n - 2}{4} \right] + 1 \right)$ $= 12n^3 - 3n^2$
8	2, 3	$p(12n - 1, 4) = \frac{1}{9} \left(4 \left[\frac{12n - 1}{4} \right]^3 + 12 \left[\frac{12n - 1}{4} \right]^2 + 9 \left[\frac{12n - 1}{4} \right] + 1 \right)$ $= 12n^3 - n$
9	0, 0	$p(12n, 4) = \frac{1}{9} \left(4 \left[\frac{12n}{4} \right]^3 + \left[\frac{12n}{4} \right]^2 \right) = 12n^3 + 3n^2$

Table 1. (Continued)

	m_3, m_4	
10	1, 1	$p(12n + 1, 4) = \frac{1}{9} \left(4 \left[\frac{12n + 1}{4} \right]^3 + 6 \left[\frac{12n + 1}{4} \right]^2 \right) = 12n^3 + 6n^2$
11	2, 2	$p(12n + 2, 4) = \frac{1}{9} \left(4 \left[\frac{12n + 2}{4} \right]^3 + 9 \left[\frac{12n + 2}{4} \right]^2 + 6 \left[\frac{12n + 2}{4} \right] \right)$ $= 12n^3 + 9n^2 + 2n$
12	0, 3	$p(12n + 3) = \frac{1}{9} \left(4 \left[\frac{12n + 3}{4} \right]^3 + 12 \left[\frac{12n + 3}{4} \right]^2 + 9 \left[\frac{12n + 3}{4} \right] \right)$ $= 12n^3 + 12n^2 + 3n$

Proof example for $m_3 = 1$ and $m_4 = 0$

We will prove the truth of the general formula using the mathematical induction method for the first case in Table 1: $p(12n - 8, 4)$, applying the previously presented formulas for 3 part partitions. For $m_3 = 1$ and $m_4 = 0$, we have

$$p(12n - 8, 4) = 12n^3 - 21n^2 + 12n - 2. \quad (19)$$

For $n = 1$ we obtain $p(4, 4) = 12 \cdot 1^3 - 21 \cdot 1^2 + 12 \cdot 1 - 2 = 1$ and for $n + 1$ $p(12(n + 1) - 8, 4) = 12n^3 + 15n^2 + 6n + 1$.

On the other hand,

$$\begin{aligned}
p(12(n + 1) - 8, 4) &= p(12n - 8, 4) + p(12n - 5, 3) + p(12n - 1, 3) \\
&+ p(12n + 3, 3) = (12n^3 - 21n^2 + 12n - 2) + (12n^2 - 10n + 2) \\
&+ (12n^2 - 2n) + (12n^2 + 6n + 1) = 12n^3 + 15n^2 + 6n + 1 \quad (20)
\end{aligned}$$

which confirms the truth of Equation (18).

By the same method we prove the truth of all remaining 11 expressions, thus the general formula for each natural number $n \geq 4$ is true.

4. Summary

The methodology presented here together with Theorems for 3 and 4 part partitions indicate the method and direction of further research covering 5, 6, and generally, k part partitions, what is the most significant achievement of this work. Obtaining patterns for the partitions of higher parts would be very difficult or even impossible applying the methods used so far.

We would like to pay special attention to the significance of the identity given in the paper, which, in our opinion, plays a key role in the method of constructing and proving theorems about k -component partitions. They are calculated as the sums of the series of the corresponding $k-1$ -component partitions, and this relationship allows these formulas to be written as finite sums of integers (using Bernoulli numbers, respectively) and the mantissas and sums of the mantissas of the Floor function, which facilitates and simplifies the summation processes. Finding the general formula for k -component partitions for each k requires further work, but the method used here indicate the direction and purpose of the research, and this in turn, would finally solve the problem of *Partitio Numerorum*.

References

- [1] J. A. Anderson and J. Bell, Number Theory with Applications, Pearson, 1996.
- [2] G. E. Andrews and K. Eriksson, Integer Partitions, Cambridge, 2004.
- [3] Byungehan Kim, On the number of partitions of n into k different parts, Journal of Number Theory 132(6) (2012) 1306–1313.
DOI: <https://doi.org/10.1016/j.jnt.2012.01.003>
- [4] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, A. W. P. C., 1994.

- [5] G. H. Hardy, Asymptotic Partition Formula, London, 1918.
- [6] J. C. Lagarias and D. H. Richman, Dilated floor functions having non-negative commutator II: Negative dilations. *Acta Arithmetica* 196 (2020), 163-186.
DOI: <https://doi.org/10.4064/aa190628-14-1>
- [7] A. Nowicki, Numbers, Functions, Sequences, Sets, Geometry, Warsaw, 2013.
- [8] M. A. Nyblon, Some curious sequences involving floor and ceiling functions, *The American Mathematical Monthly* 109(6) (2002), 559-564.
DOI: <https://doi.org/10.1080/00029890.2002.11919886>
- [9] R. Palka and Z. Trukszyn, Finite sums of the floor functions series, *Integers* 20 (2020); Article 41.