ON SKEW FIBONACCI MATRICES

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Abstract

In this paper we mainly define the skew Fibonacci sequence and skew Fibonacci matrices, and give the eigenvalues of skew Fibonacci matrices. Furthermore, we present some conditions for a matrix to be a skew Fibonacci matrix.

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1. Introduction

The current paper will focus on skew Fibonacci matrices, which is a cross research target in Fibonacci-Lucas matrices (see [1]), skew Fibonacci sequences and matrices.

Let $\mathbb{Z} = \{x | x \text{ is an integer}\}$, $\mathbb{N} = \{x | x \text{ is an integer, } x > 0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{C} = \{x | x \text{ is a complex}\}$. Throughout this paper all matrices we consider below are the n order square matrices over $\mathbb{C}$ without special announcement, $n \in \mathbb{N}$.

Let $f_1, f_2, \dots$, denote Fibonacci sequence, i.e., $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, n \geq 3$. Set $f_0 = 0, f_{-i} = (-1)^{i-1} f_i, \forall i \geq 1$, then $f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{Z}$, and $f_n = 0$ if and only if $n = 0, f_i = (-1)^{i-1} f_{-i}$ for all $i \in \mathbb{Z}$.

Let $s_1, s_2, \dots$, denote skew Fibonacci sequence, i.e., $s_1 = 1, s_2 = -1, s_n + s_{n-1} = s_{n-2}, n \geq 3$. Set $s_0 = 0, s_{-i} = (-1)^{i-1} s_i, \forall i \geq 1$, then $s_n + s_{n-1} = s_{n-2}$ for all $n \in \mathbb{Z}$, and $s_n = 0$ if and only if $n = 0, s_i = (-1)^{i-1} s_{-i}$ for all $i \in \mathbb{Z}$.

It is clear $s_i = f_{-i}$ for all $i \in \mathbb{Z}$.

Definition 1.1. Matrix F is called a skew Fibonacci matrix if $F^2 = -F + E$, where E is a unit matrix.

By Definition 1.1 one obtains $F^n + F^{n-1} = F^{n-2}$ for all $n \geq 2$ with $n \in \mathbb{N}$. Using $F^2 = E - F$ one has $E = F(F + E)$, and hence skew Fibonacci matrices are invertible and $F^{-1} = F + E$. Set $F^0 = E, F^{-n} = (F^{-1})^n$, then $F^n + F^{n-1} = F^{n-2}$ for $\forall n \in \mathbb{Z}$. One can also define skew Fibonacci matrices by $F^n + F^{n-1} = F^{n-2}$ for all $n \geq 2$ with $n \in \mathbb{N}$.

2. The Main Results

Let $r = \frac{1+\sqrt{5}}{2}$, $s = \frac{1-\sqrt{5}}{2}$, then $r^n = s_{-n}r + s_{-n+1}$, $s^n = s_{-n}s + s_{-n+1}$,

it is clear $\lim_{n \rightarrow +\infty} \frac{s_{n-1}}{s_n} = s$, $\lim_{n \rightarrow -\infty} \frac{s_{n-1}}{s_n} = r$.

Lemma 2.1. $f_n = f_m f_{n-m+1} + f_{m-1} f_{n-m}$ for $\forall n, m \in \mathbb{Z}$.

Proof. When $n = 0$, $f_m f_{-m+1} + f_{m-1} f_{-m} = f_m (-1)^{m-2} f_{m-1} + f_{m-1} (-1)^{m-1} f_m = 0 = f_0$. Assumed that $n \geq 1$. We can show this by induction on n . When $n = 1$, $f_m f_{2-m} + f_{m-1} f_{1-m} = (-1)^{m-3} \{f_m f_{m-2} - f_{m-1} f_{m-1}\} = (-1)^{m-3} (-1)^{m-1} = 1 = f_1$ by [1, Lemma 1.1(i)]. If $n \geq 2$, $f_{n+1} = f_n + f_{n-1} = f_m f_{n-m+1} + f_{m-1} f_{n-m} + f_m f_{n-m} + f_{m-1} f_{n-1-m} = f_m \{f_{n-m+1} + f_{n-m}\} + f_{m-1} \{f_{n-m} + f_{n-1-m}\} = f_m f_{n-m+2} + f_{m-1} f_{n-m+1}$. Now assume $n \leq -1$. We know $f_k = f_{-m+1} f_{k+m} + f_{-m} f_{k+m-1}$ for $\forall k \in \mathbb{N}$, then $f_{-k} = (-1)^{k-1} \{(-1)^{m-2} f_{m-1} f_{k+m} + (-1)^{m-1} f_m f_{k+m-1}\} = (-1)^{m+k-1} f_{m-1} f_{k+m} + (-1)^{m+k-2} f_m f_{k+m-1} = f_{m-1} f_{-k-m} + f_m f_{-k-m+1}$, let $n = -k$, completing the proof. $\square$

Proposition 2.2. If F is a skew Fibonacci matrix, then $F^n = f_{-n} F + f_{-(n-1)} E = s_n F + s_{n-1} E$ for $\forall n \in \mathbb{Z}$.

Proof. When $n = 0$, $E = s_{-1} E$. Assumed that $n \geq 1$. One can show this by induction on n . It is clear when $n = 1$. If $n \geq 2$, $F^{n+1} = F(s_{n-1} E + s_n F) = s_{n-1} F + s_n F^2 = s_{n-1} F + s_n (E - F) = s_n E + s_{n+1} F$. Now assume $n \leq -1$. One can show $(F^{-1})^m = s_{-(m-1)} E + s_{-m} F^{-1}$ for all $m \in \mathbb{N}$ by induction on m , let $n = -m$, then $F^n = s_{n+1} E + s_n F^{-1} = s_{n-1} E + s_n F$. This proves the Proposition 2.2. $\square$

Remark 2.3. (i) F is a skew Fibonacci matrix if and only if $-F$ is a Fibonacci-Lucas matrix (see [1, Definition 1.2]).

(ii) Assumed that $n \geq 2$, equation $F^n = s_n F + s_{n-1} E$ can be deduced that equation $F^n + F^{n-1} = F^{n-2}$ holds, so one can also define skew Fibonacci matrices by $F^n = s_n F + s_{n-1} E$.

Proposition 2.4. *Assumed that F is a skew Fibonacci matrix. Then*

(i) F^i are different for distinct $i \in \mathbb{Z}$.

(ii) $x = -r$ or $-s$ if $F = xE$, $x \in \mathbb{C}$; F^i , $i \neq 1$ are not skew Fibonacci matrices for all $i \in \mathbb{Z}$.

(iii) The eigenvalues of F are $-r$ or $-s$.

Proof. It is similar to the proof of [1, Proposition 2.3], [1, Proposition 2.4] and [2, Lemma 2.1], respectively. $\square$

Proposition 2.5. $s_{-n} = \sum_{i=0}^n C_n^i s_i$, i.e., $f_n = \sum_{i=0}^n C_n^i f_{-i}$ for $\forall n \in \mathbb{Z}_0$.

Proof. When $n = 0$, $s_0 = \sum_{i=0}^0 C_0^i s_i = 0$. Assumed that $n \geq 1$, then

$$s_{-n} F + s_{-n-1} E = F^{-n} = (F^{-1})^n = (F + E)^n = \sum_{i=0}^n C_n^i F^i E^{n-i} = \sum_{i=0}^n C_n^i (s_i F$$

$$+ s_{i-1} E) = \sum_{i=0}^n C_n^i s_i F + \sum_{i=0}^n C_n^i s_{i-1} E \text{ by Proposition 2.2. We have } (s_{-n} -$$

$$\sum_{i=0}^n C_n^i s_i) F = (\sum_{i=0}^n C_n^i s_{i-1} - s_{-n-1}) E, \text{ then } s_{-n} - \sum_{i=0}^n C_n^i s_i = 0 = \sum_{i=0}^n C_n^i s_{i-1}$$

$- s_{-n-1}$ for $\forall n \geq 1$ by r or s is an irrational number and Proposition 2.4 (ii). $\square$

Proposition 2.6. *Assumed that F is a Fibonacci-Lucas matrix. Then*

$$\sum_{i=1}^n F^i = F^{n+2} - F^2 \text{ and } \sum_{i=1}^n F^{2i-1} = F^{2n} - E \text{ for } \forall n \in \mathbb{N}.$$

Proof. One can show $\sum_{i=1}^n f_i = f_{n+2} - 1$, $\sum_{i=1}^n f_{2i-1} = f_{2n}$ and $\sum_{i=1}^n f_{2i} = f_{2n+1} - 1$ by induction on n , the other is clear by the simple verification since Proposition 2.2. $\square$

Lemma 2.7. (i) *If $F_1 + F_2 = -E$, then F_1 is a skew Fibonacci matrix if and only if F_2 is a skew Fibonacci matrix.*

(ii) *If F is a skew Fibonacci matrix, then the NI matrix ([2]) and the transpose matrix of F are skew Fibonacci matrices.*

Proof. (i) It is clear $F_1 = -E - F_2$, then $(-E - F_2)^2 + (-E - F_2) - E = F_2^2 + F_2 - E$.

(ii) Set $F_1 + F = -E$, by (i) one obtains F_1 is a Fibonacci-Lucas matrix and $F_1 = -E - F = -F^{-1}$. Consider $F', 0 = (F^2 + F - E)' = (F')^2 + F' - E$. $\square$

Let $f(F)$ be any polynomial of matrix F on $\mathbb{C}$.

Proposition 2.8. (i) *Assumed that F is a Fibonacci-Lucas matrix. Then $f(F)$ is a Fibonacci-Lucas matrix if and only if $f(F)$ appears in one of the following: (1) $f(F) = rE$ or sE ; (2) $f(F) = F$; and (3) $f(F) = -F + E$.*

(ii) *Assumed that F is a skew Fibonacci matrix. Then $f(F)$ is a skew Fibonacci matrix if and only if $f(F)$ appears in one of the following: (1) $f(F) = -rE$ or $-sE$; (2) $f(F) = F$; and (3) $f(F) = -F - E$.*

Proof. (ii) is similar to (i). (i) We only need to consider $aF + bE$ for $\forall a, b \in \mathbb{C}$ by [1, Proposition 2.1]. Let $(aF + bE)^2 - (aF + bE) - E = (a^2 + 2ab - a)F + (a^2 + b^2 - b - 1)E = 0$. If $a^2 + b^2 - b - 1 = 0$, then $a^2 + 2ab - a = 0$, we have $a = 0, b = r$ or s ; or $a = 1, b = 0$; or $a = -1, b = 1$. If $a^2 + b^2 - b - 1 \neq 0$, then $a^2 + 2ab - a \neq 0$, we have $F = rE$ or sE by [1, Proposition 2.4], the other is clear by the simple verification. $\square$

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