COHOMOLOGY AND CONFORMAL DERIVATIONS OF BIHOM-LIE CONFORMAL SUPERALGEBRAS

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Abstract

In this paper, we introduce the notion of BiHom-Lie conformal superalgebras. We develop its representation theory and define the cohomology group with coefficients in a module. Finally, we introduce conformal derivations of BiHom-Lie conformal superalgebras and study some of their properties.

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Introduction

The notion of Lie conformal (super)algebras are introduced by Kac in [10], in which he gave an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. On the other hand, it is a useful tool to study vertex (super)algebras and has many applications in the theory of Lie superalgebras. Moreover, it is equivalent to the notion of a linear Hamiltonian operator introduced in [6], and in [14], Zhao et al. developed deformation of Lie conformal superalgebras and introduced derivations of Lie conformal superalgebras and study their properties.

In [1], authors introduced the notion of Hom-Lie superalgebras and they gave a classification of Hom-Lie admissible superalgebras. Later, Makhlouf et al. studied the representation and the cohomology of Hom-Lie superalgebras in [2] and calculated the derivations and the second cohomology group of q -deformed Witt superalgebra. In [13], Yuan introduced the notion of Hom-Lie conformal superalgebra and proved that a Hom-Lie conformal superalgebra is equivalent to a Hom-Gelfand-Dorfman super bialgebra. In [15], the authors developed cohomology theory of Hom-Lie conformal algebras, discussed some applications to the study of deformations of regular Hom-Lie conformal algebras and introduced derivations of multiplicative Hom-Lie conformal algebras. Motivated by these results, authors, in [8], introduced the notion of representation theory of Hom-Lie conformal superalgebras and discuss the cases of adjoint representations. Furthermore, they developed their cohomology group and discuss some applications to the study of deformation theory.

Motivated by a categorical study of Hom-algebra and new type of categories, the authors introduced, in [5], a generalized algebraic structure dealing with two commuting multiplicative linear maps, called BiHom-algebras including BiHom-associative algebras and BiHom-Lie algebras. Recently, Zhao, Yuan and Chen developed the cohomology and deformation theory of BiHom-Lie conformal algebras and investigated the notion of conformal derivations of BiHom-Lie conformal algebras in [9].

The main goal of the present work is to introduce the notion of BiHom-Lie conformal superalgebras, study their cohomology theory and investigate the notion of generalized conformal derivations.

The paper is organized as follows. In Section 1, we recall some basic definitions and results about the notion of Lie conformal (super)algebras. In Section 2, we introduce the notion BiHom-Lie conformal superalgebras illustrated by some examples give some related results. In Section 3, we develop representation and cohomology theory of BiHom-Lie conformal superalgebras. Section 4 is devoted to the study of derivations and generalized derivations of BiHom-Lie conformal superalgebras and their properties.

1. Preliminaries

Throughout the paper, all algebraic systems are supposed to be over a field $\mathbb{C}$, of characteristic 0 and denote by $\mathbb{Z}_+$ the set of all nonnegative integers and by $\mathbb{Z}$ the set of all integers.

Let V be a superspace that is a $\mathbb{Z}_2$ -graded linear space with a direct sum $V = V_0 \oplus V_1$. The elements of V_j , $j = [0, 1]$, are said to be homogeneous and of parity j . The parity of a homogeneous element x is denoted by $|x|$. Throughout what follows, if $|x|$ occurs in an expression, then it is assumed that x is homogeneous and that the expression extends to the other elements by linearity.

Definition 1.1. A Lie conformal superalgebra $\mathcal{R}$ is a left $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module, and for any $n \in \mathbb{Z}_{\geq 0}$ there is a family of $\mathbb{C}$ -linear n -products from $\mathcal{R} \otimes \mathcal{R}$ to $\mathcal{R}$ satisfying the following conditions:

(C0) For any $a, b \in \mathcal{R}$, there is an N such that $a_{(n)}b = 0$ for $n \gg N$;

(C1) For any $a, b \in \mathcal{R}$ and $n \in \mathbb{Z}_{\geq 0}$, $(\partial a)_{(n)}b = -na_{(n)}b$;

(C2) For any $a, b \in \mathcal{R}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$a_{(n)}b = -(-1)^{|a||b|} \sum_{j=0}^{\infty} (-1)^{j+n} \frac{1}{j!} \partial^j (b_{(n+j)}a);$$

(C3) For any $a, b, c \in \mathcal{R}$ and $m, n \in \mathbb{Z}_{\geq 0}$,

$$a_{(m)}(b_{(n)}c) = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c + (-1)^{|a||b|} b_{(n)}(a_{(m)}c).$$

(Convention: $a_{(n)}b = 0$ if $n < 0$). Note that if we define λ -bracket $[-_{\lambda} -]$:

$$[a_{\lambda}b] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a_{(n)}b, \quad a, b \in \mathcal{R}. \quad (1.1)$$

That is, $\mathcal{R}$ is a Lie conformal superalgebra if and only if $[-_{\lambda} -]$ satisfies the following axioms:

$$(C0)_{\lambda} \quad [\mathcal{R}_{i\lambda} \mathcal{R}_j] \subseteq \mathcal{R}_{i+j}[\lambda], \quad \forall i, j \in \mathbb{Z}_2;$$

$$(C1)_{\lambda} \quad \text{Conformal sesquilinearity: } [(\partial a)_{\lambda}b] = -\lambda[a_{\lambda}b];$$

$$(C2)_{\lambda} \quad \text{Skew-symmetry: } [a_{\lambda}b] = -(-1)^{|a||b|}[b_{-\partial-\lambda}a];$$

$$(C3)_{\lambda} \quad \text{Jacobi identity: } [a_{\lambda}[b_{\mu}c]] = [[a_{\lambda}b]_{\lambda+\mu}c] + (-1)^{|a||b|}[b_{\mu}[a_{\lambda}c]].$$

Similarly, an associative conformal superalgebra is a left $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module $\mathcal{R}$ equipped with a $\mathbb{C}$ -linear map $\cdot_{\lambda} : \mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{R}[\lambda]$ such that $\mathcal{R}_{i\lambda} \mathcal{R}_j \subseteq \mathcal{R}_{i+j}[\lambda]$ and

$$(C1)_{\lambda} \quad \text{Conformal sesquilinearity: } [(\partial a)_{\lambda}b] = -\lambda[a_{\lambda}b];$$

$$(C2)_{\lambda} \quad \text{Associativity: } (a_{\lambda}b)_{\lambda+\mu}c = a_{\lambda}(b_{\mu}c).$$

$\mathcal{R}$ is called finite if it is finitely generated as a $\mathbb{C}[\partial]$ -module.

Proposition 1.1. *Let $(\mathcal{R}, \cdot_\lambda)$ be an associative conformal superalgebra. Then $(\mathcal{R}, [\cdot_\lambda \cdot])$ is a Lie conformal superalgebra, where for any homogeneous elements $a, b \in \mathcal{R}$*

$$[a_\lambda b] = a_\lambda b - (-1)^{|a||b|} b_{-\lambda-\partial} a.$$

Example 1.1. Let $\mathcal{R}$ be a Lie conformal superalgebra and B be a commutative associative (ordinary) superalgebra. Then $\mathcal{R} \otimes B$ carries a Lie conformal superalgebra structure defined as follows. The $\mathbb{C}[\partial]$ -module structure is given by $\partial(r \otimes b) = (\partial r) \otimes b$ ($r \in \mathcal{R}, b \in B$), and the λ -bracket by

$$[(r \otimes b)_\lambda (r' \otimes b')] = (-1)^{|b||r'|} [r_\lambda r'] \otimes (bb').$$

Notice that if $\mathcal{R}$ is finite and B is finite-dimensional, then $\mathcal{R} \otimes B$ is also finite.

A first generalization of Lie conformal superalgebras is the notion of Hom-Lie conformal superalgebras introduced in [12].

Definition 1.2. A Hom-Lie conformal superalgebra $R = R_0 \oplus R_1$ is a $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module equipped with an even linear endomorphism α such that $\alpha\partial = \partial\alpha$, and a $\mathbb{C}$ -linear map

$$R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R, \quad a \otimes b \mapsto [a_\lambda b],$$

such that $[R_{i\lambda} R_j] \subseteq R_{i+j\lambda}[\lambda]$, $i, j \in \mathbb{Z}_2$, and the following axioms hold for $a, b, c \in R$:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \quad (1.2)$$

$$[a_\lambda b] = -(-1)^{|a||b|} [b_{-\lambda-\partial} a], \quad (1.3)$$

$$[\alpha(a)_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} \alpha(c)] + (-1)^{|a||b|} [\alpha(b)_\mu [a_\lambda c]]. \quad (1.4)$$

Definition 1.3. A BiHom-Lie conformal algebra is a BiHom-conformal algebra $(A, [\cdot]_{\lambda}, \alpha, \beta)$ such that the following axioms hold for any $a, b, c \in A$:

$$\alpha([a_{\lambda} b]) = [\alpha(a)_{\lambda} \alpha(b)], \beta([a_{\lambda} b]) = [\beta(a)_{\lambda} \beta(b)], \quad (1.5)$$

$$[\beta(a)_{\lambda} \alpha(b)] = -[\beta(b)_{-\lambda-\partial} \alpha(a)], \quad (1.6)$$

$$[\alpha\beta(a)_{\lambda} [b_{\mu} c]] = [[\beta(a)_{\lambda} b]_{\lambda+\mu} \beta(c)] + [\beta(b)_{\mu} [\alpha(a)_{\lambda} c]]. \quad (1.7)$$

In particular, if α, β are algebra isomorphisms, then $(A, [\cdot]_{\lambda}, \alpha, \beta)$ is called *regular*.

Definition 1.4. A module (V, ϕ, ψ) over a BiHom-Lie conformal algebra $(A, [\cdot]_{\lambda}, \alpha, \beta)$ is a $\mathbb{C}[\partial]$ -module endowed with two commuting $\mathbb{C}$ -linear maps ϕ, ψ and a $\mathbb{C}$ -bilinear map $A \otimes V \rightarrow V[\lambda], a \otimes v \mapsto a_{\lambda} v$, such that for $a, b \in A, v \in V$:

$$\psi \circ \partial = \partial \circ \psi, \phi \circ \partial = \partial \circ \phi, \quad (1.8)$$

$$(\partial a)_{\lambda} v = -\lambda(a_{\lambda} v), \alpha_{\lambda}(\partial v) = (\partial + \lambda)\alpha_{\lambda} v, \quad (1.9)$$

$$\phi(a_{\lambda} v) = \alpha(a)_{\lambda}(\phi(v)), \psi(a_{\lambda} v) = \beta(a)_{\lambda}(\psi(v)), \quad (1.10)$$

$$[\beta(a)_{\lambda} b]_{\lambda+\mu} \psi(v) = \alpha\beta(a)_{\lambda}(b_{\mu} v) - \beta(b)_{\mu}(\alpha(a)_{\lambda} v). \quad (1.11)$$

Remark 1.1. By Definition 1.4, it is easy to see that a module over a BiHom-Lie conformal algebra A in a finite $\mathbb{C}[\partial]$ -module V is the same as a homomorphism of BiHom-Lie conformal algebra $\rho : A \rightarrow gc(V)$ which is called a representation of A . The identity (1.11) can be written as

$$\rho([\beta(a)_{\lambda} b])_{\lambda+\mu} \circ \psi = \rho(\alpha\beta(a))_{\lambda} \circ \rho(b)_{\mu} - \rho(\beta(b))_{\mu} \circ \rho(\alpha(a))_{\lambda},$$

for all $a, b \in A$. In addition, ρ satisfy $\rho(\partial a)_{\lambda} v = -\lambda\rho(a)_{\lambda} v, \forall a \in A, v \in V$.

The adjoint representation of A is denoted by ad , i.e., $ad(a)_{\lambda}(b) = [a_{\lambda} b]$, where $a, b \in A$.

Definition 1.5. A BiHom-associative conformal superalgebra R is a $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module equipped with two even linear endomorphisms α, β and endowed with a λ -product from $\mathcal{R} \otimes \mathcal{R}$ to $\mathbb{C}[\partial] \otimes \mathcal{R}$, for any $a, b, c \in \mathcal{R}$, satisfying the following conditions:

$$(\partial a)_\lambda b = -\lambda a_\lambda b, a_\lambda(\partial b) = (\partial + \lambda)a_\lambda b, \quad (1.12)$$

$$\alpha\partial = \partial\alpha, \beta\partial = \partial\beta, \quad (1.13)$$

$$\alpha(a_\lambda b) = \alpha(a)_\lambda \alpha(b), \beta(a_\lambda b) = \beta(a)_\lambda \beta(b), \quad (1.14)$$

$$\alpha(a)_\lambda (b_\mu c) = (a_\lambda b)_{\lambda+\mu} \alpha(c). \quad (1.15)$$

Definition 1.6. A subset U of a BiHom-conformal (super)-algebra $(A, \cdot_\lambda, \alpha, \beta)$ is called a *BiHom-conformal subalgebra* if $\alpha(U) \subseteq A$, $\beta(U) \subseteq A$ and $a_\lambda b \in U$, for any $a, b \in U$.

A subset $U \subseteq A$ is called a left BiHom-conformal ideal (resp., right BiHom-conformal ideal) if $\alpha(U) \subseteq A$, $\beta(U) \subseteq A$ and $a_\lambda b \in U$, for any $a \in A$, $b \in U$. (resp., for any $a \in U$, $b \in A$).

2. Definitions and Examples

In this section, we introduce the notion of BiHom-Lie conformal superalgebra, which is a BiHom-generalization of (Hom)-Lie conformal superalgebras and also a superanalogue of (Bi)Hom-Lie conformal algebras.

Definition 2.1. A BiHom-Lie conformal superalgebra $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ is a $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module equipped with two commuting linear maps α, β and a λ -bracket $[\cdot]_\lambda \cdot$ which defines a $\mathbb{C}$ -linear map from $\mathcal{R} \otimes \mathcal{R}$ to $R[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{R}$ such that $[\mathcal{R}_{i\lambda} \mathcal{R}_j] \subseteq R_{i+j}[\lambda]$, $i, j \in \mathbb{Z}_2$ and the following axioms hold for all homogeneous elements $a, b, c \in \mathcal{R}$:

$$\alpha\partial = \partial\alpha, \beta\partial = \partial\beta, \quad (2.1)$$

$$\alpha([a_\lambda b]) = [\alpha(a)_\lambda \alpha(b)], \beta([a_\lambda b]) = [\beta(a)_\lambda \beta(b)], \quad (2.2)$$

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \quad (2.3)$$

$$[\beta(a)_\lambda \alpha(b)] = -(-1)^{|a||b|}[\beta(b)_{-\lambda-\partial} \alpha(a)], \quad (2.4)$$

$$[\alpha\beta(a)_\lambda [b_\mu c]] = [[\beta(a)_\lambda b]_{\lambda+\mu} \beta(c)] + (-1)^{|a||b|}[\beta(b)_\mu [\alpha(a)_\lambda c]]. \quad (2.5)$$

A BiHom-Lie conformal superalgebra $(\mathcal{R}, \alpha, \beta)$ is called *finite* if $\mathcal{R}$ is a finitely generated $\mathbb{C}[\partial]$ -module. The rank of $\mathcal{R}$ is its rank as a $\mathbb{C}[\partial]$ -module. If α and β are algebra isomorphisms, then $(\mathcal{R}, \alpha, \beta)$ is called *regular*.

We recover Lie conformal superalgebras when $\alpha = \beta = id$. The BiHom-Lie conformal algebras are obtained when the odd part is trivial.

A linear map $\rho : \mathcal{R} \rightarrow \mathcal{R}'$ is a homomorphism of BiHom-Lie conformal superalgebras if ρ satisfies $\rho\partial = \partial\rho$, $\rho\alpha = \alpha'\rho$, $\rho\beta = \beta'\rho$ and $\rho([a_\lambda b]) = [\rho(a)_\lambda \rho(b)]$, for all $a, b \in \mathcal{R}$.

Example 2.1. Let $g = g_0 \oplus g_1$ be a BiHom-Lie superalgebra with Lie bracket $[-, -]$ and with structure maps α and β . Let $(Curg)_\theta := \mathbb{C}[\partial] \otimes g_\theta$ be the free $\mathbb{C}[\partial]$ -module. Then $Curg = (Curg)_0 \oplus (Curg)_1$ is a BiHom-Lie conformal superalgebra with λ -bracket given by

$$\alpha(f(\partial) \otimes a) = f(\partial) \otimes \alpha(a),$$

$$\beta(f(\partial) \otimes a) = f(\partial) \otimes \beta(a),$$

$$[(f(\partial) \otimes a)_\lambda (g(\partial) \otimes b)] = f(-\lambda)g(\partial + \lambda) \otimes [a, b], \forall a, b \in g.$$

Example 2.2. Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a regular BiHom-Lie superalgebra.

Denote by $\hat{L} = L \otimes \mathbb{C}[t, t^{-1}]$ the affization of L with

$$[u \otimes t^m, v \otimes t^n] = [u, v] \otimes t^{m+n}, |u \otimes t^n| = |u|,$$

for any $u, v \in L, m, n \in \mathbb{Z}$. Extend α, β to $\hat{L}$ by $\alpha(u \otimes t^m) = \alpha(u) \otimes t^m$ and $\beta(u \otimes t^m) = \beta(u) \otimes t^m$. Then $(\hat{L}, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie superalgebra. By simple verification, we get a BiHom-Lie conformal superalgebra $\mathcal{R} = \mathbb{C}[\partial]L$ with

$$[u_\lambda v] = [u, v], \alpha(f(\partial)u) = f(\partial)\alpha(u), \beta(f(\partial)u) = f(\partial)\beta(u),$$

for any $u, v \in L$.

Example 2.3. Let $\mathcal{R} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]E$ be a free $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module.

Define

$$\alpha(L) = f(\partial)L, \alpha(E) = g(\partial)E,$$

$$\beta(L) = L, \beta(E) = E,$$

$$[L_\lambda L] = (\partial + 2\lambda)L, [L_\lambda E] = (\partial + \frac{3}{2}\lambda)E,$$

$$[E_\lambda L] = (\frac{1}{2}\partial + \frac{3}{2}\lambda)E, [E_\lambda E] = 0.$$

Then $(\mathcal{R}, \alpha, \beta)$ is a BiHom-Lie conformal superalgebra, where $R_0 = \mathbb{C}[\partial]L$ and $R_1 = \mathbb{C}[\partial]E$.

Example 2.4. Let $\mathcal{R} = \mathbb{C}[\partial]e_1 \oplus \mathbb{C}[\partial](e_2 + e_3)$ be a free $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module and

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define

$$\alpha(e_1) = e_1, \alpha(e_2) = e_3, \alpha(e_3) = e_2,$$

$$\beta(e_1) = e_1, \beta(e_2) = -e_3, \beta(e_3) = -e_2,$$

$$[e_{1\lambda}e_1] = [e_{2\lambda}e_2] = [e_{3\lambda}e_3] = 0, [e_{1\lambda}e_2] = [e_{1\lambda}e_3] = 0, [e_{2\lambda}e_3] = e_1.$$

One may check directly that $(\mathcal{R}, \alpha, \beta)$ is a BiHom-Lie conformal superalgebra.

Proposition 2.1. *Let $\mathcal{R}$ be a Lie conformal superalgebra and $\alpha, \beta : \mathcal{R} \rightarrow \mathcal{R}$ two even commuting linear maps such that*

$$\alpha\partial = \partial\alpha, \beta\partial = \partial\beta, \alpha([a_\lambda b]) = [\alpha(a)_\lambda \alpha(b)], \beta([a_\lambda b]) = [\beta(a)_\lambda \beta(b)].$$

Define a $\mathbb{C}$ -linear map from $\mathcal{R} \otimes \mathcal{R}$ to $\mathcal{R}[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{R}$ by $[a_\lambda b]' = [\alpha(a)_\lambda \beta(b)]$. Then $(\mathcal{R}, [\cdot]_\lambda', \alpha, \beta)$ is a BiHom-Lie conformal superalgebra.

Proof. We will just verify axioms (2.4) and (2.5). Let $a, b, c \in \mathcal{R}$. Then

$$[\beta(a)_\lambda \alpha(b)]' = [\alpha\beta(a)_\lambda \alpha\beta(b)] = -[\alpha\beta(b)_{-\lambda-\partial} \alpha\beta(a)] = -[\beta(b)_{-\lambda-\partial} \alpha(a)]'.$$

On the other hand, we have

$$\begin{aligned} [\alpha\beta(a)_\lambda [b_\mu c]]' &= [\alpha^2\beta(a)_\lambda [\alpha\beta(b)_\mu \beta^2(c)]] \\ &= [[\alpha^2\beta(a)_\lambda \alpha\beta(b)]_{\lambda+\mu} \beta^2(c)] + (-1)^{|a||b|} [\alpha\beta(b)_\mu [\alpha^2\beta(a)_\lambda \beta^2(c)]] \\ &= [[\beta(a)_\lambda b]'_{\lambda+\mu} \beta(c)]' + (-1)^{|a||b|} [\beta(b)_\mu [\alpha(a)_\lambda c]]'. \end{aligned}$$

□

The following result can be proved similarly to the previous one.

Proposition 2.2. *Let $(\mathcal{R}, [\cdot_\lambda \cdot], \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. Let $\alpha', \beta' : \mathcal{R} \rightarrow \mathcal{R}$ be two even homomorphisms of conformal algebras such that any two of the maps $\alpha, \beta, \alpha', \beta'$ commute. Then $(\mathcal{R}, [\cdot_\lambda \cdot]^\bullet = [\cdot_\lambda \cdot] \circ (\alpha' \otimes \beta'), \alpha \circ \alpha', \beta \circ \beta')$ is a BiHom-Lie conformal superalgebra.*

Corollary 2.1. *Let $(\mathcal{R}, [\cdot_\lambda \cdot], \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. Then $(\mathcal{R}, [\cdot_\lambda \cdot]^\bullet = [\cdot_\lambda \cdot] \circ (\alpha^k \otimes \beta^k), \alpha^{k+1}, \beta^{k+1})$ is a BiHom-Lie conformal superalgebra.*

Proposition 2.3. *Let $(\mathcal{R}, \alpha, \beta)$ and $(\mathcal{R}', \alpha', \beta')$ be two BiHom-Lie conformal superalgebras, Then $(\mathcal{R} \oplus \mathcal{R}', \alpha + \alpha', \beta + \beta')$ is a BiHom-Lie conformal superalgebra with*

$$[(u_1 + v_1)_\lambda (u_2 + v_2)] = [u_{1\lambda} v_1] + [u_{2\lambda} v_2], \forall u_1, u_2 \in \mathcal{R}, \forall v_1, v_2 \in \mathcal{R}',$$

$$(\alpha + \alpha')(u + v) = \alpha(u) + \alpha'(v), (\beta + \beta')(u + v) = \beta(u) + \beta'(v), \forall u \in \mathcal{R}, \forall v \in \mathcal{R}'.$$

Proof. Straightforward. □

Definition 2.2. Let (M, α, β) and (N, ϕ, ψ) be $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -modules. A BiHom-conformal linear map of degree θ from M to N is a sequence $f = f_{(n)}_{n \in \mathbb{Z}_+}$ of $f_{(n)} \in \text{Hom}_{\mathbb{C}}(M, N)$ satisfying that

$$\partial_N f_{(n)} - f_{(n)} \partial_M = -n f_{(n-1)}, f_\lambda(M_\mu) \subseteq N_{\mu+\theta}, n \in \mathbb{Z}_+, \mu, \theta \in \mathbb{Z}_2,$$

$$\partial_M \alpha = \alpha \partial_M, \partial_M \beta = \beta \partial_M, \partial_N \phi = \phi \partial_N, \partial_N \psi = \psi \partial_N,$$

$$f_{(n)} \alpha = \phi f_{(n)}, f_{(n)} \beta = \psi f_{(n)}.$$

Set $f_\lambda = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_{(n)}$. Then f is a BiHom-conformal linear map of degree θ if and only if

$$f_\lambda \partial_M = (\partial_N + \lambda) f_\lambda, f_\lambda(M_\mu) \subseteq N_{\mu+\theta}[\lambda],$$

$$\partial_M \alpha = \alpha \partial_M, \partial_M \beta = \beta \partial_M, \partial_N \phi = \phi \partial_N, \partial_N \psi = \psi \partial_N,$$

$$f_\lambda \alpha = \phi f_\lambda, f_\lambda \beta = \psi f_\lambda.$$

Let $Chom(M, N)_\theta$ denote the set of BiHom-conformal linear maps of degree θ from M to N . Then $Chom(M, N) = Chom(M, N)_0 \oplus Chom(M, N)_1$ is a $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module via:

$$\partial f_{(n)} = -n f_{(n-1)}, \text{ equivalently, } \partial f_\lambda = -\lambda f_\lambda.$$

The composition $f_\lambda g : M \rightarrow N \otimes \mathbb{C}[\lambda]$ of BiHom-conformal linear maps $f : M \rightarrow N$ and $g : L \rightarrow M$ is given by

$$(f_\lambda g)_{\lambda+\mu} = f_\lambda g_\mu, \forall f, g \in Chom(M, N).$$

If M is a finitely generated $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module, then $Cend(M) := Chom(M, M)$ is an associative conformal superalgebra with respect to the above composition. Thus, $Cend(M)$ becomes a Lie conformal superalgebra, denoted as $gc(M)$, with respect to the following λ -bracket:

$$[f_\lambda g]_\mu = f_\lambda g_{\mu-\lambda} - (-1)^{|f||g|} g_{\mu-\lambda} f_\lambda. \quad (2.6)$$

Hereafter all $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module are supposed to be finitely generated.

Proposition 2.4. *Let $(\mathcal{R}, \alpha, \beta)$ be a regular BiHom-associative conformal superalgebra with two even morphisms α and β . Then the λ -bracket*

$$[a_\lambda b] = \alpha_\lambda b - (-1)^{|a||b|} \alpha^{-1} \beta(b)_{-\lambda-\partial} \alpha \beta^{-1}(a), \quad \forall a, b \in \mathcal{R}.$$

defines a BiHom-Lie conformal superalgebra structure on $\mathcal{R}$.

Proof. For any $a, b \in \mathcal{R}$, we have

$$\begin{aligned} [\beta(a)_\lambda \alpha(b)] &= \beta(a)_\lambda \alpha(b) - (-1)^{|a||b|} \beta(b)_{-\lambda-\partial} \alpha(a) \\ &= -(-1)^{|a||b|} (\beta(b)_{-\lambda-\partial} \alpha(a) - (-1)^{|a||b|} \beta(a)_\lambda \alpha(b)) \\ &= -(-1)^{|a||b|} [\beta(b)_\lambda \alpha(a)]. \end{aligned}$$

On the other hand, given $a, b, c \in \mathcal{R}$. Then

$$\begin{aligned} [\alpha \beta(a)_\lambda [b_\mu c]] &= [\alpha \beta(a)_\lambda (b_\mu c - (-1)^{|b||c|} \alpha^{-1} \beta(c)_{-\mu-\partial} \alpha \beta^{-1}(b))] \\ &= \alpha \beta(a)_\lambda (b_\mu c) - (-1)^{|a|(|b|+|c|)} \alpha^{-1} \beta(b_\mu c)_{-\lambda-\partial} \alpha^2(a) \\ &\quad - (-1)^{|b||c|} \alpha \beta(a)_\lambda (\alpha^{-1} \beta(c)_{-\mu-\partial} \alpha \beta^{-1}(b)) \\ &\quad + (-1)^{|a||c|+|a||b|+|b||c|} \alpha^{-1} \beta(\alpha^{-2} \beta^2(c)_{-\mu-\partial} b)_{-\lambda-\partial} \alpha^2(a). \end{aligned}$$

Similarly, we have

$$\begin{aligned} [[\beta(a)_\lambda b]_{\lambda+\mu} \beta(c)] &= [(\beta(a)_\lambda b - (-1)^{|a||b|} \alpha^{-1} \beta(b)_{-\lambda-\partial} \alpha(a))_{\lambda+\mu} \beta(c)] \\ &= (\beta(a)_\lambda b)_{\lambda+\mu} \beta(c) - (-1)^{|c|(|a|+|c|)} \alpha^{-1} \beta^2(c)_{-\lambda-\mu-\partial} (\alpha(a)_\lambda \alpha \beta^{-1}(b)) \\ &\quad - (-1)^{|a||b|} (\alpha^{-1} \beta(b)_{-\lambda-\partial} \alpha(a))_{\lambda+\mu} \beta(c) \\ &\quad + (-1)^{|a||c|+|a||b|+|b||c|} \alpha^{-1} \beta^2(c)_{-\lambda-\mu-\partial} (b_{-\lambda-\partial} \alpha^2 \beta^{-1}(a)), \end{aligned}$$

and

$$\begin{aligned}
& (-1)^{|a||b|} [\beta(b)_\mu, [\alpha(a)_\lambda c]] \\
&= (-1)^{|a||b|} \beta(b)_\mu (\alpha(a)_\lambda c) - (-1)^{|a|(|c|+|b|)} \beta(b)_\mu (\alpha^{-1} \beta(c)_{-\lambda-\partial} \alpha^2 \beta^{-1}(a)) \\
&\quad - (-1)^{|a||c|} (\beta(a)_\lambda \alpha^{-1} \beta(c))_{-\mu-\partial} \alpha(b) \\
&\quad + (-1)^{|a||b|} (\alpha^{-2} \beta^2(c)_{-\lambda-\partial} \alpha(a))_{-\mu-\partial} \alpha(b).
\end{aligned}$$

Using the associativity, it is not hard to check that

$$[\alpha \beta(a)_\lambda [b_\mu c]] = [[\beta(a)_\lambda b]_{\lambda+\mu} \beta(c)] + (-1)^{|a||b|} [\beta(b)_\mu [\alpha(a)_\lambda c]]$$

as desired. This finishes the proof. $\square$

3. Cohomology of BiHom-Lie Conformal Superalgebra

In this section, we develop representation and cohomology theory of BiHom-Lie conformal superalgebras.

Definition 3.1. A $\mathbb{C}[\partial]$ -module (M, ϕ, ψ) is a BiHom-conformal module of a BiHom-Lie conformal superalgebra $\mathcal{R}$ if there is a $\mathbb{C}$ -linear map $\rho : \mathcal{R} \rightarrow \text{Cend}(M)$ satisfying the following conditions (for all $a, b \in \mathcal{R}$):

$$\rho(\partial(a))_\lambda = -\lambda \rho(a)_\lambda, \quad \rho(a)_\lambda \partial = (\lambda + \partial) \rho(a)_\lambda, \quad (3.1)$$

$$\phi \psi = \psi \phi, \quad \psi \partial = \partial \psi, \quad \phi \partial = \partial \phi, \quad (3.2)$$

$$\phi \rho(a)_\lambda = \rho(\alpha(a))_\lambda \phi, \quad \psi \rho(a)_\lambda = \rho(\beta(a))_\lambda \psi, \quad (3.3)$$

$$\rho([\beta(a)_\lambda b])_{\lambda+\mu} \psi = \rho(\alpha \beta(a))_\lambda \rho(b)_\mu - (-1)^{|a||b|} \rho(\beta(b))_\mu \rho(\alpha(a))_\lambda. \quad (3.4)$$

The map ρ is called the corresponding representation.

Example 3.1. Let $(\mathcal{R}, \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. Then $(\mathcal{R}, \alpha, \beta)$ is an $\mathcal{R}$ -module under the adjoint diagonal action, $\rho(a)_\lambda b = [a_\lambda b]$, $\forall a, b \in \mathcal{R}$.

Example 3.2. Let $(g, [\cdot, \cdot], \alpha, \beta)$ be a finite BiHom-Lie superalgebra, (M, ϕ, ψ) be a BiHom-conformal module of g with corresponding representation map $\varrho : g \rightarrow gl(M)$. Then the $\mathbb{Z}_2$ -graded $\mathbb{C}[\partial]$ -module $(\mathbb{C}[\partial] \otimes M, \tilde{\phi}, \tilde{\psi})$ is a BiHom-conformal module of $Cur\ g$, where

$$\tilde{\phi}(f(\partial) \otimes m) = f(\partial) \otimes \phi(m), \quad \tilde{\psi}(f(\partial) \otimes m) = f(\partial) \otimes \psi(m),$$

with a module structure $\rho : Cur\ g \rightarrow Cend(\mathbb{C}[\partial] \otimes M)$ given by

$$\rho(f(\partial) \otimes a)_\lambda (g(\partial) \otimes m) = f(-\partial)g(\partial + \lambda) \otimes \varrho(a)(m), \quad \forall f, g \in \mathbb{C}[\partial], a \in g, m \in M.$$

Proposition 3.1. Let $(\mathcal{R}, \alpha, \beta)$ be a regular BiHom-Lie conformal superalgebra, (M, ϕ, ψ) a BiHom-conformal module of $\mathcal{R}$ and $\rho : \mathcal{R} \rightarrow \mathbb{C}[\lambda] \otimes Cend(M) : a \mapsto \rho(a)_\lambda$ the corresponding representation. Define a λ -bracket $[-_\lambda -]_M$ on $\mathcal{R} \oplus M = (\mathcal{R} \oplus M)_0 \oplus (\mathcal{R} \oplus M)_1$ by

$$[(r + m)_\lambda (r' + m')]_M = [r_\lambda r'] + \rho(r)_\lambda m' - (-1)^{|r'| |m|} \rho(\alpha^{-1} \beta(r'))_{-\partial - \lambda} \phi \psi^{-1}(m), \quad (3.5)$$

$\forall r + m, r' + m' \in \mathcal{R} \oplus M$, where $(\mathcal{R} \oplus M)_\theta = \mathcal{R}_\theta \oplus M_\theta$, $\theta \in \mathbb{Z}_2$. Then $(\mathcal{R} \oplus M, \alpha + \phi, \beta + \psi)$ is a BiHom-Lie conformal superalgebra, called the semidirect product of $\mathcal{R}$ and M , and denote by $\mathcal{R} \ltimes_\rho M$.

Proof. $\forall r + m, r' + m', r'' + m'' \in \mathcal{R} \oplus M$, note that $\mathcal{R} \oplus M$ is equipped with a $\mathbb{C}[\partial]$ -module structure via

$$\partial(r + m) = \partial(r) + \partial(m).$$

In addition, it is immediate that $\partial(\alpha + \phi) = (\alpha + \phi)\partial$ and $\partial(\beta + \psi) = (\beta + \psi)\partial$.

A direct computation gives

$$\begin{aligned}
[\partial(r + m)_\lambda(r' + m')]_M &= [(\partial r + \partial m)_\lambda(r' + m')]_M \\
&= [(\partial r)_\lambda r'] + \rho(\partial r)_\lambda(m') - (-1)^{|r'| |m|} \rho(\alpha^{-1} \beta(r'))_{-\partial-\lambda}(\partial \phi \psi^{-1}(m)) \\
&= -\lambda[r_\lambda r'] - \lambda \rho(r)_\lambda(m') - (-1)^{|r'| |m|} (\partial - \lambda - \partial) \rho(\alpha^{-1} \beta(r'))_{-\partial-\lambda}(\phi \psi^{-1}(m)) \\
&= -\lambda([r_\lambda r'] + \rho(r)_\lambda(m') - (-1)^{|r'| |m|} \rho(\alpha^{-1} \beta(r'))_{-\partial-\lambda}(\phi \psi^{-1}(m))) \\
&= -\lambda[(r + m)_\lambda(r' + m')]_M,
\end{aligned}$$

and

$$\begin{aligned}
[(r + m)_\lambda \partial(r' + m')]_M &= [(r + m)_\lambda(\partial r' + \partial m')]_M \\
&= [r_\lambda \partial r'] + \rho(r)_\lambda(\partial m') - (-1)^{|r'| |m|} \rho(\partial \alpha^{-1} \beta(r'))_{-\lambda-\partial} \phi \psi^{-1}(m) \\
&= (\lambda + \partial)([r_\lambda r'] + \rho(r)_\lambda(m') - (-1)^{|r'| |m|} \rho(\alpha^{-1} \beta(r'))_{-\lambda-\partial} \phi \psi^{-1}(m)) \\
&= (\lambda + \partial)[(r + m)_\lambda(r' + m')]_M.
\end{aligned}$$

Thus (2.3) holds. (2.4) follows from

$$\begin{aligned}
&[(\beta(r) + \psi(m))_\lambda(\alpha(r') + \phi(m'))]_M \\
&= [\beta(r)_\lambda \alpha(r')] + \rho(\beta(r))_\lambda(\phi(m')) - (-1)^{|r'| |m|} \rho(\beta(r'))_{-\partial-\lambda}(\phi(m)) \\
&= -(-1)^{|r'| |m|} ([\beta(r')_{-\partial-\lambda} \alpha(r)] + \rho(\beta(r'))_{-\partial-\lambda}(\phi(m)) \\
&\quad - (-1)^{|r'| |m|} \rho(\beta(r))_{-\partial-(-\partial-\lambda)}(\phi(m'))) \\
&= -(-1)^{|r'| |m|} [(\beta(r') + \psi(m'))_{-\partial-\lambda}(\alpha(r) + \phi(m))]_M.
\end{aligned}$$

To check the BiHom-Jacobi conformal identity (2.5), we compute

$$\begin{aligned}
& [(\alpha\beta(r) + \phi\psi(m))_\lambda [(r' + m')_\mu (r'' + m'')]_M]_M \\
&= [(\alpha\beta(r) + \phi\psi(m))_\lambda ([r'_\mu r''] + \rho(r')_\mu (m'')) \\
&\quad - (-1)^{|r''||m'|} \rho(\alpha^{-1}\beta(r'))_{-\partial-\mu} (\phi\psi^{-1}(m'))]_M \\
&= [\alpha\beta(r)_\lambda [r'_\mu r'']] + \rho(\alpha\beta(r))_\lambda \rho(r')_\mu (m'') \\
&\quad - (-1)^{|r''||m'|} \rho(\alpha\beta(r))_\lambda \rho(\alpha^{-1}\beta(r'))_{-\partial-\mu} \phi\psi^{-1}(m') \\
&\quad - (-1)^{(|r''|+|r'|)|m'|} \rho([\alpha^{-1}\beta(r')_\mu \alpha^{-1}\beta(r'')])_{-\partial-\lambda} \phi^2(m), \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
& (-1)^{|r||r'|} [(\beta(r') + \psi(m'))_\mu [(\alpha(r) + \phi(m))_\lambda (r'' + m'')]_M]_M \\
&= (-1)^{|r||r'|} [\beta(r')_\mu [\alpha(r)_\lambda r'']] + (-1)^{|r||r'|} \rho(\beta(r'))_\mu (\rho(\alpha(r))_\lambda (m'')) \\
&\quad - (-1)^{|r||r'|+|r''||m'|} \rho(\beta(r'))_\mu (\rho(\alpha^{-1}\beta(r''))_{-\partial-\lambda} (\phi^2\psi^{-1}(m))) \\
&\quad - (-1)^{|r''||m'|} \rho([\beta(r)_\lambda \alpha^{-1}\beta(r'')])_{-\partial-\mu} \phi(m'), \tag{3.7}
\end{aligned}$$

and

$$\begin{aligned}
& [(\beta(r) + \psi(m))_\lambda (r' + m')]_{M(\lambda+\mu)} (\beta(r'') + \psi(m''))_M \\
&= [([\beta(r)_\lambda r'] + \rho(\beta(r))_\lambda (m') - (-1)^{|r''||m'|} \rho(\alpha^{-1}\beta(r'))_{-\partial-\lambda} \\
&\quad \times \phi\psi^{-1}(m))_{\lambda+\mu} (\beta(r'') + \psi(m''))]_M \\
&= [(\beta(r)_\lambda \beta(r'))_{\lambda+\mu} r''] + \rho([\beta(r)_\lambda r'])_{\lambda+\mu} (\psi(m'')) \\
&\quad - (-1)^{|r''|(|r|+|m'|)} \rho(\alpha^{-1}\beta^2(r''))_{-\partial-\lambda-\mu} \phi\psi^{-1}(\rho(\beta(r))_\lambda m') \\
&\quad + (-1)^{|r''|(|r'|+|m|)+|r'| |m'|} \rho(\alpha^{-1}\beta^2(r''))_{-\partial-\lambda-\mu} \phi\psi^{-1}(\rho(\alpha^{-1}\beta(r'))_{-\partial-\lambda} \phi(m)). \tag{3.8}
\end{aligned}$$

Now, since (M, ρ, ϕ, ψ) is a representation of $\mathcal{R}$, then we can easily conclude that $\mathcal{R} \oplus M$ is a BiHom-Lie conformal superalgebra. $\square$

In the following we aim to develop the cohomology theory of regular BiHom-Lie conformal superalgebras. To do this, we need the following concept.

Definition 3.2. An n -cochain ($n \in \mathbb{Z}_+$) of a BiHom-Lie conformal superalgebra $\mathcal{R}$ with coefficients in a module M is a $\mathbb{C}$ -linear map of degree θ

$$\gamma : \mathcal{R}^n \rightarrow M[\lambda_1, \dots, \lambda_n],$$

$$(a_1, \dots, a_n) \mapsto \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n),$$

where $M[\lambda_1, \dots, \lambda_n]$ denotes the space of polynomials with coefficients in M , satisfying the following conditions:

Conformal antilinearity:

$$\gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, \partial a_i, \dots, a_n) = -\lambda_i \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_i, \dots, a_n).$$

Skew-symmetry:

$$\begin{aligned} & \gamma_{\lambda_1, \dots, \lambda_i, \lambda_{i+1}, \dots, \lambda_n}(a_1, \dots, \beta(a_i), \alpha(a_{i+1}), \dots, a_n) \\ &= -(-1)^{|a_i||a_{i+1}|} \gamma_{\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots, \lambda_n}(a_1, \dots, \beta(a_{i+1}), \alpha(a_i), \dots, a_n). \end{aligned}$$

Commutativity: $\gamma \circ \alpha = \phi \circ \gamma$, $\gamma \circ \beta = \psi \circ \gamma$.

A 0-cochain is just an element in M . Define a differential d of a cochain γ by

$$\begin{aligned} & (d\gamma)_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}) \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} (-1)^{(|\gamma|+|a_1|+\dots+|a_{i-1}|)|a_i|} \rho(\alpha \beta^{n-1}(a_i))_{\lambda_i} \\ & \quad \times \gamma_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i < j}^{n+1} (-1)^{i+j} (-1)^{(|a_1| + \dots + |a_{i-1}|)|a_i| + (|a_1| + \dots + |a_{j-1}|)|a_j| + |a_i||a_j|} \\
& \times \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}} ([\alpha^{-1} \beta(a_i)_{\lambda_i} a_j], \\
& \beta(a_1), \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, \beta(a_{n+1})),
\end{aligned}$$

where ρ is the corresponding representation of M , and γ is extended linearly over the polynomials in λ_i . In particular, if γ is a 0-cochain, then $(d\gamma)_\lambda a = a_\lambda \gamma$.

Proposition 3.2. *$d\gamma$ is a cochain and $d^2 = 0$.*

Proof. Let γ be an n -cochain. It is not hard to see that $d\gamma$ satisfies conformal antilinearity, skew-symmetry and commutativity. That is, $d\gamma$ is an $(n+1)$ -cochain. It remains to show that $d^2 = 0$. In fact, we have

$$\begin{aligned}
& (d^2\gamma)_{\lambda_1, \dots, \lambda_{n+2}}(a_1, \dots, a_{n+2}) \\
& = \sum_{i=1}^{n+1} (-1)^{i+1} (-1)^{i+1+|\gamma||a_i|+A_i} \rho(\alpha\beta^n(a_i))_{\lambda_i} (d\gamma)_{\lambda_1, \dots, \lambda_i, \dots, \lambda_{n+2}} \\
& \quad \times (a_1, \dots, \hat{a}_i, \dots, a_{n+2}) \\
& + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+A_i+A_j+|a_i||a_j|} (d\gamma)_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+2}} \\
& \quad \times ([\alpha^{-1} \beta(a_i)_{\lambda_i} a_j], \beta(a_1), \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, \beta(a_{n+2})) \\
& = \sum_{i=1}^{n+2} \sum_{j=1}^{i-1} (-1)^{i+j+|\gamma|(|a_i|+|a_j|)+A_i+A_j} \rho(\alpha\beta^n(a_i))_{\lambda_i} (\alpha\beta^{n-1}(a_j)_{\lambda_j} \\
& \quad \gamma_{\lambda_1, \dots, \hat{\lambda}_{j,i}, \dots, \lambda_{n+2}}(a_1, \dots, \hat{a}_{j,i}, \dots, a_{n+2}))
\end{aligned} \tag{3.9}$$

$$\sum_{i=1}^{n+2} \sum_{j=i+1}^{n+2} (-1)^{i+j+1+|\gamma|(|a_i|+|a_j|)+A_i+(A_j-|a_i|)} \rho(\alpha\beta^n(a_i))_{\lambda_i}$$

$$(\alpha\beta^{n-1}(a_j)_{\lambda_j} \gamma_{\lambda_1, \dots, \hat{\lambda}_{i,j}, \dots, \lambda_{n+2}}(a_1, \dots, \hat{a}_{i,j}, \dots, a_{n+2})) \quad (3.10)$$

$$+ \sum_{i=1}^{n+2} \sum_{1 \leq j < k < i}^{n+2} (-1)^{i+j+k+1+|\gamma|(|a_i|+A_i+A_j+A_k+|a_j||a_k|)} \rho(\alpha\beta^n(a_i))_{\lambda_i}$$

$$\gamma_{\lambda_j+\lambda_k, \lambda_1, \dots, \hat{\lambda}_{j,k,i}, \dots, \lambda_{n+2}}([\alpha^{-1}\beta(a_j)_{\lambda_j} a_k], \beta(a_1), \dots, \hat{a}_{j,k,i}, \dots, \beta(a_{n+2})) \quad (3.11)$$

$$+ \sum_{i=1}^{n+2} \sum_{1 \leq j < i < k}^{n+2} (-1)^{i+j+k+|\gamma|(|a_i|+A_i+A_j+(A_k-|a_i|)+|a_j||a_k|)} \rho(\alpha\beta^n(a_i))_{\lambda_i}$$

$$\gamma_{\lambda_j+\lambda_k, \lambda_1, \dots, \hat{\lambda}_{j,i,k}, \dots, \lambda_{n+2}}([\alpha^{-1}\beta(a_j)_{\lambda_j} a_k], \beta(a_1), \dots, \hat{a}_{j,i,k}, \dots, \beta(a_{n+2})) \quad (3.12)$$

$$+ \sum_{i=1}^{n+2} \sum_{1 \leq i < j < k}^{n+2} (-1)^{i+j+k+1+|\gamma|(|a_i|+A_i+(A_j-|a_i|)+(A_k-|a_i|)+|a_j||a_k|)} \rho(\alpha\beta^n(a_i))_{\lambda_i}$$

$$\gamma_{\lambda_j+\lambda_k, \lambda_1, \dots, \hat{\lambda}_{i,j,k}, \dots, \lambda_{n+2}}([\alpha^{-1}\beta(a_j)_{\lambda_j} a_k], \beta(a_1), \dots, \hat{a}_{i,j,k}, \dots, \beta(a_{n+2})) \quad (3.13)$$

$$+ \sum_{1 \leq i < j}^{n+2} \sum_{k=1}^{i-1} (-1)^{i+j+k+A_i+A_j+A_k+|a_i||a_j|+(|\gamma|+|a_i|+|a_j|)|a_k|} \rho(\alpha\beta^n(a_k))_{\lambda_k}$$

$$\gamma_{\lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_{k,i,j}, \dots, \lambda_{n+2}}([\alpha^{-1}\beta(a_i)_{\lambda_i} a_j], \beta(a_1), \dots, \hat{a}_{k,i,j}, \dots, \beta(a_{n+2})) \quad (3.14)$$

$$\begin{aligned}
& + \sum_{1 \leq i < j}^{n+2} \sum_{k=i+1}^{j-1} (-1)^{i+j+k+1+A_i+A_j+A_k+|a_i||a_j|+(|\gamma|+|a_j|)|a_k|} \rho(\alpha \beta^n(a_k))_{\lambda_k} \\
& \gamma_{\lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_{i,k,j}, \dots, \lambda_{n+2}} ([\alpha^{-1} \beta(a_i)_{\lambda_j} a_j], \beta(a_1), \dots, \hat{a}_{i,k,j}, \dots, \beta(a_{n+2}))
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
& + \sum_{1 \leq i < j}^{n+2} \sum_{k=j+1}^{n+2} (-1)^{i+j+k+A_i+A_j+A_k+|a_i||a_j|+|\gamma||a_k|} \rho(\alpha \beta^n(a_k))_{\lambda_k} \\
& \gamma_{\lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_{i,j,k}, \dots, \lambda_{n+2}} ([\alpha^{-1} \beta(a_i)_{\lambda_i} a_j], \beta(a_1), \dots, \hat{a}_{i,j,k}, \dots, \beta(a_{n+2}))
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& + \sum_{1 \leq i < j}^{n+2} (-1)^{i+j+A_i+A_j+|a_i||a_j|+|\gamma|(|a_i|+|a_j|)} \rho(\alpha \beta^{n-1}([\alpha^{-1} \beta(a_i)_{\lambda_i} a_j]))_{\lambda_i+\lambda_j} \\
& \gamma_{\lambda_1, \dots, \hat{\lambda}_j, \dots, \hat{\lambda}_i, \dots, \lambda_{n+2}} (\beta(a_1), \dots, \hat{a}_j, \dots, \hat{a}_i, \dots, \beta(a_{n+2}))
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& + \sum_{\text{distinct } i, j, k, l, i < j, k < l}^{n+2} (-1)^{i+j+k+l} sg\{i, j, k, l\} \\
& (-1)^{A_i+A_j+|a_i||a_j|+(|a_i|+|a_j|)(|a_k|+|a_l|)} \gamma_{\lambda_k+\lambda_l, \lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_{i,j,k,l}, \dots, \lambda_{n+2}} \\
& (\beta[\alpha^{-1} \beta(a_k)_{\lambda_k} a_l], \beta[\alpha^{-1} \beta(a_i)_{\lambda_i} a_j], \dots, \hat{a}_{i,j,k,l}, \dots, \beta^2(a_{n+2}))
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
& \sum_{i, j, k=1, i < j, k \neq i, j}^{n+2} (-1)^{i+j+k+l} sg\{i, j, k\} (-1)^{A_i+A_j+|a_i||a_j|+(A_k-|a_i|-|a_j|)} \\
& \gamma_{\lambda_i+\lambda_k+\lambda_j, \lambda_1, \dots, \hat{\lambda}_{i,j,k}, \dots, \lambda_{n+2}} ([\alpha^{-1} \beta(a_i)_{\lambda_i} a_j]_{\lambda_i+\lambda_j}, \beta(a_k)), \\
& \beta^2(a_1), \dots, \hat{a}_{i,j,k}, \dots, \beta^2(a_{n+2})),
\end{aligned} \tag{3.19}$$

where $sg\{i_1, \dots, i_p\}$ is the sign of the permutation putting the indices in increasing order and $\hat{a}_{i,j}, a_i, a_j, \dots$ are omitted.

It is obvious that (3.11) and (3.22) summations cancel each other. The same is true for (3.13) and (3.15), (3.13) and (3.14). The BiHom-Jacobi identity implies (3.19) = 0 and the skew-symmetry implies (3.18) = 0. Because (M, ρ, ϕ, ψ) is an $\mathcal{R}$ -module, it follows that

$$\rho([\beta(a)_\lambda b])_{\lambda+\mu} \psi(v) = \rho(\alpha\beta(a))_\lambda \rho(b)_\mu(v) - (-1)^{|a||b|} \rho(\beta(b))_\mu \rho(\alpha(a))_\lambda(v).$$

Since $\gamma \circ \alpha = \alpha_M \circ \gamma$, $\gamma \circ \beta = \beta_M \circ \gamma$, we have (3.9), (3.10) and (3.17) summations vanish. Therefore $d^2\gamma = 0$ and the proof is completed. $\square$

Thus the cochains of a BiHom-Lie conformal superalgebra $(\mathcal{R}, \alpha, \beta)$ with coefficients in a module M form a complex, which is denoted by

$$C_{\alpha,\beta}^\bullet = C_{\alpha,\beta}^\bullet(\mathcal{R}, M) = \bigoplus_{n \in \mathbb{Z}_+} C_{\alpha,\beta}^n(\mathcal{R}, M),$$

where $C_{\alpha,\beta}^n(\mathcal{R}, M) = C_{\alpha,\beta}^n(\mathcal{R}, M)_0 \oplus C_{\alpha,\beta}^n(\mathcal{R}, M)_1$, $C_{\alpha,\beta}^n(\mathcal{R}, M)_\theta$ is the set of n -cochain of degree θ ($\theta \in \mathbb{Z}_2$). The cohomology group $H_{\alpha,\beta}^\bullet(\mathcal{R}, M)$ of a BiHom-Lie conformal superalgebra $\mathcal{R}$ with coefficients in a module M is the cohomology of complex $C_{\alpha,\beta}^\bullet$.

Definition 3.3. Let $(\mathcal{R}, \alpha, \beta)$ be a regular BiHom-Lie conformal superalgebra, (M, ϕ, ψ) a BiHom-conformal module of $\mathcal{R}$ and $\rho : \mathcal{R} \rightarrow \mathbb{C}[\lambda] \otimes \text{Cend}(M) : a \mapsto \rho(a)_\lambda$ the corresponding representation. If a $C[\partial]$ -module homomorphism $T : M \rightarrow A$ satisfies

$$T \circ \phi = \alpha \circ T, \quad T \circ \psi = \beta \circ T, \quad (3.20)$$

$$[Tu_\lambda Tv] = T(\rho(Tu)_\lambda v - (-1)^{|u||v|} \rho(T\phi^{-1}\psi(v))_{-\lambda-\partial} \phi\psi^{-1}(u)), \quad \forall u, v \in M, \quad (3.21)$$

then T is called an *O-operator* associated with ρ .

Proposition 3.3. *Let $(\mathcal{R}, \alpha, \beta)$ be a regular BiHom-Lie conformal superalgebra, (M, ρ, ϕ, ψ) be a representation and $T : M \rightarrow A$ be an O-operator. Then the λ -product*

$$[u_\lambda v]_T = \rho(Tu)_\lambda v - (-1)^{|u||v|} \rho(T\phi^{-1}\psi(v))_{-\lambda-\partial} \phi\psi^{-1}(u), \quad u, v \in M, \quad (3.22)$$

defines a BiHom-Lie superalgebra structure on M .

Proof. It can be checked by a direct computation, so we omit details. □

The following result is obvious.

Corollary 3.1. *Let T be an O-operator on a BiHom-Lie superalgebra $(\mathcal{R}, \alpha, \beta)$ with respect to a representation (M, ρ, ϕ, ψ) . Then T is a homomorphism from the BiHom-Lie superalgebra $(M, [\cdot_\lambda \cdot]_T, \phi, \psi)$ to the BiHom-Lie superalgebra $(\mathcal{R}, \alpha, \beta)$.*

4. Generalized Derivations of BiHom-Lie Conformal Superalgebras

In this section, we investigate derivations and generalized derivations of BiHom-Lie conformal superalgebras and study some of their properties.

Definition 4.1. Let $(\mathcal{R}, \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. Then a BiHom conformal linear map $f_\lambda : \mathcal{R} \rightarrow \mathcal{R}$ is called an $\alpha^k \beta^l$ -derivation of $(\mathcal{R}, \alpha, \beta)$ if

$$f_\lambda \circ \alpha = \alpha \circ f_\lambda, \quad f_\lambda \circ \beta = \beta \circ f_\lambda,$$

$$f_\lambda([a_\mu b]) = [f_\lambda(a)_{\lambda+\mu} \alpha^k \beta^l(b)] + (-1)^{|a||f|} [\alpha^k \beta^l(a)_\mu f_\lambda(b)], \quad (4.1)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

Denote by $\text{Der}_{\alpha^k \beta^l}$ the set of $\alpha^k \beta^l$ -derivations of the BiHom-Lie conformal superalgebra $(\mathcal{R}, \alpha, \beta)$. For any $a \in \mathcal{R}$ satisfying $\alpha(a) = a$, $\beta(a) = a$, define $f_{k,l} : \mathcal{R} \rightarrow \mathcal{R}$ by

$$f_{k,l}(a)_\lambda(b) = [a_\lambda \alpha^{k+1} \beta^{l-1}(b)], \quad \forall b \in \mathcal{R}.$$

Then $f_{k,l}(a)$ is an $\alpha^{k+1} \beta^l$ -derivation, which is called an inner $\alpha^{k+1} \beta^l$ -derivation. In fact,

$$\begin{aligned} f_{k,l}(a)_\lambda(\partial b) &= [a_\lambda \alpha^{k+1} \beta^{l-1}(\partial b)] = [a_\lambda \partial \alpha^{k+1} \beta^{l-1}(b)] = (\partial + \lambda) f_{k,l}(a)_\lambda(b), \\ f_{k,l}(a)_\lambda(\alpha(b)) &= [a_\lambda \alpha^{k+2} \beta^{l-1}(b)] = \alpha[a_\lambda \alpha^{k+1} \beta^{l-1}(b)] = \alpha \circ f_{k,l}(a)_\lambda(b), \\ f_{k,l}(a)_\lambda(\beta(b)) &= [a_\lambda \alpha^{k+1} \beta^l(b)] = \beta[a_\lambda \alpha^{k+1} \beta^{l-1}(b)] = \beta \circ f_{k,l}(a)_\lambda(b), \\ f_{k,l}(a)_\lambda([b_\mu c]) &= [a_\lambda \alpha^{k+1} \beta^{l-1}([b_\mu c])] = [\alpha \beta(a)_\lambda [\alpha^{k+1} \beta^{l-1}(b)_\mu \alpha^{k+1} \beta^{l-1}(c)]] \\ &= [\beta(a)_\lambda \alpha^{k+1} \beta^{l-1}(b)]_{\lambda+(-1)^{|a||b|}\mu} \alpha^{k+1} \beta^l(c)] \\ &\quad + [\alpha^{k+1} \beta^l(b)_\mu [\alpha(a)_\lambda \alpha^{k+1} \beta^{l-1}(c)]] \\ &= [a_\lambda \alpha^{k+1} \beta^{l-1}(b)]_{\lambda+\mu} \alpha^{k+1} \beta^l(c)] \\ &\quad + (-1)^{|a||b|} [\alpha^{k+1} \beta^l(b)_\mu [a_\lambda \alpha^{k+1} \beta^{l-1}(c)]] \\ &= [f_{k,l}(a)_\lambda(b)_{\lambda+\mu} \alpha^{k+1} \beta^l(c)] \\ &\quad + (-1)^{|a||b|} [\alpha^{k+1} \beta^l(b)_\mu (f_{k,l}(a)_\lambda(c))]. \end{aligned}$$

Denote by $\text{Inn}_{\alpha^k \beta^l}(\mathcal{R})$ the set of inner $\alpha^k \beta^l$ -derivations. For $f_\lambda \in \text{Der}_{\alpha^k \beta^l}(\mathcal{R})$ and $g_{\mu-\lambda} \in \text{Der}_{\alpha^s \beta^t}(R)$, define their commutator $[f_\lambda g]_\mu$ by

$$[f_\lambda g]_\mu(a) = f_\lambda(g_{\mu-\lambda}a) - (-1)^{|f||g|} g_{\mu-\lambda}(f_\lambda a), \quad \forall a \in \mathcal{R}. \quad (4.2)$$

Lemma 4.1. *For any $f_\lambda \in \text{Der}_{\alpha^k \beta^l}(\mathcal{R})$ and $g_{\mu-\lambda} \in \text{Der}_{\alpha^s \beta^t}(\mathcal{R})$, we have*

$$[f_\lambda g] \in \text{Der}_{\alpha^{k+s} \beta^{l+t}}(\mathcal{R})[\lambda].$$

Proof. For any $a, b \in \mathcal{R}$, we have

$$\begin{aligned} [f_\lambda g]_\mu(\partial a) &= f_\lambda(g_{\mu-\lambda} \partial a) - (-1)^{|f||g|} g_{\mu-\lambda}(f_\lambda \partial a) \\ &= f_\lambda((\partial + \mu - \lambda)g_{\mu-\lambda} a) + (-1)^{|f||g|} g_{\mu-\lambda}((\mu + \lambda)f_\lambda a) \\ &= (\partial + \mu)f_\lambda(g_{\mu-\lambda} a) - (-1)^{|f||g|}(\partial + \mu)g_{\mu-\lambda}(f_\lambda a) \\ &= (\partial + \mu)[f_\lambda g]_\mu(a). \end{aligned}$$

Moreover, we have

$$\begin{aligned} [f_\lambda g]_\mu([a_\gamma b]) &= f_\lambda(g_{\mu-\lambda}[a_\gamma b]) - (-1)^{|f||g|} g_{\mu-\lambda}(f_\lambda[a_\gamma b]) \\ &= f_\lambda([g_{\mu-\lambda}(a)]_{\mu-\lambda+\gamma} \alpha^s \beta^t(b)) + (-1)^{|a||g|} [\alpha^s \beta^t(a)_\gamma g_{\mu-\lambda}(b)] \\ &\quad - (-1)^{|f||g|} g_{\mu-\lambda}([f_\lambda(a)]_{\lambda+\gamma} \alpha^k \beta^l(b)) \\ &\quad + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\gamma f_\lambda(b)] \\ &= [f_\lambda(g_{\mu-\lambda}(a))]_{\mu+\gamma} \alpha^{k+s} \beta^{l+t}(b) \\ &\quad + (-1)^{|f||g||a|} [\alpha^k \beta^l(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma} f_\lambda(\alpha^s \beta^t(b))] \\ &\quad + (-1)^{|a||g|} [f_\lambda(\alpha^s \beta^t(a))_{\lambda+\gamma} \alpha^k \beta^l(g_{\mu-\lambda}(b))] \\ &\quad + (-1)^{|f||g||a|} [\alpha^{k+s} \beta^{l+t}(a)_\gamma (f_\lambda(g_{\mu-\lambda}(b)))] \\ &\quad - (-1)^{|f||g|} [(g_{\mu-\lambda} f_\lambda(a))_{\mu+\gamma} \alpha^{k+s} \beta^{l+t}(b)] \\ &\quad - [\alpha^s \beta^t(f_\lambda(a))_{\lambda+\gamma} (g_{\mu-\lambda}(\alpha^k \beta^l(b)))] \end{aligned}$$

$$\begin{aligned}
& - (-1)^{|f||g||a|} [(g_{\mu-\lambda}(\alpha^k \beta^l(a)))_{\mu-\lambda+\gamma} \alpha^s \beta^t(f_\lambda(b))] \\
& - (-1)^{|f||g||a|} [\alpha^{k+s} \beta^{l+t}(a)_\lambda (g_{\mu-\lambda}(f_\lambda(b)))] \\
& = [([f_\lambda g]_\mu a)_{\mu+\gamma} \alpha^{k+s} \beta^{l+t}(b)] \\
& + (-1)^{|f||g||a|} [\alpha^{k+s} \beta^{l+t}(a)_\gamma ([f_\lambda g]_\mu b)].
\end{aligned}$$

Therefore, $[f_\lambda g] \in \text{Der}_{\alpha^{k+s} \beta^{l+t}}(\mathcal{R})[\lambda]$. $\square$

Define

$$\text{Der}(\mathcal{R}) = \bigoplus_{k, l \geq 0} \text{Der}_{\alpha^k \beta^l}(\mathcal{R}). \quad (4.3)$$

Proposition 4.1. *(Der($\mathcal{R}$), α' , β') is a BiHom-Lie conformal superalgebra with respect to (4.2) where $\alpha'(f) = f \circ \alpha$ and $\beta'(f) = f \circ \beta$.*

Proof. It is straightforward. So we omit details. $\square$

Let $\mathcal{R}$ be a BiHom-Lie conformal superalgebra. Define $\text{Der}(\mathcal{R})_\theta$ is the set of all derivations of degree θ , then it is obvious that $\text{Der}(\mathcal{R}) = \text{Der}(\mathcal{R})_0 \oplus \text{Der}(\mathcal{R})_1$ is a subalgebra of $\text{Cend}(\mathcal{R})$.

Define

$$\Omega = \{f_\lambda \in \text{gc}(\mathcal{R}) \mid f_\lambda \circ \alpha = \alpha \circ f_\lambda, f_\lambda \circ \beta = \beta \circ f_\lambda\}.$$

Let $\alpha', \beta' : \Omega \rightarrow \Omega$, $\alpha'(f_\lambda) = f_\lambda \circ \alpha$, $\beta'(f_\lambda) = f_\lambda \circ \beta$. Then $(\Omega, \alpha', \beta')$ is a BiHom-Lie conformal algebra with respect to (2.6) and $\text{Der}(\mathcal{R})$ is a BiHom-conformal subalgebra of Ω .

Definition 4.2. An element f in Ω is called

(a) an $\alpha^k \beta^l$ -generalized derivation of $\mathcal{R}$, if there exist $f', f'' \in \Omega$ such that $|f| = |f'| = |f''|$ and

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f'_\lambda(b))] = f''_\lambda([a_\mu b]), \quad (4.4)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

(b) an $\alpha^k \beta^l$ -quasiderivation of $\mathcal{R}$, if there is $f' \in \Omega$ such that $|f| = |f'|$ and

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f'_\lambda(b))] = f'_\lambda([a_\mu b]), \quad (4.5)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

(c) an $\alpha^k \beta^l$ -centroid of $\mathcal{R}$, if it satisfies

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] = (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f_\lambda(b))] = f_\lambda([a_\mu b]), \quad (4.6)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

(d) an $\alpha^k \beta^l$ -quasicentroid of $\mathcal{R}$, if it satisfies

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] = (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f_\lambda(b))], \quad (4.7)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

(e) an $\alpha^k \beta^l$ -central derivation of $\mathcal{R}$, if it satisfies

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] = f_\lambda([a_\mu b]) = 0, \quad (4.8)$$

for all homogeneous elements $a, b \in \mathcal{R}$.

Denote by $\text{GDer}_{\alpha^k\beta^l}(\mathcal{R})$, $\text{QDer}_{\alpha^k\beta^l}(\mathcal{R})$, $C_{\alpha^k\beta^l}(\mathcal{R})$, $QC_{\alpha^k\beta^l}(\mathcal{R})$ and $\text{ZDer}_{\alpha^k\beta^l}(\mathcal{R})$ the sets of all $\alpha^k\beta^l$ -generalized derivations, $\alpha^k\beta^l$ quasi-derivations, $\alpha^k\beta^l$ centroids, $\alpha^k\beta^l$ quasiceintroids and $\alpha^k\beta^l$ central derivations of $\mathcal{R}$. Set

$$\text{GDer}(\mathcal{R}) := \bigoplus_{k \geq 0, l \geq 0} \text{GDer}_{\alpha^k\beta^l}(\mathcal{R}), \quad \text{QDer}(\mathcal{R}) := \bigoplus_{k \geq 0, l \geq 0} \text{QDer}_{\alpha^k\beta^l}(\mathcal{R}).$$

$$C(\mathcal{R}) := \bigoplus_{k \geq 0, l \geq 0} C_{\alpha^k\beta^l}(\mathcal{R}), \quad QC_{\alpha^k\beta^l}(\mathcal{R}) := \bigoplus_{k \geq 0, l \geq 0} QC_{\alpha^k\beta^l}(\mathcal{R}),$$

$$\text{ZDer}(\mathcal{R}) := \bigoplus_{k \geq 0, l \geq 0} \text{ZDer}_{\alpha^k\beta^l}(\mathcal{R}).$$

It is easy to see that

$$\text{ZDer}(\mathcal{R}) \subseteq \text{Der}(\mathcal{R}) \subseteq \text{QDer}(\mathcal{R}) \subseteq \text{GDer}(\mathcal{R}) \subseteq \text{Cend}(\mathcal{R}),$$

$$C(\mathcal{R}) \subseteq QC(\mathcal{R}) \subseteq \text{GDer}(\mathcal{R}). \quad (4.9)$$

Proposition 4.2. *Let $(\mathcal{R}, \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. Then*

- (i) *$\text{GDer}(\mathcal{R})$, $\text{QDer}(\mathcal{R})$ and $C(\mathcal{R})$ are BiHom-Lie conformal subalgebras of Ω ,*
- (ii) *$\text{ZDer}(\mathcal{R})$ is a BiHom-Lie conformal ideal of $\text{Der}(\mathcal{R})$.*

Proof. (i) We only prove that $\text{GDer}(\mathcal{R})$ is a BiHom-conformal subalgebra of Ω . The proof for the other two cases can be done similarly.

For $f_\lambda \in \text{GDer}_{\alpha^k\beta^l}(\mathcal{R})$, $g_\lambda \in \text{GDer}_{\alpha^s\beta^t}(\mathcal{R})$, $a, b \in \mathcal{R}$, there exist $f', f'' \in \Omega$ (resp., $g', g'' \in \Omega$) such that Equation (4.4) holds for f (resp., g). We only need to show

$$\begin{aligned} [f''_\lambda g''_\lambda]_\theta([a_\mu b]) &= [([f_\lambda g]_\theta(a))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ &\quad + (-1)^{(|f|+|g|)|a|} [\alpha^{k+s} \beta^{l+t}(a)_\mu ([f'_\lambda g'_\lambda]_\theta(b))]. \end{aligned} \quad (4.10)$$

In fact, we have

$$[(f_\lambda g)_\theta(a))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] = [(f_\lambda(g_{\theta-\lambda}(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ - (-1)^{|f||g|} [(g_{\theta-\lambda}(f_\lambda(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)]. \quad (4.11)$$

On the other hand,

$$[(f_\lambda(g_{\theta-\lambda}(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ = f_\lambda''([(g_{\theta-\lambda}(a))_{\mu+\theta-\lambda} \alpha^s \beta^t(b)]) - (-1)^{|f|(|g|+|a|)} \\ \times [\alpha^k \beta^l(g_{\theta-\lambda}(a))_{\mu+\theta-\lambda} (f_\lambda'(\alpha^s \beta^t(b)))] \\ = f_\lambda''(g_{\theta-\lambda}''([a_\mu b])) - (-1)^{|g||a|} f_\lambda''([\alpha^s \beta^t(a)]_\mu (g_{\theta-\lambda}'(b))) \\ - (-1)^{|f|(|g|+|a|)} g_{\theta-\lambda}''([\alpha^s \beta^t(a)]_\mu (f_\lambda'(b))) \\ + (-1)^{|f|(|g|+|a|)+|g||a|} [\alpha^{k+s} \beta^{l+t}(a)]_\mu (g_{\theta-\lambda}'(f_\lambda'(b))), \quad (4.12)$$

and

$$[(g_{\theta-\lambda}(f_\lambda(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ = g_{\theta-\lambda}''([(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)]) - (-1)^{|g|(|f|+|a|)} \\ \times [f_\lambda(\alpha^s \beta^t(a))_{\lambda+\mu} (g_{\theta-\lambda}'(\alpha^k \beta^l(b)))] \\ = g_{\theta-\lambda}''(f_\lambda''([a_\mu b])) - (-1)^{|f||a|} g_{\theta-\lambda}''([\alpha^k \beta^l(a)]_\mu (f_\lambda'(b))) \\ - (-1)^{|g|(|f|+|a|)} f_\lambda''([\alpha^s \beta^t(a)]_\mu (g_{\theta-\lambda}'(b))) \\ + (-1)^{|g|(|f|+|a|)+|f||a|} [\alpha^{k+s} \beta^{l+t}(a)]_\mu f_\lambda'(g_{\theta-\lambda}'(b))). \quad (4.13)$$

Substituting Equations (4.12) and (4.13) into Equation (4.11), we obtain Equation (4.10). In addition, It is obvious that $\alpha'(f) \in \text{GDer}(\mathcal{R})$ and $\beta'(f) \in \text{GDer}(\mathcal{R})$. Hence $[f_\lambda g] \in \text{GDer}(\mathcal{R})[\lambda]$, and $\text{GDer}(\mathcal{R})$ is a BiHom-conformal subalgebra of Ω .

(ii) For $f \in \text{ZDer}_{\alpha^k \beta^l}(\mathcal{R})$, $g \in \text{Der}_{\alpha^s \beta^t}(\mathcal{R})$, and $a, b \in \mathcal{R}$, we have

$$\begin{aligned} [f_\lambda g]_\theta([a_\mu b]) &= f_\lambda(g_{\theta-\lambda}([a_\mu b])) - (-1)^{|f||g|} g_{\theta-\lambda}(f_\lambda([a_\mu b])) \\ &= f_\lambda(g_{\theta-\lambda}([a_\mu b])) = f_\lambda([(g_{\theta-\lambda}(a))_{\mu+\theta-\lambda} \alpha^s \beta^t(b)]) \\ &\quad + (-1)^{|f||g|} [\alpha^s \beta^t(a)_\mu (g_{\theta-\lambda}(b))] = 0, \end{aligned}$$

and

$$\begin{aligned} [[f_\lambda g]_\theta(a)_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] &= [(f_\lambda(g_{\theta-\lambda}(a)) - (-1)^{|f||g|} g_{\theta-\lambda}(f_\lambda(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ &= [-(-1)^{|f||g|} (g_{\theta-\lambda}(f_\lambda(a)))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ &= -(-1)^{|f||g|} g_{\theta-\lambda}^{\prime\prime}([f_\lambda(a)_{\lambda+\mu} \alpha^k \beta^{l+t}(b)]) \\ &\quad + (-1)^{|g||a|} [f_\lambda(\alpha^s \beta^t(a))_{\lambda+\mu} g_{\theta-\lambda}'(\alpha^k \beta^t(b))] \\ &= 0. \end{aligned}$$

This means that $[f_\lambda g] \in \text{ZDer}(\mathcal{R})[\lambda]$. Thus $\text{ZDer}(\mathcal{R})$ is an ideal of $\text{Der}(\mathcal{R})$. $\square$

Proposition 4.3. *Let $f \in \text{QC}_{\alpha^k \beta^l}(\mathcal{R})$ and $g \in \text{QC}_{\alpha^s \beta^t}(\mathcal{R})$. Then $[f_\lambda g]$ is an $\alpha^{k+s} \beta^{l+t}$ -generalized derivation of degree $|f| + |g|$.*

Proof. Assume that $f \in QC_{\alpha^k \beta^l}(\mathcal{R})$ and $g \in QC_{\alpha^s \beta^t}(\mathcal{R})$. Then for any homogeneous elements $a, b \in \mathcal{R}$, we have

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] = (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f_\lambda(b))],$$

and

$$[(g_\lambda(a))_{\lambda+\mu} \alpha^s \beta^t(b)] = (-1)^{|g||a|} [\alpha^s \beta^t(a)_\mu (g_\lambda(b))].$$

Hence

$$\begin{aligned} [[f_\lambda g]_\theta(a)_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] &= [f_\lambda(g_{\theta-\lambda}(a))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ &\quad - (-1)^{|f||g|} [g_{\theta-\lambda}(f_\lambda(a))_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] \\ &= (-1)^{(|f|+|g|)|a|} [\alpha^{k+s} \beta^{l+t}(a)_\mu g_{\theta-\lambda}(f_\lambda(b))] \\ &\quad - (-1)^{(|f|+|g|)|a|} (-1)^{|f||g|} [\alpha^{k+s} \beta^{l+t}(a)_\mu f_\lambda(g_{\theta-\lambda}(b))] \\ &= -(-1)^{(|f|+|g|)|a|} [\alpha^{k+s} \beta^{l+t}(a)_\mu (-1)^{|f||g|} [f_\lambda g]_\theta(b)], \end{aligned}$$

which implies that

$$[[f_\lambda g]_\theta(a)_{\mu+\theta} \alpha^{k+s} \beta^{l+t}(b)] + (-1)^{(|f|+|g|)|a|} [\alpha^{k+s} \beta^{l+t}(a)_\mu (-1)^{|f||g|} [f_\lambda g]_\theta(b)] = 0.$$

Then $[f_\lambda g]$ is an $\alpha^{k+s} \beta^{l+t}$ -generalized derivation of degree $|f| + |g|$. $\square$

Proposition 4.4. Let $(\mathcal{R}, \alpha, \beta)$ be a BiHom-Lie conformal superalgebra. If $f \in QDer_{\alpha^k \beta^l}(\mathcal{R})$ and $g \in QC_{\alpha^k \beta^l}(\mathcal{R})$ such that $|f| = |g|$, then $f + g \in GDer_{\alpha^k \beta^l}(\mathcal{R})$ with degree $|f|$.

Proof. Let $f_\lambda \in QDer_{\alpha^k \beta^l}(\mathcal{R})$. Then there exist $f' \in \Omega$ such that

$$[(f_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f_\lambda(b))] = f'_\lambda([a_\mu b]), \quad \forall a, b \in \mathcal{R}.$$

(4.14)

And let $g \in \mathcal{QC}_{\alpha^k \beta^l}(\mathcal{R})$. Then

$$[(g_\lambda(a))_{\lambda+\mu} \alpha^k \beta^l(b)] = (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (g_\lambda(b))], \quad \forall a, b \in \mathcal{R}. \quad (4.15)$$

We will prove that $f + g$ satisfies Equation (4.4). In fact,

$$\begin{aligned} [(f + g)_\lambda(a)_{\lambda+\mu} \alpha^k \beta^l(b)] &= [f_\lambda(a)_{\lambda+\mu} \alpha^k \beta^l(b)] + [g_\lambda(a)_{\lambda+\mu} \alpha^k \beta^l(b)] \\ &= f'_\lambda([a_\mu b]) - (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu f_\lambda(b)] \\ &\quad + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu g_\lambda(b)] \\ &= f'_\lambda([a_\mu b]) - (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f - g)_\lambda(b)]. \end{aligned}$$

Then

$$[(f + g)_\lambda(a)_{\lambda+\mu} \alpha^k \beta^l(b)] + (-1)^{|f||a|} [\alpha^k \beta^l(a)_\mu (f - g)_\lambda(b)] = f'_\lambda([a_\mu b]).$$

Therefore $f + g$ satisfies Equation (4.4). This completes the proof. $\square$

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