INTEGRABLE CONDITIONS FOR DIRAC EQUATION AND SCHRÖDINGER EQUATION

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Abstract

By constructing the commutative operators chain, we derive the integrable conditions for solving the eigenfunctions of Dirac equation and Schrödinger equation. These commutative relations correspond to the intrinsic symmetry of the physical system, which are equivalent to the original partial differential equation can be solved by separation of variables. Detailed calculation shows that, only a few cases can be completely solved by separation of variables. In general cases, we have to solve the Dirac equation and the Schrödinger equation by effective perturbation or approximation methods, especially in the cases including nonlinear potential or self-interactive potentials.

1. Introduction

Fermions with $\text{spin} - \frac{1}{2}$ are described by Dirac equations. Many physicists such as H. Weyl, W. Heisenberg, once proposed using nonlinear spinor equations to establish unified field theory for elementary particles

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[1, 2]. Some rigorous solutions for the simplest dark nonlinear spinor models were obtained and analyzed [3]-[11]. The nonlinear spinor coupling with self-electromagnetic field was calculated in [12]-[15], but only roughly approximate results were obtained due to the complexity of the interaction. In contrast, for the linear Schrödinger equation, Pauli equation and Dirac equation in spherical coordinate system (r, θ, φ) coupling with external potential $V(r)$ and $\vec{A}(r, \theta)$, there are a lot of rigorous eigen solutions obtained by separation of variables [16]-[29]. Some theoretical analyses on the existence of stationary solutions to nonlinear Dirac equations were provided in [30]-[36].

In the resolution of eigenfunctions to the linear Dirac equation, we find that the commutative relations [16]

$$[\hat{H}, \hat{J}_z] = 0, \quad [\hat{H}, \hat{K}] = 0, \quad [\hat{K}, \hat{J}_z] = 0 \quad (1.1)$$

play an important role, where the operators $(\hat{J}_z, \hat{K}, \hat{H})$ stand for angular momentum, spin-orbit coupling, and total energy operators, respectively. They are good quantum numbers guaranteeing the common eigenfunctions exist. Noticing the definition of operators chain

$$\hat{J}_z = J_z(\partial_\varphi), \quad \hat{K} = K(\partial_\varphi, \partial_\theta), \quad \hat{H} = H(\partial_\varphi, \partial_\theta, \partial_r), \quad (1.2)$$

which enable us to solve the eigen solutions by separation of variables, and then the original problem reduces to ordinary differential equations. So the existence of complete commutative operator chain forms the integrable condition for Dirac equation or Schrödinger equation. Since the construction of commutative operator chain is quite simple and elementary, and the existence conditions are easily converted into some algebraic requirements for potentials as shown below, then the integrable condition of a quantum system is equivalently obtained.

Denote the Minkowski metric by $\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1]$, where $\mu, \nu \in \{0, 1, 2, 3\}$. Pauli matrices by

$$\vec{\sigma} = (\sigma^j) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad j \in \{1, 2, 3\}. \quad (1.3)$$

Define 4×4 Hermitian matrices as follows:

$$\alpha^\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right\}, \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \quad (1.4)$$

In this paper, we adopt the Hermitian matrices (1.4) instead of Dirac matrices γ^μ for convenience of calculation.

The basic problem in this paper is to examine a nonlinear spinor field ϕ moving in 4-vector potential A^μ . The corresponding Lagrangian is generally given by [13]

$$\mathcal{L} = \phi^+ [\alpha^\mu (i\hbar \partial_\mu - eA_\mu) - mc\gamma] \phi + F(\check{\gamma}, \check{\beta}), \quad (1.5)$$

where $m > 0$ is a constant mass, the index ‘+’ stands for transpose conjugation, F is the nonlinear coupling potential such as $F = \frac{1}{2} \omega \check{\gamma}^2$, which is usually the polynomial of scalar $\check{\gamma}$ and pseudo-scalar $\check{\beta}$ defined by

$$\check{\gamma} = \phi^+ \gamma \phi, \quad \check{\beta} = \phi^+ \beta \phi. \quad (1.6)$$

The variation of (1.5) with respect to ϕ^+ gives the dynamic equation

$$i\hbar \partial_0 \phi = \hat{H} \phi, \quad \hat{H} \equiv \vec{\alpha} \cdot (-i\hbar \nabla - e\vec{A}) + eA_0 + (mc - F_\gamma) \gamma - F_\beta \beta, \quad (1.7)$$

where $F_\gamma = \frac{\partial F}{\partial \check{\gamma}}$ and $F_\beta = \frac{\partial F}{\partial \check{\beta}}$.

Let coordinate $x^3 = z$ along the direction of magnetic field $\vec{B}$, then we locally have [8, 9, 13]

$$\vec{A} = A(r, \theta)(-\sin \varphi, \cos \varphi, 0), \quad (1.8)$$

which satisfies the Coulomb gauge $\nabla \cdot \vec{A} = 0$. In the spherical coordinate system (r, θ, φ) , we have

$$\vec{\sigma} \cdot \nabla = \sigma_r \partial_r + \frac{1}{r} \sigma_\theta \partial_\theta + \frac{1}{r \sin \theta} \sigma_\varphi \partial_\varphi, \quad (1.9)$$

where $(\sigma_r, \sigma_\theta, \sigma_\varphi)$ is given by

$$\left\{ \begin{pmatrix} \cos \theta & \sin \theta e^{-\varphi i} \\ \sin \theta e^{\varphi i} & -\cos \theta \end{pmatrix}, \begin{pmatrix} \sin \theta & \cos \theta e^{-\varphi i} \\ \cos \theta e^{\varphi i} & \sin \theta \end{pmatrix}, \begin{pmatrix} 0 & ie^{-\varphi i} \\ ie^{\varphi i} & 0 \end{pmatrix} \right\}. \quad (1.10)$$

Let $\hat{J}$ be the angular momentum operator for the spinor field, we have

$$\hat{J} = \vec{r} \times \hat{p} + \frac{1}{2} \hbar \vec{S}, \quad \hat{p} = -i\hbar \nabla, \quad \vec{S} = \text{diag}(\vec{\sigma}, \vec{\sigma}). \quad (1.11)$$

However, it should be pointed out that, here $\hat{p} = -i\hbar \nabla$ is only geometrical momentum rather the physical one $\hat{p} = -i\hbar \nabla - e\vec{A}$. Then the angular momentum $\hat{J}_3 = \hat{J}_z$ commutates with the nonlinear Hamilton operator (1.7), and the eigenfunctions of $\hat{J}_z = -i\hbar \partial_\varphi + \frac{1}{2} \hbar S_3$ are equivalent to the following form:

$$\phi = (u_1, u_2 e^{\varphi i}, -iv_1, -iv_2 e^{\varphi i})^T \exp\left(\kappa \varphi i - \frac{mc^2}{\hbar} ti\right) \quad (1.12)$$

with $(\kappa = 0, \pm 1, \pm 2, \dots)$, where $u_k, v_k (k = 1, 2)$ are real functions of (r, θ) but independent on φ and t , the index T stands for transposed matrix.

2. Commutative Algebras for Dirac Equation

In spherical coordinate system, we have the following explicit operator relations [16]. For orbit angular momentum operator $\hat{L} = \vec{r} \times \vec{p}$, we have

$$\hat{L}_x = i\hbar(\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi), \quad (2.1)$$

$$\hat{L}_y = i\hbar(-\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi), \quad (2.2)$$

$$\hat{L}_z = -i\hbar \partial_\varphi, \quad (2.3)$$

$$\hat{L}^2 = -\hbar^2(\partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2). \quad (2.4)$$

For the nonlinear equation (1.7), detailed calculation shows that

$$[\hat{H}, \hat{J}_z] = 0. \quad (2.5)$$

(2.5) means $\hat{J}_z$ is still a good quantum number for the nonlinear Dirac equation. But we have $[\hat{H}, \hat{J}_x] \neq 0$ and $[\hat{H}, \hat{J}_y] \neq 0$. Define the spin-orbit coupling operator by

$$\hat{K} \equiv \gamma(\hat{L} \cdot \vec{S} + \hbar) = K_0 - i\hbar(K_\theta \partial_\theta + K_\varphi \partial_\varphi), \quad K_0 \equiv \hbar\gamma, \quad (2.6)$$

where K_θ and K_φ defined by

$$K_\theta = \text{diag} \left[\begin{pmatrix} 0 & -ie^{-\varphi i} \\ ie^{\varphi i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & ie^{-\varphi i} \\ -ie^{\varphi i} & 0 \end{pmatrix} \right], \quad (2.7)$$

$$K_\varphi = \text{diag} \left[\begin{pmatrix} 1 & -\cot \theta e^{-\varphi i} \\ -\cot \theta e^{\varphi i} & -1 \end{pmatrix}, \begin{pmatrix} -1 & \cot \theta e^{-\varphi i} \\ \cot \theta e^{\varphi i} & 1 \end{pmatrix} \right], \quad (2.8)$$

then for (1.7), we can check

$$[\hat{H}, \hat{K}] = K_V + K_A + K_\beta, \quad (2.9)$$

in which

$$K_V = -i\hbar(e\partial_\theta V - \partial_\theta F_\gamma \gamma)(\sin \varphi S_1 - \cos \varphi S_2), \quad (2.10)$$

$$K_A = -e\hbar(A\gamma(\sin \varphi \alpha^1 - \cos \varphi \alpha^2) - (\partial_\theta A + \cot \theta A + 2A\partial_\theta)\beta), \quad (2.11)$$

$$\begin{aligned} K_\beta &= 2\hbar\gamma\beta F_\beta + \hbar(\partial_\theta F_\beta + 2F_\beta\partial_\theta)(\sin \varphi \alpha^1 - \cos \varphi \alpha^2) \\ &\quad + 2\hbar F_\beta(\cot \theta(\cos \varphi \alpha^1 + \sin \varphi \alpha^2) - \alpha^3)\partial_\varphi. \end{aligned} \quad (2.12)$$

Relations (2.9)-(2.12) reflect the influence of parameters on the integrability.

3. Conditions for Separation of Variables

For the nonlinear Hamiltonian (1.7), the commutative relation (2.5) shows $\hat{J}_z$ is still a good quantum number, but (2.9) shows $\hat{K}$ is not. In what follows, we look for a new operator $\hat{T}$, such that the systems (1.7) can be solved by separation of variables. This is equivalent to the existence of operator $\hat{T} = T(\partial_\theta, \partial_\varphi)$ or $\hat{T}' = T'(\partial_r, \partial_\varphi)$, which satisfies commutative relations

$$[\hat{J}_z, \hat{T}] = 0, \quad [\hat{H}, \hat{T}] = 0. \quad (3.1)$$

We only consider the following first order operator

$$\hat{T} = T_0 - i\hbar(T_\theta\partial_\theta + T_\varphi\partial_\varphi), \quad (3.2)$$

where $(T_0, T_\theta, T_\varphi)$ are all Hermitian matrices, and their components are smooth functions of (r, θ, φ) . For Dirac equation, it is enough to look for the linear operators similar to (3.2). But for the Schrödinger and Pauli equations, we should consider the second order operator similar to $\hat{L}^2$ as shown below. Different from $\hat{p}$ and $\hat{J}$, the operators constructed from the procedure may have not manifest physical meanings.

According to condition $[\hat{J}_z, \hat{T}] = 0$, we can solve

$$T_0 = \begin{pmatrix} P_{11} & P_{12} \cot \theta e^{-i\phi} & P_{13} & P_{14} \cot \theta e^{-i\phi} \\ P_{21} \cot \theta e^{i\phi} & P_{22} & P_{23} \cot \theta e^{i\phi} & P_{24} \\ P_{31} & P_{32} \cot \theta e^{-i\phi} & P_{33} & P_{34} \cot \theta e^{-i\phi} \\ P_{41} \cot \theta e^{i\phi} & P_{42} & P_{43} \cot \theta e^{i\phi} & P_{44} \end{pmatrix}, \quad (3.3)$$

$$T_\theta = \begin{pmatrix} M_{11} & M_{12} e^{-i\phi} & M_{13} & M_{14} e^{-i\phi} \\ M_{21} e^{i\phi} & M_{22} & M_{23} e^{i\phi} & M_{24} \\ M_{31} & M_{32} e^{-i\phi} & M_{33} & M_{34} e^{-i\phi} \\ M_{41} e^{i\phi} & M_{42} & M_{43} e^{i\phi} & M_{44} \end{pmatrix}, \quad (3.4)$$

$$T_\phi = \begin{pmatrix} N_{11} & N_{12} \cot \theta e^{-i\phi} & N_{13} & N_{14} \cot \theta e^{-i\phi} \\ N_{21} \cot \theta e^{i\phi} & N_{22} & N_{23} \cot \theta e^{i\phi} & N_{24} \\ N_{31} & N_{32} \cot \theta e^{-i\phi} & N_{33} & N_{34} \cot \theta e^{-i\phi} \\ N_{41} \cot \theta e^{i\phi} & N_{42} & N_{43} \cot \theta e^{i\phi} & N_{44} \end{pmatrix}, \quad (3.5)$$

where (P_{kl}, M_{kl}, N_{kl}) are functions of (r, θ) , the factor ‘ $\cot \theta$ ’ is introduced for convenience of the following calculation.

The relation $[\hat{H}, \hat{T}]$ can be expressed as

$$\begin{aligned} [\hat{H}, \hat{T}] &= H_{\theta\theta} \partial_\theta^2 + H_{r\theta} \partial_{r\theta} + (H_\theta + H_{\theta\phi} \partial_\phi) \partial_\theta + (H_r + H_{r\phi} \partial_\phi) \partial_r \\ &\quad + (H_0 + H_\phi \partial_\phi + H_{\phi\phi} \partial_\phi^2), \end{aligned} \quad (3.6)$$

where the coefficient matrices can be obtained by straightforward calculation. $(H_{\theta\theta}, H_{r\theta}, H_{\theta\varphi}, H_{r\varphi}, H_{\varphi\varphi})$ are only functions of (P_{kl}, M_{kl}, N_{kl}) , but the others depend on their first order derivatives. The operator ∂_φ should be replaced by matrix D_φ due to solution (1.12),

$$\partial_\varphi \leftrightarrow D_\varphi = i \operatorname{diag}(\kappa, \kappa + 1, \kappa, \kappa + 1), \quad (3.7)$$

$$\partial_\varphi^2 \leftrightarrow D_\varphi^2 = - \operatorname{diag}(\kappa^2, (\kappa + 1)^2, \kappa^2, (\kappa + 1)^2). \quad (3.8)$$

Then $[\hat{H}, \hat{T}] = 0$ is equivalent to the following equations:

$$H_{\theta\theta} = H_{r\theta} = 0, \quad (3.9)$$

$$H_\theta + H_{\theta\varphi}D_\varphi = 0, \quad (3.10)$$

$$H_r + H_{r\varphi}D_\varphi = 0, \quad (3.11)$$

$$H_0 + H_\varphi D_\varphi + H_{\varphi\varphi}D_\varphi^2 = 0. \quad (3.12)$$

The differential equations including in (3.9)-(3.12) are linear, and one component usually satisfies two independent equations, which lead to constant solutions. So they can be easily solved as shown below.

By (3.9), we can solve

$$T_\theta = \begin{pmatrix} W_1 & -iW_2e^{-i\phi} & W_3 & W_4e^{-i\phi} \\ iW_2e^{i\phi} & W_1 & -W_4e^{i\phi} & W_3 \\ W_3 & -W_4e^{-i\phi} & W_1 & iW_2e^{-i\phi} \\ W_4e^{i\phi} & W_3 & -iW_2e^{i\phi} & W_1 \end{pmatrix}, \quad (3.13)$$

in which $W_k = W_k(r, \theta, \kappa)$, $(k = 1, 2, 3, 4)$. Again by (3.10), we find that $W_1 = W_3 = W_4 = 0$, and $W_2 \neq 0$ is a constant. Obviously $\hat{T}$ can be only

determined to a constant factor and constant translation, so we choose $W_2 = 1$ to fix the solution. Then in equivalent sense, we have

$$T_\theta = K_\theta. \quad (3.14)$$

For the other equations in (3.10)-(3.12), detailed calculation shows that the solutions for P_{kl} and N_{kl} are underdetermined. Expressing P_{kl} by N_{kl} , we can get the following results by simple calculation:

$$V = V(r), \quad A = r^{-1}U(\theta), \quad \partial_\theta F_\gamma = 0, \quad F_\beta = 0, \quad (3.15)$$

$$T_0 - i\hbar T_\phi D_\phi = (K_0 - i\hbar K_\phi D_\phi) + e\hbar U \text{diag}(\sigma_\theta, -\sigma_\theta) + \lambda_0, \quad (3.16)$$

where λ_0 is a constant. If we set $\hat{T} = \hat{K}$ when $U = 0$, we get $\lambda_0 = 0$. Substituting $D_\phi \rightarrow \partial_\phi$ into (3.16), in equivalent sense, we finally get

$$T_0 = K_0 + e\hbar U \text{diag}(\sigma_\theta, -\sigma_\theta), \quad T_\theta = K_\theta, \quad T_\phi = K_\phi. \quad (3.17)$$

(3.17) shows $\hat{T}$ is almost the spin-orbit coupling term $\hat{K}$. Only if the vector potential $\vec{A}$ takes some special shape, we can exactly solve the Dirac equation. The physical meanings of the additional term $e\hbar U \text{diag}(\sigma_\theta, -\sigma_\theta)$ should be clarified by Clifford algebra as shown below.

In the spherical coordinate system, the above procedure is invertible, so Equations (3.15) actually form the sufficient and necessary conditions of the separation of variables to Dirac equation with axisymmetry. Noticing the relation (3.17), conditions (3.15) can also be verified by the commutative relations (2.9)-(2.12). The following cases were studied in previous works by many researchers: $\{V = A = F_\beta = \partial_\theta F_\gamma = 0\}$ is the simplest nonlinear case discussed in [3]-[9]. $\{V = V(r), A = F_\beta = 0, F_\gamma = G(r)\}$ are the cases solved in [16] and [17]-[28]. $\{V = \frac{Ze}{r}, A = \frac{e}{r \sin \theta} (a - b \cos \theta), F_\beta = F_\gamma = 0\}$ is the case solved in [29].

For the general case of (3.15), although the separation of variables is valid for the Dirac equation (1.7), but the solutions to the reduced ordinary differential equations usually can not be expressed by elementary functions. So some auxiliary numerical computation is still necessary.

For the Schrödinger equation

$$i\hbar\partial_0\phi = \hat{H}\phi, \quad \hat{H} \equiv -\frac{\hbar^2}{2m} \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{\hbar^2 r^2} \hat{L}^2 \right) + V(r, \theta), \quad (3.18)$$

the calculation is simple. Obviously, for $\hat{L}_z = -\hbar i \partial_\varphi$, we have

$$[\hat{H}, \partial_\varphi] = 0, \quad \partial_\varphi \leftrightarrow \kappa i. \quad (3.19)$$

We look for the following second order operator:

$$\hat{Y}(\theta, \varphi) = Y_0 + Y_\theta \partial_\theta + \hat{L}^2. \quad (3.20)$$

By $[\hat{Y}, \partial_\varphi] = 0$, we get

$$\partial_\varphi Y_0 = \partial_\varphi Y_\theta = 0. \quad (3.21)$$

By $[\hat{H}, \hat{Y}] = 0$, we find

$$V = W(r) + \frac{1}{r^2} U(\theta), \quad (3.22)$$

$$Y_0 = \lambda_0 - 2mU(\theta), \quad Y_\theta = 0, \quad (3.23)$$

where $U(\theta)$ and $W(r)$ are arbitrary functions. (3.22) is the condition to solve the eigenfunctions of (3.18) by separating variables. Different from the Dirac equation, (3.22) shows V can vary with θ .

4. The Relations in the Formalism of Clifford Algebra

The above discussion is based on the conventional notations of quantum mechanics. Now we make some general considerations. We express the main contents in the formalism of Clifford algebra, because this algebra faithfully and exactly reflects the intrinsic symmetry and property of space-time and fields [10, 37]. In this formalism, the calculations such as (3.3)-(3.6) become much simple and clear, and we may get some new insights into the symmetries of the integrable conditions for the Dirac equations.

Denote the generators of $Cl(1, 3)$ by

$$\gamma^a = \begin{pmatrix} 0 & \tilde{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (4.1)$$

$$\sigma^a = (I, \vec{\sigma}), \quad \tilde{\sigma}^a = (\sigma^0, -\vec{\sigma}), \quad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \quad (4.2)$$

For dynamics (1.7), making transformation

$$\phi' = C\phi, \quad C = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^5) = C^{-1}, \quad (4.3)$$

we get Dirac equation with coefficient in standard Clifford algebra as follows:

$$\gamma^a(i\hbar\partial_a - eA_a)\phi' + i\gamma^5 F_\beta \phi' = (mc - F_\gamma)\phi', \quad (4.4)$$

The quadratic scalar and pseudo scalar become

$$\check{\gamma} = \phi'^+ \gamma^0 \phi', \quad \check{\beta} = \phi'^+ i\gamma^0 \gamma^5 \phi' = \phi'^+ \beta \phi'. \quad (4.5)$$

By transformation (4.3), we get

$$\hat{T}' = \gamma^0 C \hat{T} C^{-1} = \hat{K}' + e\hbar U S^\theta. \quad (4.6)$$

(4.6) shows $\hat{T}'$ includes a spin-potential coupling term $e\hbar r A S^\theta$, which reflects the spin-vector potential interaction. Its physical meaning can be clarified only by Clifford algebra. In what follows we directly use ϕ to replace ϕ' for simplicity.

To get rigorous solution of Dirac equation, using coordinate system reflects the symmetry of the interaction potential is important. The results derived above are only relevant to the potentials near the spherical symmetry. In general cases, we should discuss the problem in curved space-time. Clifford algebra is the perfect language. Then Dirac equation (4.4) becomes [10, 37, 38]

$$\gamma^\mu (\hat{p}_\mu + \frac{1}{2} \hbar \gamma^5 \Omega_\mu) \phi + i \gamma^5 F_\beta \phi = (mc - F_\gamma) \phi, \quad (4.7)$$

in which the coefficient belongs to $\Lambda^0 \cup \Lambda^1 \cup \Lambda^3 \cup \Lambda^4 \subset C\ell(1, 3)$, and parameters and operators are defined as

$$\gamma^\mu = h^\mu{}_a \gamma^a, \quad \gamma_\mu = l_\mu{}^a \gamma_a, \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \quad (4.8)$$

$$\hat{p}_\mu = i\hbar(\partial_\mu + \Upsilon_\mu) - eA_\mu, \quad S_{ab}^{\mu\nu} \equiv \frac{1}{2}(h^\mu{}_a h^\nu{}_b + h^\nu{}_a h^\mu{}_b) \text{sgn}(a-b), \quad (4.9)$$

$$\Upsilon_\mu = \frac{1}{2} h^\nu{}_a (\partial_\mu l_\nu^a - \partial_\nu l_\mu^a), \quad \Omega^\alpha = \frac{1}{4} \epsilon^{dabc} h^\alpha{}_d h^\beta{}_a S_{bc}^{\mu\nu} \partial_\beta g_{\mu\nu}. \quad (4.10)$$

Here we use Greek index stands for curved space-time, Latin index for Minkowski space-time, and (j, k, l, m, n) for spatial indices. If the metric can be diagonalized, we have $\Omega_\mu \equiv 0$.

Obviously, for eigenstates of spinor, the time t is special, which should be global simultaneity. That is to say, we must discuss the Hamiltonian formalism of (4.7) in the natural coordinate system $ds^2 = g_{00}dt^2 - g_{kl}dx^k dx^l$ with global realistic time [39] or in local Gauss normal coordinate system (LGCS) $ds^2 = dt^2 - g_{kl}dx^k dx^l$. We take LGCS for simplicity.

In LGCS, to lift and lower the index of a vector means

$$\Upsilon^0 = \Upsilon_0 = \frac{1}{2} \partial_t (\ln \sqrt{g}), \quad \Upsilon^k = -g^{kl} \Upsilon_l, \quad g_{kl} g^{ln} = \delta_k^n. \quad (4.11)$$

Then (4.7) can be rewritten in the following Hamiltonian formalism:

$$i(\partial_t + \Upsilon_t)\phi = \hat{H}\phi, \quad \hat{H} = -\alpha^k \hat{p}_k + eA_0 + (mc - F_\gamma)\gamma^0 - \beta F_\beta - \Omega_\mu \hat{S}^\mu, \quad (4.12)$$

where $\hat{H}$ is the Hamiltonian of dynamics in curved space-time, and $(\alpha^\mu, \hat{S}^\mu)$ are respectively current and spin operators defined by

$$\alpha^\mu \equiv \gamma^0 \gamma^\mu = h^\mu_a \text{diag}(\sigma^a, \tilde{\sigma}^a), \quad \hat{S}^\mu \equiv \frac{1}{2} \hbar \alpha^\mu \gamma^5 = \frac{1}{2} \hbar h^\mu_a \text{diag}(\sigma^a, -\tilde{\sigma}^a). \quad (4.13)$$

$\tilde{S} \in \Lambda^3$ is the usual spin of Dirac bispinor. The eigenstate of energy means $i(\partial_t + \Upsilon_t)\phi = E\phi$.

The Hamiltonian $\hat{H}$ has the following properties:

- (1) $\hat{H}$ is Hermitian, so its eigen values are all real.
- (2) $\hat{H}$ is a linear operator with respect to derivatives $\nabla_k = \partial_k + \Upsilon_k$.
- (3) $\hat{H} \in C\ell(4, 0)$, and the generators of $C\ell(4, 0)$ are (γ^0, α^k) .
- (4) $\hat{H}$ does not include component in Λ^2 , i.e., we have not $\gamma^{\alpha\beta}$ term in $\hat{H}$.

From the calculations in the above sections, we learn that, to solve the rigorous eigen solution to (4.12) is equivalent to construct an Abelian Lie algebra $\{H_1, H_2, H_3 = \hat{H}\}$, where $\forall H_k$ are linear independent. Besides, $\forall H_k$ should satisfy all listed properties of $\hat{H}$. Under these constraints, we can solve the other two operators (H_1, H_2) for any dynamics of spinor in principle. And then, by a linear transformation with respect to derivatives ∇_k , we can get the Abelian Lie group with proper elements $\{\hat{H}_1(\nabla_1), \hat{H}_2(\nabla_1, \nabla_2), \hat{H}\}$. Thus the solvability of Dirac

equation is thoroughly resolved in theory. Obviously this procedure can be generalized for high dimensional problems. However, to derive the elements of Abelian Lie group $\{\hat{H}_1, \hat{H}_2\}$ for a concrete problem is not an easy task. It should be realized by computing program.

5. Discussion and Conclusion

From the above calculation, we find the method is quite simple and direct. We get the following conclusion and remarks. The above procedure has general significance to solve the rigorous eigen solution of Dirac equation. The separation of variables is equivalent to the existence of the following Hermitian operators chain:

$$\hat{H}_1 = H_1(\partial_1), \quad \hat{H}_2 = H_2(\partial_1, \partial_2), \dots, \quad \hat{H}_n = H_n(\partial_1, \partial_2, \dots, \partial_n), \quad (5.1)$$

which form an Abelian Lie algebra

$$[\hat{H}_j, \hat{H}_k] = 0, \quad (j, k = 1, \dots, n). \quad (5.2)$$

(5.2) forms the integrable conditions of the equation, and can be obtained via simple calculation. In general, a complete chain similarly to the case of (1.7) may be absent, but the incomplete chain is still helpful to simplify the dynamic equation.

The commutative relations (5.2) are conditional relations, that is, it is enough to hold only for the eigenfunctions, rather than hold in the whole function space of the solutions. The solution of the nonlinear spinors is an example [3]-[9]. So we can use the information of solutions for solving the operators chain. By Clifford algebra, we can consider the problems in curved space-time.

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