CONGRUENCES OF THE CARDINALITIES OF RATIONAL POINTS OF LOG FANO VARIETIES AND LOG CALABI-YAU VARIETIES OVER THE LOG POINTS OF FINITE FIELDS

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Abstract

In this article, we give the definitions of log Fano varieties and log Calabi-Yau varieties in the framework of theory of log schemes of Fontaine-Illusie-Kato and give congruences of the cardinalities of rational points of them over the log points of finite fields.

1. Introduction

In this article, we discuss a new topic—rational points of the underlying schemes of log schemes in the sense of Fontaine-Illusie-Kato over the log point of a finite field—for interesting log schemes. First let us recall results on rational points of (proper smooth) schemes over a finite field.

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The following is famous Ax' and Katz' theorem:

Theorem 1.1 ([1], [33]). *Let $\mathbb{F}_q$ be the finite field with $q = p^e$ elements, where p is a prime number. Let n and r be positive integers. Let D_i ($1 \leq i \leq r$) be a hypersurface of $\mathbb{P}_{\mathbb{F}_q}^n$ of degree d_i . If $\sum_{i=1}^r d_i \leq n$, then $\#(\bigcap_{i=1}^r D_i)(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k}$.*

In [16], Esnault has proved the following theorem generalizing this theorem in the case where $\bigcap_{i=1}^r D_i$ is smooth over $\mathbb{F}_q$ and geometrically connected:

Theorem 1.2 ([16], Corollary 1.3). *Let X be a geometrically connected projective smooth scheme over $\mathbb{F}_q$. If $X/\mathbb{F}_q$ is a Fano variety (i.e., the inverse of the canonical sheaf $\omega_{X/\mathbb{F}_q}^{-1}$ of $X/\mathbb{F}_q$ is ample), then $\#X(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k}$ ($k \in \mathbb{Z}_{\geq 1}$).*

In [34], Kim has proved the following theorem and he has reproved Esnault's theorem as a corollary of his theorem by using the Lefschetz trace formula for the crystalline cohomology of $X/\mathbb{F}_q$:

Theorem 1.3 ([34], Theorem 1). *Let κ be a perfect field of characteristic $p > 0$. Set $\mathcal{W} := \mathcal{W}(\kappa)$ and $K_0 := \text{Frac}(\mathcal{W})$. Let X be a projective smooth scheme over κ . If X/κ is a Fano variety, then $H^i(X, \mathcal{W}(\mathcal{O}_X)) \otimes_{\mathcal{W}} K_0 = 0$ for $i > 0$.*

In [24], Gongyo et al. have proved the following theorem generalizing (1.2) for the 3-dimensional case by using methods of MMP(=minimal model program) in characteristic $p \geq 7$:

Theorem 1.4 ([24], Theorems 1.2, 1.3). *Let κ be as in Theorem 1.3. Assume that $p \geq 7$. Let X be a geometrically connected proper variety over κ . Let Δ be an effective $\mathbb{Q}$ -Cartier divisor on X . Assume that (X, Δ) is *klt*(=Kawamata log terminal) pair over κ and that $-(K_X + \Delta)$ is a $\mathbb{Q}$ -Cartier ample divisor on X , where K_X is the canonical divisor on X . Then the following hold:*

$$(1) \ H^i(X, \mathcal{W}(\mathcal{O}_X)) \otimes_{\mathcal{W}} K_0 = 0 \text{ for } i > 0.$$

$$(2) \text{ Assume that } \kappa = \mathbb{F}_q. \text{ Then } \# X(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k} \ (k \in \mathbb{Z}_{\geq 1}).$$

See [54] for the case where $-(K_X + \Delta)$ is nef and big and (X, Δ) is log canonical.

In this article, we give other generalizations of the Theorems 1.2 and 1.3 under the assumption of certain finiteness: we give the definition of a log Fano variety and we prove a log and stronger version (Theorem 1.5) below of Kim's theorem under the assumption as a really immediate good application of a recent result: Nakkajima-Yobuko's Kodaira vanishing theorem for a quasi- F -split projective log smooth scheme of vertical type ([55]). In this vanishing theorem, we use theory of log structures due to Fontaine-Illusie-Kato ([31, 32]) essentially. (See Section 3 for the precise statement of this vanishing theorem.) As a corollary of Theorem 1.5, we obtain the congruence of the cardinality of rational points of a log Fano variety over the log point of $\mathbb{F}_q$ (Corollary 1.6 below).

To state our result (Theorem 1.5), we first recall the notion of the quasi-Frobenius splitting height due to Yobuko, which plays an important role for log Fano varieties in this article.

Let Y be a scheme of characteristic $p > 0$. Let $F_Y : Y \rightarrow Y$ be the Frobenius endomorphism of Y . Set $F := \mathcal{W}_n(F_Y^*) : \mathcal{W}_n(\mathcal{O}_Y) \rightarrow F_{Y*}(\mathcal{W}_n(\mathcal{O}_Y))$. This is a morphism of $\mathcal{W}_n(\mathcal{O}_Y)$ -modules. In [69], Yobuko has introduced the notion of the quasi-Frobenius splitting height $h^F(Y)$ for Y . (In [loc. cit.] he has denoted it by $\text{ht}^S(Y)$.) It is the minimum of positive integers n 's such that there exists a morphism $\rho : F_{Y*}(\mathcal{W}_n(\mathcal{O}_Y)) \rightarrow \mathcal{O}_Y$ of $\mathcal{W}_n(\mathcal{O}_Y)$ -modules such that $\rho \circ F : \mathcal{W}_n(\mathcal{O}_Y) \rightarrow \mathcal{O}_Y$ is the natural projection. (If there does not exist such n , then we set $h^F(Y) = \infty$.) This is a highly nontrivial generalization of the notion of the Frobenius splitting by Mehta and Ramanathan in [44] because they have said that, for a scheme Z of characteristic $p > 0$, Z is a Frobenius splitting(=F-split) scheme if $F : \mathcal{O}_Z \rightarrow F_{Z*}(\mathcal{O}_Z)$ has a section of $\mathcal{O}_Z$ -modules. Because the terminology “quasi Frobenius splitting height” is too long, we call this *Yobuko height*.

Let κ be a perfect field of characteristic $p > 0$. Let s be a log scheme whose underlying scheme is $\text{Spec}(\kappa)$ and whose log structure is associated to a morphism $\mathbb{N} \ni 1 \mapsto a \in \kappa$ for some $a \in \kappa$. That is, s is the log point of κ or $(\text{Spec}(\kappa), \kappa^*)$. Let X/s be a proper (not necessarily projective) log smooth scheme of pure dimension d of vertical type with log structure $(M_X, \alpha : M_X \rightarrow \mathcal{O}_X)$. Here “vertical type” means that $\alpha(\mathcal{I}_{X/s})\mathcal{O}_X = \mathcal{O}_X$, where $\mathcal{I}_{X/s}$ is Tsuji’s ideal sheaf of the log structure M_X of X defined in [66] and denoted by I_f in [loc. cit.], where $f : X \rightarrow s$ is the structural morphism. (In Section 3 below we recall the definition of $\mathcal{I}_{X/s}$.) For example, the product of (locally) simple normal crossing log schemes over s defined in [46], [55] and [51] is of vertical type. Let $\overset{\circ}{X}$ be the underlying scheme of X . Let $\Omega_{X/s}^i$ be the sheaf of log

differential forms of degree i on $\overset{\circ}{X}$, which has been denoted by $\omega_{X/s}^i$ in [31]. Set $\omega_{X/s} := \Omega_{X/s}^d$. We say that X/s is a *log Fano scheme* if $\omega_{X/s}^{-1}$ is ample. Moreover, if $\overset{\circ}{X}$ is geometrically connected, then we say that X/s is a *log Fano variety*.

In this article we prove the following:

Theorem 1.5. *Let X/s be a log Fano scheme. Assume that $h^F(\overset{\circ}{X}) < \infty$. Then $H^i(X, \mathcal{W}_n(\mathcal{O}_X)) = 0$ for $i > 0$ and for $n > 0$. Consequently, $H^i(X, \mathcal{W}(\mathcal{O}_X)) = 0$ for $i > 0$.*

As mentioned above, we obtain this theorem immediately by using Nakajima-Yobuko's Kodaira vanishing theorem for a quasi- F -split projective log smooth scheme of vertical type ([55]). As a corollary of this theorem, we obtain the following:

Corollary 1.6. *Let X/s be a log Fano variety. Assume that $\kappa = \mathbb{F}_q$ and that $h^F(\overset{\circ}{X}) < \infty$. Then*

$$\# \overset{\circ}{X}(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k} \quad (k \in \mathbb{Z}_{\geq 1}). \quad (1.6.1)$$

In particular $\overset{\circ}{X}(\mathbb{F}_q) \neq \emptyset$.

This is a generalization of Esnault's Theorem 1.2 under the assumption of the finiteness of the Yobuko height. To derive Corollary 1.6 from Theorem 1.5, we use

(A) Étess-Le Stum's Lefschetz trace formula for rigid cohomology (with compact support) ([17]).

(B) Berthelot-Bloch-Esnault's calculation of the slope < 1 -part of the rigid cohomology (with compact support) via Witt sheaves ([4]) as in [5], [24] and [54]. However our proofs of Theorem 1.5 and Corollary 1.6 are very different from Esnault's, Kim's and Gongyo-Nakamura-Tanaka's proofs of Theorems 1.2, 1.3, and 1.4 in their articles because we do not use the rational connectedness of a Fano variety which has been used in them.

We guess that the assumption of the finiteness of the Yobuko height is not a strong one for log Fano schemes. However, this assumption is not always satisfied for smooth Fano schemes because the Kodaira vanishing holds if the Yobuko height is finite and because the Kodaira vanishing does not hold for certain Fano varieties ([27, 39, 65]); the Yobuko heights of them are infinity. Hence to calculate the Yobuko heights of (log) Fano schemes is a very interesting problem.

The conclusion of Corollary 1.6 holds for a proper scheme $Y / \mathbb{F}_q$ such that $H^i(Y, \mathcal{O}_Y) = 0$ ($i > 0$). Tanaka has kindly told me that it is not known whether there exists an example of a smooth Fano variety over κ for which this vanishing of the cohomologies does not hold. (In [30], Joshi has already pointed out this; Shepherd-Barron has already proved that this vanishing holds for a smooth Fano variety of dimension 3 ([59, (1.5)])).

On the other hand, it is not clear at all that there is a precise rule as above about congruences of the cardinalities of the rational points of varieties except Fano varieties. One may think that there is no rule for them. In this article, we show that this is not the case for log Calabi-Yau varieties over s of any dimension when $\mathring{s} = \text{Spec}(\mathbb{F}_q)$; we are more interested in the cardinalities of the rational points of log Calabi-Yau varieties than those of log Fano varieties.

First let us recall the following suggestive observation, which seems well-known ([3]).

Let E be an elliptic curve over $\mathbb{F}_p$. It is well-known that E is nonordinary if and only if

$$\# E(\mathbb{F}_p) = p + 1, \quad (1.6.2)$$

if $p \geq 5$. By the purity of the weight for $E / \mathbb{F}_p$:

$$|\# E(\mathbb{F}_p) - (p + 1)| \leq 2\sqrt{p}, \quad (1.6.3)$$

this equality is equivalent to a congruence

$$\# E(\mathbb{F}_p) \equiv 1 \pmod{p} \quad (1.6.4)$$

since $\sqrt{p} > 2$.

In this article, we generalize the congruence (1.6.4) for higher dimensional (log) varieties as follows. (We also generalize (1.6.2) for any nonordinary elliptic curve over $\mathbb{F}_q$ when $p \geq 5$.)

Let X/s be a proper (not necessarily projective) simple normal crossing log scheme of pure dimension d . Recall that, in [55], we have said that X/s is a log Calabi-Yau scheme of pure dimension d if $H^i(X, \mathcal{O}_X) = 0$ ($0 < i < d$) and $\omega_{X/s} \simeq \mathcal{O}_X$. Moreover, if $\overset{\circ}{X}$ is geometrically connected, then we say that X/s is a log Calabi-Yau variety of pure dimension d . (This is a generalization of a log K3 surface defined in [46].) Note that $H^d(X, \mathcal{O}_X) = H^d(X, \omega_{X/s}) \simeq \kappa$. The last isomorphism is obtained by log Serre duality of Tsuji ([66, (2.21)]). More generally, we consider a proper scheme Y of pure dimension d satisfying only the following four conditions:

- (a) $H^0(Y, \mathcal{O}_Y) = \kappa$,
- (b) $H^i(Y, \mathcal{W}(\mathcal{O}_Y))_{K_0} = 0$ for $0 < i < d - 1$,
- (c) $H^{d-1}(Y, \mathcal{O}_Y) = 0$ if $d \geq 2$,
- (d) $H^d(Y, \mathcal{O}_Y) \simeq \kappa$.

Let $\Phi_{Y/\kappa}$ be the Artin-Mazur formal group of Y/κ in degree d , that is, $\Phi_{Y/\kappa}$ is the following functor:

$$\Phi_{Y/\kappa}(A) := \Phi_{Y/\kappa}^d(A) := \text{Ker}(H_{\text{et}}^d(Y \otimes_{\kappa} A, \mathbb{G}_m) \rightarrow H_{\text{et}}^d(Y, \mathbb{G}_m)) \in (\text{Ab}),$$

for artinian local κ -algebras A 's with residue fields κ . Then $\Phi_{Y/\kappa}$ is pro-represented by a commutative formal Lie group over κ ([2]). Denote the height of $\Phi_{Y/\kappa}$ by $h(Y/\kappa)$. We prove the following:

Theorem 1.7. *Let Y/κ be as above. Assume that $\kappa = \mathbb{F}_q$. Set $h := h(Y/\mathbb{F}_q)$. Then the following hold:*

(1) *Assume that $h = \infty$. Then*

$$\# Y(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k} \quad (k \in \mathbb{Z}_{\geq 1}). \quad (1.7.1)$$

In particular, $Y(\mathbb{F}_q) \neq \emptyset$.

(2) *Assume that $2 \leq h < \infty$. Let $\lceil \cdot \rceil$ be the ceiling function: $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}$. Then*

$$\# Y(\mathbb{F}_{q^k}) \equiv 1 \pmod{p^{\lceil ek(1-h^{-1}) \rceil}} \quad (k \in \mathbb{Z}_{\geq 1}). \quad (1.7.2)$$

In particular, $Y(\mathbb{F}_q) \neq \emptyset$ (recall that $e = \log_p q$).

(3) *Assume that $h = 1$. Then*

$$\# Y(\mathbb{F}_{q^k}) \not\equiv 1 \pmod{p} \quad (k \in \mathbb{Z}_{\geq 1}). \quad (1.7.3)$$

(In particular $Y(\mathbb{F}_{q^k})$ can be empty.)

To give the statement (1.7) is a nontrivial work. However the proof of Theorem 1.7 is not difficult. (It does not matter whether the proof is not difficult.) As far as we know, Theorem 1.7 even in the 2-dimensional trivial logarithmic and smooth case, i.e., the case of K3 surfaces over finite fields, is a new result. Even in the case $d = 1$, Y need not be assumed to be an elliptic curve over $\mathbb{F}_q$.

The heights of Artin-Mazur formal groups describe the different phenomena about the congruences of rational points for schemes satisfying four conditions (a), (b), (c), and (d).

By using Theorem 1.7, we raise an important problem how the certain supersingular prime ideals are distributed for a smooth Calabi-Yau variety of dimension less than or equal to 2 over a number field. (I think that this problem has no relation with Sato-Tate conjecture in non-CM cases.)

To obtain Theorem 1.7, we use the Theorems (A) and (B) explained after Corollary 1.6 again and the determination of the slopes of the Dieudonné module $D(\Phi_{Y/\kappa})$ of $\Phi_{Y/\kappa}$.

Theorem 1.7 (3) also follows from a formula of ([7], (3.7.2)) immediately.

The contents of this article are as follows.

In Section 2, we recall Étess-Le Stum's Lefschetz trace formula for rigid cohomology, Berthelot-Bloch-Esnault's theorem and the congruence of the cardinality of rational points of a separated scheme of finite type over a finite field.

In Section 3, we prove Theorem 1.5 and Corollary 1.6.

In Section 4, we prove (1.7). We also raise the important problem about the distribution of supersingular primes already mentioned.

In Section 5, we give the formulas of two kinds of zeta functions of a few projective SNCL(=simple normal crossing log) schemes over the log point of a finite field. One kind of them gives us examples of the conclusions of the congruences in Corollary 1.6 and Theorem 1.7.

In Section 6, we give a remark on Van der Geer and Katsura's characterization of the height $h(Y/\kappa)$ ([67]).

Notations: (1) For an element a of a commutative ring A with unit element and for an A -modules M , M / a denotes M / aM .

(2) For a finite field $\mathbb{F}_q$, $s_{\mathbb{F}_q}$ denotes the log point whose underlying scheme is $\text{Spec}(\mathbb{F}_q)$.

2. Preliminaries

In this section, we recall Étess-Le Stum's Lefschetz trace formula for rigid cohomology with compact support ([17]) and Berthelot-Bloch-Esnault's calculation of the slope < 1 -part of the rigid cohomology with compact support via Witt sheaves with compact support ([4]).

Let $K_0(\mathbb{F}_q)$ be the fraction field of the Witt ring $\mathcal{W}(\mathbb{F}_q)$ of $\mathbb{F}_q$. Let Y be a separated scheme of finite type over $\mathbb{F}_q$ of dimension d . Let $F_q : Y \rightarrow Y$ be the q -th power Frobenius endomorphism of Y . The following is Étess-Le Stum's Lefschetz trace formula proved in [17, Théorème II]:

$$\# Y(\mathbb{F}_q) = \sum_{i=0}^{2d} (-1)^i \text{Tr}(F_q^* | H_{\text{rig}, c}^i(Y / K_0(\mathbb{F}_q))). \quad (2.0.1)$$

Let $\{\alpha_{ij}\}_j$ be the set of the eigenvalues of F_q^* on $H_{\text{rig}, c}^i(Y / K_0(\mathbb{F}_q))$. Then

$$\# Y(\mathbb{F}_q) = \sum_{i=0}^{2d} (-1)^i \left(\sum_j \alpha_{ij} \right). \quad (2.0.2)$$

By [10, (3.1.2)] (see also [49, (17.2)]),

$$\max\{0, i - d\} \leq \text{ord}_q(\alpha_{ij}) \leq \min\{i, d\}. \quad (2.0.3)$$

Henceforth we consider the equalities (2.0.1) and (2.0.2) as the equalities in the integer ring $\overline{\mathcal{W}(\mathbb{F}_q)}$ of an algebraic closure $\overline{K_0(\mathbb{F}_q)}$ of $K_0(\mathbb{F}_q)$.

Let κ be a perfect field of characteristic $p > 0$. Let Y/κ be a separated scheme of finite type. Let K_0 be the fraction field of the Witt ring $\mathcal{W}$ of κ . Let $H_{\text{rig},c}^i(Y/K_0)_{[0,1]}$ be the slope < 1 -part of the rigid cohomology $H_{\text{rig},c}^i(Y/K_0)$ with compact support with respect to the absolute Frobenius endomorphism of Y . Let $H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0}))$ be the cohomology of the Witt sheaf with compact support of Y/K_0 defined by Berthelot et al. in [4]:

$$H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0})) := H^i(Y, \mathcal{W}(\mathcal{I}_{K_0})),$$

where $\mathcal{W}(\mathcal{I}_{K_0}) := \text{Ker}(\mathcal{W}(\mathcal{O}_Z)_{K_0} \rightarrow \mathcal{W}(\mathcal{O}_Z/\mathcal{I})_{K_0})$ and $\mathcal{I}$ is a coherent ideal sheaf of $\mathcal{O}_Z$ for an open immersion $Y \hookrightarrow Z$ into a proper scheme over κ such that $V(\mathcal{I}) = Z \setminus Y$. They have proved that $H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0}))$ is independent of the choice of the closed immersion. By the definition of $H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0}))$, we have the following exact sequence:

$$\begin{aligned} H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0})) &\rightarrow H^i(Z, \mathcal{W}(\mathcal{O}_{Z,K_0})) \rightarrow H^i(Z, \mathcal{W}(\mathcal{O}_Z/\mathcal{I})_{K_0}) \\ &\rightarrow H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0})) \rightarrow \cdots \end{aligned} \quad (2.0.4)$$

By replacing Z by the closure of Y in Z , we see that

$$H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0})) = 0 \quad (2.0.5)$$

if $i > d$. Then they have proved that there exists the following contravariantly functorial isomorphism

$$H_{\text{rig},c}^i(Y/K_0)_{[0,1]} \xrightarrow{\sim} H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0})) \quad (2.0.6)$$

([4, Theorem 1.1]).

Now let us come back to the case $\kappa = \mathbb{F}_q$. Since

$$\begin{aligned}
H_{\text{rig},c}^i(Y / K_0(\mathbb{F}_q)) &= \sum_{j=0}^{d-1} H_{\text{rig},c}^i(Y / K_0(\mathbb{F}_q))_{[j,j+1)} \oplus H_{\text{rig},c}^i(Y / K_0(\mathbb{F}_q))_{[d]}, \\
\# Y(\mathbb{F}_q) &= \sum_{i=0}^{2d} (-1)^i \sum_{j=0}^{d-1} \text{Tr}(F_q^* | H_{\text{rig},c}^i(Y / K_0(\mathbb{F}_q))_{[j,j+1)}) \\
&\quad + \sum_{i=d}^{2d} (-1)^i \text{Tr}(F_q^* | H_{\text{rig},c}^i(Y / K_0(\mathbb{F}_q))_{[d]}). \tag{2.0.7}
\end{aligned}$$

Hence we have the following congruence by (2.0.5) and (2.0.6):

$$\# Y(\mathbb{F}_q) \equiv \sum_{i=0}^d (-1)^i \text{Tr}(F_q^* | H_c^i(Y, \mathcal{W}(\mathcal{O}_{Y,K_0}))) \pmod{q} \tag{2.0.8}$$

in $\overline{\mathcal{W}(\mathbb{F}_q)}$.

Remark 2.1. In [4, (1.4)], the following zeta function:

$$Z^{\mathcal{W}}(Y / \mathbb{F}_q, t) := \prod_{i=0}^d \det(1 - tF_q^* | H_c^i(Y, \mathcal{W}(\mathcal{O}_Y))_{K_0})^{(-1)^{i+1}},$$

which is equal to the zeta function

$$Z^{<1}(Y / \mathbb{F}_q, t) := \prod_{\text{ord}_q(\alpha_{ij}) < 1} (1 - \alpha_{ij}t)^{(-1)^{i+1}},$$

has been considered. In this article we do not need this zeta function. We do not need Ax's theorem in [1] (see [4, Proposition 6.3]) either.

3. Proofs of Theorem 1.5 and Corollary 1.6

It is well-known that the analogue of Kodaira's vanishing theorem for projective smooth schemes over a field of characteristic 0 ([35]) do not hold in characteristic $p > 0$ in general ([57]). However, in [55], we have

proved the Kodaira vanishing theorem in characteristic $p > 0$ under the assumption of the finiteness of the Yobuko height. To state this theorem precisely, we recall the definition of the vertical type for a relative log scheme.

For a commutative monoid P with unit element, an ideal is, by definition, a subset I of P such that $PI \subset I$. An ideal $\mathfrak{p}$ of P is called a prime ideal if $P \setminus \mathfrak{p}$ is a submonoid of P ([32, (5.1)]). For a prime ideal $\mathfrak{p}$ of P , the height $ht(\mathfrak{p})$ is the maximal length of sequence's $\mathfrak{p} \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r$ of prime ideals of P . Let $h : Q \rightarrow P$ be a morphism of monoids. A prime ideal $\mathfrak{p}$ of P is said to be horizontal with respect to h if $h(Q) \subset P \setminus \mathfrak{p}$ ([66, (2.4)]).

Let $Y \rightarrow Z$ be a morphism of fs(=fine and saturated) log schemes. Let $h : Q \rightarrow P$ be a local chart of g such that P and Q are saturated. Set $I := \{a \in P \mid a \in \mathfrak{p} \text{ for any horizontal prime ideal of } P \text{ of height 1 with respect to } h\}$.

Let $\mathcal{I}_{Y/Z}$ be the ideal sheaf of M_Y generated by $\text{Im}(I \rightarrow M_Y)$. In [66, (2.6)], Tsuji has proved that $\mathcal{I}_{Y/Z}$ is independent of the choice of the local chart h . Let $\mathcal{I}_{Y/Z}\mathcal{O}_Y$ be the ideal sheaf of $\mathcal{O}_Y$ generated by the image of $\mathcal{I}_{Y/Z}$.

Definition 3.1. We say that Y/Z is of vertical type if $\mathcal{I}_{Y/Z}\mathcal{O}_Y = \mathcal{O}_Y$.

In [55, (1.9)] we have proved the following theorem:

Theorem 3.2 (Log Kodaira Vanishing theorem). *Let $Y \rightarrow s$ be a projective log smooth morphism of Cartier type of fs log schemes. Assume that $\overset{\circ}{Y}$ is of pure dimension d . Let $\mathcal{L}$ be an ample invertible sheaf on $\overset{\circ}{Y}$. Assume that $h^F(\overset{\circ}{Y}) < \infty$. Then $H^i(Y, \mathcal{I}_{Y/s}\omega_{Y/s} \otimes_{\mathcal{O}_Y} \mathcal{L}) = 0$ for $i > 0$. In particular, if Y/s is of vertical type, then $H^i(Y, \omega_{Y/s} \otimes_{\mathcal{O}_Y} \mathcal{L}) = 0$ for $i > 0$.*

Now let us prove Theorem 1.5 and Corollary 1.6 quickly. Let the notations be as in Corollary 1.6. Since $\omega_{X/s}^{-1}$ is ample, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ by Theorem 3.2. Hence, by the following exact sequence

$$0 \rightarrow \mathcal{W}_{n-1}(\mathcal{O}_X) \xrightarrow{V} \mathcal{W}_n(\mathcal{O}_X) \rightarrow \mathcal{O}_X \rightarrow 0, \quad (3.2.1)$$

$H^i(X, \mathcal{W}_n(\mathcal{O}_X)) = 0$ for $i > 0$ and $n > 0$. Hence

$$H^i(X, \mathcal{W}(\mathcal{O}_X)) = \lim_{\leftarrow n} H^i(X, \mathcal{W}_n(\mathcal{O}_X)) = 0. \quad (3.2.2)$$

Thus we have proved Theorem 1.5.

Next let us prove Corollary 1.6. It suffices to prove Corollary 1.6 for the case $k = 1$ by considering the base change $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^k}$. Because $H^0(X, \mathcal{W}(\mathcal{O}_X)) = \mathcal{W}(\mathbb{F}_q)$ and $F_q^* = \text{id}$ on $H^0(X, \mathcal{W}(\mathcal{O}_X))$, we obtain the following by (2.0.8):

$$\# \overset{\circ}{X}(\mathbb{F}_q) \equiv 1 \pmod{q} \quad (3.2.3)$$

in $\overline{\mathcal{W}(\mathbb{F}_q)}$. This shows Corollary 1.6.

Remark 3.3. (1) If X is a Fano variety over $\mathbb{Q}$, then the reduction $\mathcal{X} \bmod p$ of a flat model $\mathcal{X}$ over $\mathbb{Z}$ of X for $p \gg 0$ is a Fano variety and F -split ([6, Exercise 1.6. E5]). In particular, $h^F(\mathcal{X} \bmod p) < \infty$ for $p \gg 0$.

(2) As pointed out in [6, p. 58], a Fano variety X is not necessarily F -split.

The Kodaira vanishing theorem does not hold for certain Fano varieties ([27, 39, 65]). By Theorem 3.2 we see that the Yobuko heights of them are infinity.

(3) Let X/s be an SNCL Fano scheme of pure dimension d . Then any irreducible component of $\overset{\circ}{X}_i$ of $\overset{\circ}{X}$ is Fano. Indeed, since $\omega_{X/s}^{-1}$ is ample, $\omega_{X/s}^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i} = \omega_{X_i/\kappa}^{-1} (-\sum_j \log D_j)$ is also ample. Here $\{D_j\}_j$ is the set of the double varieties in $\overset{\circ}{X}_i$. Hence $-K_{X_i} - \sum_j D_j$ is ample. Consequently $-K_{X_i}$ is ample.

Remark 3.4. Let $X/\mathbb{F}_q$ be a separated scheme of finite type. Assume that X is geometrically connected. By the argument in this section, it is obvious that, if $H^i(X, \mathcal{O}_X) = 0 (\forall i > 0)$, then the congruence (1.6.1) holds for $X/\mathbb{F}_q$. In particular, if $d = 2$, if $X/\mathbb{F}_q$ is smooth, if $H^1(X, \mathcal{O}_X) = 0$ and if $H^0(X, \Omega_{X/s}^2) = 0$, then the congruence (1.6.1) holds for $X/\mathbb{F}_q$. Such an example can be given by a proper smooth Godeaux surface.

Other examples are given by proper smooth unirational threefolds because $H^i(X, \mathcal{O}_X) = 0 (\forall i > 0)$ by [53, Introduction, (2.5)].

Let X/s be an SNCL(=simple normal crossing log) classical Enriques surface X/s for $p \neq 2$, i.e., $(\Omega_{X/s}^2)^{\otimes 2}$ is trivial and the corresponding étale covering X' to $\Omega_{X/s}^2$ is an SNCL K3 surface. (In [46, (7.1)] we have proved that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.) Hence the congruence (1.6.1) also holds for $X/s_{\mathbb{F}_q}$. See Corollary 5.4 below for the zeta function of this example. By the formulas for the zeta function ((5.4.1), (5.4.2)), we can easily verify that $\#\overset{\circ}{X}(\mathbb{F}_q)$ indeed satisfies the congruence (1.6.1).

More generally, if $H^i(X, \mathcal{W}(\mathcal{O}_X))_{K_0} = 0$ ($i > 0$), then the congruence (1.6.1) holds for $X/\mathbb{F}_q$ by the proof of Theorem 1.5. By the main theorem of [4], one obtains such examples which are special fibers of regular proper flat schemes over discrete valuation rings of mixed characteristics whose generic fibers are geometrically connected and of Hodge type ≥ 1 in positive degrees. See also [15] for a generalization of the main theorem in [4].

Example 3.5. Let n and N be positive integers. Set $\mathcal{X}_1 := \mathbb{P}_{\mathcal{W}(\kappa)}^N$. Blow up $\mathcal{X}_1$ along an κ -rational hyperplane of $\mathbb{P}_{\kappa}^N$ and let $\mathcal{X}_2$ be the resulting scheme. Let $\overset{\circ}{X}_1$ and $\overset{\circ}{X}_n$ be the irreducible components of the special fiber $\mathcal{X}_2$. Blow up $\mathcal{X}_2$ again along $\overset{\circ}{X}_1 \cap \overset{\circ}{X}_n$ and let $\mathcal{X}_3$ be the resulting scheme. Let $\overset{\circ}{X}_1, \overset{\circ}{X}_n$, and $\overset{\circ}{X}_{n-1}$ be the irreducible components of the special fiber $\mathcal{X}_3$. Blow up $\mathcal{X}_3$ again along $\overset{\circ}{X}_1 \cap \overset{\circ}{X}_{n-1}$. Continuing this process $(n-1)$ -times, we have a projective semistable family $\mathcal{X}_n$ over $\text{Spec}(\mathcal{W}(\kappa))$. Let $\overset{\circ}{X}_i$ ($1 \leq i \leq n$) be the irreducible components of the special fiber $\mathcal{X}_n$. Let X be the log special fiber of $\mathcal{X}_n$. Then X is a projective SNCL scheme over s . Let $\overset{\circ}{X}^{(i)}$ ($i = 0, 1$) be the disjoint union of $(i+1)$ -fold intersections of the irreducible components of $\overset{\circ}{X}$.

Then $\overset{\circ}{X}^{(0)} = \mathbb{P}_{\kappa}^N \underbrace{\coprod (\mathbb{P}_{\kappa}^{N-1} \times_{\kappa} \mathbb{P}_{\kappa}^1) \cdots \coprod (\mathbb{P}_{\kappa}^{N-1} \times_{\kappa} \mathbb{P}_{\kappa}^1)}_{(n-1)\text{-times}}$ and $\overset{\circ}{X}^{(1)} = \mathbb{P}_{\kappa}^{N-1} \underbrace{\coprod \cdots \coprod \mathbb{P}_{\kappa}^{N-1}}_{(n-1)\text{-times}}$. Using the following spectral sequence:

$$E_1^{ij} = H^j(X^{(i)}, \mathcal{O}_{X^{(i)}}) \Rightarrow H^{i+j}(X, \mathcal{O}_X), \quad (3.5.1)$$

and noting that the dual graph of $\overset{\circ}{X}$ is a segment, we see that $H^i(X, \mathcal{O}_X) = 0$ ($i > 0$). If $s = s_{\mathbb{F}_q}$, then it is easy to check that

$$\begin{aligned} \overset{\circ}{X}(\mathbb{F}_q) &= \frac{q^{N+1} - 1}{q - 1} + (n - 1) \frac{q^N - 1}{q - 1} \frac{q^2 - 1}{q - 1} - (n - 1) \frac{q^N - 1}{q - 1} \\ &= \frac{q^{N+1} - 1}{q - 1} + (n - 1)q \frac{q^N - 1}{q - 1}. \end{aligned}$$

In particular, $\# \overset{\circ}{X}(\mathbb{F}_q) \equiv 1 \pmod{q}$.

The restriction of $\omega_{X/s}$ to $\overset{\circ}{X}_i$ is isomorphic to $\mathcal{O}_{\overset{\circ}{X}_i}(-N)$ for $i = 0, N$ and $\mathcal{O}_{\overset{\circ}{X}_i}(-(N - 1))$ for $0 < i < N$. Hence $\omega_{X/s}^{-1}$ is ample if $N \geq 2$. Since each $\overset{\circ}{X}_i$ is F -split (the F -splitting is given by the “ p^{-1} -th power” of the canonical coordinate of $\overset{\circ}{X}_i$ (see [6, (1.1.5)])) and because we have the following exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{i=1}^N \mathcal{O}_{X_i} \rightarrow \bigoplus_{i=1}^{N-1} \mathcal{O}_{X_i \cap X_{i+1}},$$

$\overset{\circ}{X}$ is F -split.

4. Proof of Theorem 1.7

Let the notations be as in Theorem 1.7. In this section we prove Theorem 1.7. We may assume that $k = 1$.

Since $H^i(Y, \mathcal{O}_Y) = 0$, we see that

$$H^i(Y, \mathcal{W}(\mathcal{O}_Y)) = \lim_{\leftarrow n} H^i(Y, \mathcal{W}_n(\mathcal{O}_Y)) = 0 \quad (4.0.1)$$

by the same proof as that of Theorem 1.5. Set $\bar{Y} := Y \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and $e := \log_p q$. By [2, II (4.3)], the Dieudonné module $M := D(\Phi_{Y/\kappa})$ of $\Phi_{Y/\kappa}$ is equal to $H^d(Y, \mathcal{W}(\mathcal{O}_Y))$. Let h be the height of $\Phi_{Y/\kappa}$. Hence $\Phi_{Y/\kappa}$ is a commutative formal Lie group over κ of dimension 1 and the Dieudonné module M is a free $\mathcal{W}$ -module of rank h if $h < \infty$ ([25, V (28.3.10)]). Let $F : M \rightarrow M$ be the operator “ F ” on the Dieudonné module M . By abuse of notation, we denote the induced morphism $M_{K_0} \rightarrow M_{K_0}$ by F . By (2.0.8), we have the following congruence:

$$\# Y(\mathbb{F}_q) \equiv 1 + \text{Tr}(F^e | M_{K_0}) \pmod{q} \quad (4.0.2)$$

in $\overline{\mathcal{W}(\mathbb{F}_q)}$. Set $m := \text{ord}_p(\text{Tr}(F^e | M_{K_0}))$. If $m \leq e = \text{ord}_p(q)$, then we obtain the following congruence by (4.0.2):

$$\# Y(\mathbb{F}_q) \equiv 1 \pmod{p^{\lceil m \rceil}} \quad (4.0.3)$$

in $\mathbb{Z}$.

First we give the proof of Theorem 1.7 (1).

Proof of Theorem 1.7 (1)

Assume that $h = \infty$. Then $D(\Phi_{\bar{Y}/\bar{\mathbb{F}}_q})$ is $\mathcal{W}(\bar{\mathbb{F}}_q)$ -torsion. By [2, II (4.3)], $H^d(\bar{Y}, \mathcal{W}(\mathcal{O}_{\bar{Y}}))_{K_0} = D(\Phi_{\bar{Y}/\bar{\mathbb{F}}_q})_{K_0} = 0$. By [28, I (1.9.2)], $\mathcal{W}(\mathcal{O}_{\bar{Y}}) = \mathcal{W}(\mathcal{O}_Y) \otimes_{\mathcal{W}(\mathbb{F}_q)} \mathcal{W}(\bar{\mathbb{F}}_q)$. Since $\overset{\circ}{Y}$ is separated, we obtain the following equality $H^d(Y, \mathcal{W}(\mathcal{O}_{\bar{Y}})) = H^d(Y, \mathcal{W}(\mathcal{O}_Y)) \otimes_{\mathcal{W}(\mathbb{F}_q)} \mathcal{W}(\bar{\mathbb{F}}_q)$ by using Čech cohomologies. Hence

$$H^d(Y, \mathcal{W}(\mathcal{O}_Y))_{K_0(\mathbb{F}_q)} = 0.$$

(To obtain this vanishing, one may use the fact that the Dieudonné module commutes with base change (cf. the description of $D(\Phi_{Y/\mathbb{F}_q})$ in [43, p. 309]).) By (2.0.8) this means the congruence (1.7.1).

Now assume that $h < \infty$. Next we give the proof Theorem 1.7 (2).

Proof of Theorem 1.7 (2)

Let us recall the following well-known observation ([38, Exercise 6.13]):

Proposition 4.1. *Let G be a commutative formal Lie group of dimension 1 over a perfect field κ of characteristic $p > 0$. Assume that the height h of G is finite. Then the slopes of the Dieudonné module of $D(G)$ is $1 - h^{-1}$.*

Proof. Let $D(\kappa)$ be the Cartier-Dieudonné algebra over κ . We may assume that κ is algebraically closed. In this case, the height is the only invariant which determines the isomorphism class of a 1-dimensional commutative formal group law over κ ([25, (19.4.1)]). Hence $D(G) \simeq D(\kappa)/D(\kappa)(F - V^{h-1})$ ([67, p. 266]). Express $F(1, V, \dots, V^{h-1}) = (1, V, \dots, V^{h-1})A$, where $A \in M_h(\mathcal{W})$ (as if F were $\mathcal{W}$ -linear). Then $\det(tI - A) = t^h - p^{h-1}$. Hence the slopes of $D(G)$ is $\text{ord}_p((p^{h-1})^{h-1}) = 1 - h^{-1}$.

By Proposition 4.1 and (4.0.3), we obtain the following congruence:

$$\# Y(\mathbb{F}_q) \equiv 1 \pmod{p^{\lceil e(1-h^{-1}) \rceil}} \quad (4.1.1)$$

in $\mathbb{Z}$.

Lastly we give the proof of Theorem 1.7 (3) in the following.

Proof of Theorem 1.7 (3)

Let κ be a perfect field of characteristic $p > 0$. Let Y be a proper scheme over κ of pure dimension $d \geq 1$. (We do not assume that Y is smooth over κ .) Assume that $H^d(Y, \mathcal{O}_Y) \simeq \kappa$ and that $H^{d-1}(Y, \mathcal{O}_Y) = 0$ if $d \geq 2$. Then the following morphism:

$$H^d(Y, \mathcal{W}(\mathcal{O}_Y)) / p \rightarrow H^d(Y, \mathcal{O}_Y)$$

is an isomorphism. Indeed, this is surjective and

$$\dim_{\kappa}(H^d(Y, \mathcal{W}(\mathcal{O}_Y)) / p) = \dim_{\kappa}(M / p) = 1 = \dim_{\kappa} H^d(Y, \mathcal{O}_Y).$$

Since $h = 1$, F on $H^d(Y, \mathcal{W}(\mathcal{O}_Y)) \otimes_{\mathcal{W}(\kappa)} \mathcal{W}(\bar{\kappa})$ is an isomorphism.

Hence $F : H^d(Y, \mathcal{O}_Y) \rightarrow H^d(Y, \mathcal{O}_Y)$ is an isomorphism. Hence $\# Y(\mathbb{F}_q) \equiv 1 + \alpha \pmod{q}$ for a unit $\alpha \in \mathcal{W}(\mathbb{F}_q)^*$. Now Theorem 1.7 (3) follows.

Remark 4.2. (1) If $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < d - 1$ (this is stronger than (b) and (c) in the Introduction), then Theorem 1.7 (3) also follows from Fulton's trace formula ([19]):

$$\# Y(\mathbb{F}_q) \pmod{p} \equiv \sum_{i=0}^d (-1)^i \text{Tr}(F_q^* | H^i(Y, \mathcal{O}_Y)) \in \mathbb{F}_q$$

(cf. [3, Proposition 5.6]).

(2) Let X / s be a log Calabi-Yau scheme. In [55, (10.1)] we have proved a fundamental equality $h^F(X / \kappa) = h(X / \kappa)$. Hence X is quasi- F -split (resp., F -split) if and only if $h(X / \kappa) < \infty$ (resp., $h(X / \kappa) = 1$).

Though the following corollary immediately follows from [4, (1.6)], we state it for the convenience of our remembrance.

Corollary 4.3. *Let Y be as in Theorem 1.7. Let $f : Z_1 \rightarrow Z_2$ be a morphism of proper schemes over $\mathbb{F}_q$. Assume that Z_1 or Z_2 is isomorphic to Y over $\mathbb{F}_q$. Assume that $\Phi_{Z_i/\mathbb{F}_q}(i = 1, 2)$ is representable. If the pull-back $f^* : H^i(Z_2, \mathcal{O}_{Z_2}) \rightarrow H^i(Z_1, \mathcal{O}_{Z_1})$ is an isomorphism, then the natural morphism $\Phi_{Z_2/\mathbb{F}_q} \rightarrow \Phi_{Z_1/\mathbb{F}_q}$ is an isomorphism. In particular, $h(Z_1/\mathbb{F}_q) = h(Z_2/\mathbb{F}_q)$ and Theorem 1.7 for $\# Z_i(\mathbb{F}_q)$ holds.*

Proof. By the assumption, we have an isomorphism $f^* : H^i(Z_2, \mathcal{W}(\mathcal{O}_{Z_2})) \xrightarrow{\sim} H^i(Z_1, \mathcal{W}(\mathcal{O}_{Z_1}))$. Hence the natural morphism $D(\Phi_{Z_2/\mathbb{F}_q}) \rightarrow D(\Phi_{Z_1/\mathbb{F}_q})$ is an isomorphism. By Cartier theory, the natural morphism $\Phi_{Z_2/\mathbb{F}_q} \rightarrow \Phi_{Z_1/\mathbb{F}_q}$ is an isomorphism. This implies that $h(Z_2/\mathbb{F}_q) = h(Z_1/\mathbb{F}_q)$ and Theorem 1.7 for $\# Z_i(\mathbb{F}_q)$ holds. $\square$

The following corollary immediately follows from the proof of [4, (6.12)].

Corollary 4.4. *Let Y be as in (1.7). Let G be a finite group acting on $Y/\mathbb{F}_q$ such that each orbit of G is contained in an affine open subscheme of Y . If $\#G$ is prime to p and the induced action on $H^d(Y, \mathcal{O}_Y)$ is trivial, then $h((Y/G)/\mathbb{F}_q) = h(Y/\mathbb{F}_q)$ and Theorem 1.7 for $\#(Y/G)(\mathbb{F}_q)$ holds.*

Example 4.5. We give examples of the Theorem 1.7 in trivial logarithmic cases.

(1) Let $E/\mathbb{F}_p$ be an elliptic curve. It is very well-known that $E/\mathbb{F}_p$ is supersingular if and only if $\#E(\mathbb{F}_p) = p + 1$ if $p \geq 5$ ([60, V Exercises 5.9]). As observed in [3, Example 5.11], this also follows from the purity of the weight for an elliptic curve over $\mathbb{F}_p : |\#E(\mathbb{F}_p) - (p + 1)| \leq 2\sqrt{p}$ and Fulton's trace formula. In fact, we can say more in Proposition 4.8 below.

(2) Let $d \geq 3$ be a positive integer such that $d \not\equiv 0 \pmod{p}$. Consider a smooth Calabi-Yau variety $\mathcal{X} / \mathcal{W}(\mathbb{F}_q)$ in $\mathbb{P}_{\mathcal{W}(\mathbb{F}_q)}^{d-1}$ defined by the following equation:

$$a_0 T_0^d + \cdots + a_{d-1} T_{d-1}^d = 0 \quad (a_0, \dots, a_{d-1} \in \mathcal{W}(\mathbb{F}_q)^*).$$

Set $a := a_0 \cdots a_{d-1} \in \mathcal{W}(\mathbb{F}_q)$. Let $X / \mathbb{F}_q$ be the reduction mod p of $\mathcal{X} / \mathcal{W}(\mathbb{F}_q)$. By [62, Theorem 1] (see also [loc. cit., Example 4.13]), the logarithm $l(t)$ of $\Phi_{\mathcal{X}/\mathcal{W}(\mathbb{F}_q)}$ is given by the following formula:

$$l(t) = \sum_{m=0}^{\infty} a^m \frac{(md)!}{(m!)^d} \frac{t^{md+1}}{md+1}.$$

(a) If $p \equiv 1 \pmod{d}$, then

$$pl(t) = pt + p \sum_{i=2}^{p-1} c_i t^i + (\text{unit}) t^p + (\text{higher terms than } t^p),$$

for some $c_i \in \mathcal{W}(\mathbb{F}_q)$ in $\mathcal{W}(\mathbb{F}_q)[[t]]$. Hence $l^{-1}(pl(t)) \pmod{p} \equiv t^p + \cdots$ and the height of $\Phi_{X/\mathbb{F}_q}$ is equal to 1.

(b) If $p \not\equiv 1 \pmod{d}$, then $pl(t) \in p\mathcal{W}(\mathbb{F}_q)[[t]]$. Hence the height of $\Phi_{X/\mathbb{F}_q}$ is equal to ∞ .

(a) and (b) above are much easier and much more direct proofs of [68, Theorem 5.1].

(3) Especially consider the case $d = 4$ in (2) and let $X / \mathbb{F}_p$ be a closed subscheme of $\mathbb{P}_{\mathbb{F}_p}^3$ defined by the following equation:

$$T_0^4 + T_1^4 + T_2^4 + T_3^4 = 0.$$

(a) If $p = 3$, then $\# X(\mathbb{F}_3) \equiv 1 \pmod{3}$ by Theorem 1.7 (1). In fact, it is easy to see that $\# X(\mathbb{F}_3) = 4 = 1 + 3^2 - 3 \times 2$. (This X and X in (c) are Tate's examples in [63] of a supersingular K3-surface (in the sense of Shioda) over $\mathbb{F}_3$ and $\mathbb{F}_7$), respectively.)

(b) If $p = 5$, then $\# X(\mathbb{F}_5) \not\equiv 1 \pmod{5}$ by Theorem 1.7 (3). In fact, it is easy to see that $\# X(\mathbb{F}_5) = 0$. More generally, for a power q of a prime number p , let X_q be a closed subscheme of $\mathbb{P}_{\mathbb{F}_q}^{q-1}$ defined by the following equation:

$$a_0 T_0^{q-1} + \dots + a_{q-2} T_{q-2}^{q-1} = 0 \quad (a_0, \dots, a_{q-2} \in \mathbb{F}_q^*, (a_0, \dots, a_{q-2}) \neq (0, \dots, 0)),$$

where $a_0, \dots, a_{q-2}$ satisfying the following condition: for any nonempty set I of $\{0, \dots, q-2\}$, $\sum_{j \in I} a_j \neq 0$ in $\mathbb{F}_q$. Then $\# X_q(\mathbb{F}_q) = 0$.

(c) If $p = 7$, then $\# X(\mathbb{F}_7) \equiv 1 \pmod{7}$ by Theorem 1.7 (1). In fact, one can check that $\# X(\mathbb{F}_7) = 64 = 1 + 7^2 + 7 \times 2$. In general, if $\Phi(X/\mathbb{F}_q)$ is supersingular, then $\# X(\mathbb{F}_q) = 1 + q^2 + q\alpha$ for some $|\alpha| \leq 22$ by the purity of the weight and by $b_2(\overline{X}/\overline{\mathbb{F}_q}) = 22$. Here $b_2(\overline{X}/\overline{\mathbb{F}_q})$ is the second Betti number of $\overline{X}/\overline{\mathbb{F}_q}$. (We do not know an example of the big $|\alpha|$.)

(4) See [70, (4.8)] for explicit examples of $X/\mathbb{F}_q$'s such that $h(\Phi_{X/\mathbb{F}_q}) = 2$. See also [68, Section 6].

Example 4.6. (1) Let n be a positive integer. Let X be an n -gon over $\mathbb{F}_q$. Then, by [50, (6.7) (1)], X is F -split. In particular, $h^F(X) = h(X/\mathbb{F}_q) = 1$. Then, by Theorem 1.7 (3), $\# X(\mathbb{F}_q) \not\equiv 1 \pmod{p}$. In fact, it is easy to see that $\# X(\mathbb{F}_q) = n(q+1) - n = nq$. Compare this example with the Example 3.5.

(2) Let κ be a perfect field of characteristic $p > 0$. Let X be an SNCL(= simple normal crossing log) K3-surface over κ , that is, an SNCL Calabi-Yau variety of dimension 2 ([46]). In [50, (6.7) (2)] we have proved the following:

(a) If $\overset{\circ}{X}$ is of Type II ([46, Section 3]), then X is F -split if and only if the isomorphic double elliptic curve is ordinary. In this case, $h(X / \kappa) = 1$. If this is not the case, $h(X / \kappa) = 2$.

(b) If $\overset{\circ}{X}$ is of Type III ([loc. cit.]), then X is F -split and $h(X / \kappa) = 1$.

See Corollary 5.2 below for the zeta function of these examples. By the formulas for the zeta functions ((5.2.1) and (5.2.2)), we can easily verify that $\# \overset{\circ}{X}(\mathbb{F}_q)$ indeed satisfies the congruences (1.7.3) and (1.7.2).

Remark 4.7. (1) Let $X / \mathbb{F}_q$ and $X^* / \mathbb{F}_q$ be a strong mirror Calabi-Yau pair in the sense of Wan ([72]), whose strict definition has not been given. Then he conjectures that $\# X(\mathbb{F}_q) \equiv \# X^*(\mathbb{F}_q) \pmod{q}$ ([72, (1.3)]). Hence the following question seems natural: does the equality $h(\Phi_{X/\mathbb{F}_q}) = h(\Phi_{X^*/\mathbb{F}_q})$ hold? If his conjecture is true, only one of $h(\Phi_{X/\mathbb{F}_q})$ and $h(\Phi_{X^*/\mathbb{F}_q})$ cannot be 1 by Theorem 1.7. This is compatible with Wan's generically ordinary conjecture in [loc. cit., (8.3)].

(2) If X satisfies the conditions (a), (c) and (d) in the Introduction and if X is a special fiber of a regular proper flat scheme over a discrete valuation ring of mixed characteristics whose generic fibers are geometrically connected and of Hodge type ≥ 1 in degrees in $[1, d - 2]$, then we see that X satisfies the condition (b) in the Introduction by [4].

We conclude this section by generalizing Example (4.5) (1) by using Theorem 1.7 and raise an important question:

Proposition 4.8. *Let C be a proper smooth curve over $\mathbb{F}_q$ such that $H^0(C, \mathcal{O}_C) \simeq \mathbb{F}_q \simeq H^1(C, \mathcal{O}_C)$. Recall that $e = \log_p q$. Then the following hold:*

(1) *Assume that e is odd and $p \geq 5$. Then $h_{C/\mathbb{F}_q} = 2$ if and only if $\#C(\mathbb{F}_q) = 1 + q$.*

(2) *Assume that e is odd and $p = 3$ or 2 . Then $h_{C/\mathbb{F}_q} = 2$ if and only if $\#C(\mathbb{F}_q) = 1 + q$ or $1 + q \pm p^{\frac{e+1}{2}}$.*

(3) *Assume that e is even. Then $h_{C/\mathbb{F}_q} = 2$ if and only if $\#C(\mathbb{F}_q) = 1 + q + \alpha p^{\frac{e}{2}}$, where $\alpha \in \mathbb{N}$ and $|\alpha| \leq 2$.*

Proof. By the purity of weight, we have the following inequality:

$$|\#C(\mathbb{F}_q) - (1 + q)| \leq 2\sqrt{q}. \quad (4.8.1)$$

(1) Assume that $h_{C/\mathbb{F}_q} = 2$. By (1.7.2), $\#C(\mathbb{F}_q) \equiv 1 \pmod{p^{\lceil \frac{e}{2} \rceil}} = 1 \pmod{p^{\frac{e+1}{2}}}$.

Hence $\#C(\mathbb{F}_q) = 1 + mp^{\frac{e+1}{2}}$ for $m \in \mathbb{N}$. By (4.8.1) we have the following inequality:

$$p^{\frac{1}{2}} \left| m - p^{\frac{e-1}{2}} \right| \leq 2. \quad (4.8.2)$$

Since $p \geq 5$, $m = p^{\frac{e-1}{2}}$. Hence $\#C(\mathbb{F}_q) = 1 + q$.

Conversely, assume that $\#C(\mathbb{F}_q) = 1 + q$. Then C can be an elliptic curve over $\mathbb{F}_q$. Hence $h_{C/\mathbb{F}_q} = 1$ or 2 ([60, IV (7.5)]). By (1.7.3) and (1.7.2), $h_{C/\mathbb{F}_q} = 2$.

(2) Assume that $h_{C/\mathbb{F}_q} = 2$. Then, by (4.8.2), $m = p^{\frac{e-1}{2}}$ or $m = \pm 1 + p^{\frac{e-1}{2}}$. Hence $\#C(\mathbb{F}_q) = 1 + q$ or $\#C(\mathbb{F}_q) = 1 + (\pm 1 + p^{\frac{e-1}{2}})p^{\frac{e+1}{2}} = 1 + q \pm p^{\frac{e+1}{2}}$.

The proof of the converse implication is the same as that in (1).

(3) Assume that $h_{C/\mathbb{F}_q} = 2$. By (1.7.2), $\#C(\mathbb{F}_q) \equiv 1 \pmod{p^{\frac{e}{2}}}$. Hence $\#C(\mathbb{F}_q) = 1 + mp^{\frac{e}{2}}$ for $m \in \mathbb{N}$. By (4.8.1),

$$|m - p^{\frac{e}{2}}| \leq 2. \quad (4.8.3)$$

Hence, by (4.8.1), $m = \alpha + p^{\frac{e}{2}}$ with $|\alpha| \leq 2$. Hence $\#C(\mathbb{F}_q) = 1 + (\alpha + p^{\frac{e}{2}})p^{\frac{e}{2}} = 1 + q + \alpha p^{\frac{e}{2}}$.

The proof of the converse implication is the same as that in (1). $\square$

Remark 4.9. Assume that e is even. By Honda-Tate's theorem for elliptic curves over finite fields (Theorem 4.10 below), the case $|\alpha| = 1$ occurs only when $p \not\equiv 1 \pmod{3}$; the case $\alpha = 0$ occurs only when $p \not\equiv 1 \pmod{4}$.

Theorem 4.10 (Honda-Tate's theorem for elliptic curves ([73, (4.1)], [56, (4.8)])). *For an elliptic curve $E/\mathbb{F}_q$, set $t_E := 1 + q - \#E(\mathbb{F}_q)$. Consider the following well-defined injective map:*

$$\{\text{isogeny classes of elliptic curves } E/\mathbb{F}_q\} \ni E \rightarrow t_E \in \{t \in \mathbb{Z} \mid |t| \leq 2\sqrt{q}\}.$$

(This map is indeed injective by Tate's theorem ([64, Main Theorem]).) The image of this map consists of the following values:

- (1) t is coprime to p .

(2) e is even and $t = \pm 2\sqrt{q}$.

(3) e is even and $p \not\equiv 1 \pmod{3}$ and $t = \pm\sqrt{q}$.

(4) e is odd and $p = 2$ or 3 and $t = \pm p^{\frac{e+1}{2}}$.

(5) e is odd, or e is even and $p \not\equiv 1 \pmod{4}$ and $t = 0$.

The case (1) arises from ordinary elliptic curves over $\mathbb{F}_q$. The case (2) arises from supersingular elliptic curves over $\mathbb{F}_q$ having all their endomorphisms defined over $\mathbb{F}_q$; the rest cases arises from supersingular elliptic curves over $\mathbb{F}_q$ not having all their endomorphisms defined over $\mathbb{F}_q$.

Problem 4.11. Let K be an algebraic number field and $\mathcal{O}_K$ the integer ring of K . Let x be a positive real number.

(1) Consider the following set:

$$\mathcal{P}(x) := \{\mathfrak{p} \in \text{Spec}(\mathcal{O}_K) \mid N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x \text{ and } \log_p(\#(\mathcal{O}_K / \mathfrak{p})) \text{ is even}\},$$

where $p = \text{ch}(\mathcal{O}_K / \mathfrak{p})$.

Assume that $p \geq 5$. Let E / K be an elliptic curve. Let α be an integer such that $|\alpha| \leq 2$. Consider the following set:

$$\mathcal{P}'(x; E / K, \alpha) := \{\mathfrak{p} \in \mathcal{P}(x) \mid E \text{ has a good reduction } \mathcal{E}_0 \text{ at } \mathfrak{p}$$

$$\text{and } \# \mathcal{E}_0(\mathbb{F}_q) = 1 + q + \alpha\sqrt{q}\}.$$

Set

$$\mathcal{P}(x; E / K, \alpha) := \begin{cases} \mathcal{P}'(x; E / K, \alpha) & (|\alpha| = 2), \\ \{\mathfrak{p} \in \mathcal{P}(x; E / K, \alpha) \mid p \not\equiv 1 \pmod{3}\} & (|\alpha| = 1), \\ \{\mathfrak{p} \in \mathcal{P}(x; E / K, \alpha) \mid p \not\equiv 1 \pmod{4}\} & (\alpha = 0). \end{cases}$$

Then, what is the function

$$x \mapsto \frac{\# \mathcal{P}(x; E / K, \alpha)}{\# \mathcal{P}(x)}$$

when $x \rightarrow \infty$? (I do not know whether $\lim_{x \rightarrow \infty} \mathcal{P}(x; E / K, \alpha) = \infty$ for each α such that $|\alpha| \leq 2$ for any non-CM elliptic curve over K (see [61, p. 185, Exercise 2.33 (a), (b)] for a CM elliptic curve over $\mathbb{Q}(\sqrt{-1})$: in this example, $\lim_{x \rightarrow \infty} \mathcal{P}(x; E / K, 2) = \infty$, but $\mathcal{P}(x; E / K, \alpha) = 0$ for $\alpha \neq 2$ and for any x). If $[K : \mathbb{Q}]$ is odd or if K has a real embedding, then $\lim_{x \rightarrow \infty} \sum_{|\alpha| \leq 2} \mathcal{P}(x; E / K, \alpha) = \infty$ by Elkies' theorems ([13, Theorem 2], [14, Theorem]).)

When $p = 2$ or 3 , we can give a similar problem to the problem above by using Proposition 4.8 (2).

(2) Consider the following set:

$$\mathcal{P}(x) := \{\mathfrak{p} \in \text{Spec}(\mathcal{O}_K) \mid N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x\}.$$

Let S / K be a K3 surface. Let α be an integer such that $|\alpha| \leq 22$. Consider the following set:

$$\mathcal{P}'(x; S / K, \alpha) := \{\mathfrak{p} \in \mathcal{P}(x) \mid S \text{ has a good reduction } \mathcal{S}_0 \text{ at } \mathfrak{p}$$

$$\text{and } \# \mathcal{S}_0(\mathbb{F}_q) = 1 + q^2 + \alpha q\}.$$

Then, what is the function

$$x \mapsto \frac{\# \mathcal{P}'(x; S / K, \alpha)}{\# \mathcal{P}(x)}$$

when $x \rightarrow \infty$? (I do not know even whether $\lim_{x \rightarrow \infty} \sum_{|\alpha| \leq 22} \mathcal{P}'(x; S / K, \alpha) = \infty$.)

5. Two Kinds of Zeta Functions of Degenerate SNCL Schemes over the Log Point of $\mathbb{F}_q$

In this section, we give a few examples of two kinds of local zeta functions of a separated scheme Y of finite type over $\mathbb{F}_q$: one of them is defined by rational points of Y ; the other is defined by the Kummer étale cohomology of Y when Y is the underlying scheme of a proper log smooth scheme over the log point $s_{\mathbb{F}_q}$.

First we introduce a Grothendieck group which is convenient in this section.

Let F be a field. Consider a Grothendieck group $\mathcal{K}(F)$ with the following generators and relations: the generators of $\mathcal{K}(F)$ are $[(V, \beta)]$'s, where V is a finite-dimensional vector space over F and β is an endomorphism of V over F . The relations are as follows: $[(V, \beta)] = [(U, \alpha)] + [(W, \gamma)]$ for a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0. \end{array}$$

Let t be a variable. Note that $\det(1 - t\beta|V) = \det(1 - t\alpha|U) \det(1 - t\gamma|W)$. If $V = \{0\}$, we set $\det(1 - t0|V) = 1$ ($1 \in F$). We have a natural map

$$\det(1 - t \bullet | \bullet) : \mathcal{K}(F) \rightarrow F(t)^* \cap (1 + tF[[t]])^* \quad (5.0.1)$$

of abelian groups. Here the intersection in the target of (5.0.1) is considered in the ring of Laurent power series in one variable with coefficients in F . Set $Z((V, \alpha), t) = \det(1 - t\alpha|V)$.

Let Y be a separated scheme of finite type over $\mathbb{F}_q$. Set

$$[(E_p(Y / \mathbb{F}_q), F_q^*)] := \sum_{i=0}^{\infty} (-1)^i [(H_{\text{rig}, c}^i(Y / K_0(\mathbb{F}_q)), F_q^*)] \in \mathcal{K}(K_0(\mathbb{F}_q)),$$

where E_p means the Euler-characteristic. Let

$$Z(Y / \mathbb{F}_q, t) := \exp \left(\sum_{n=0}^{\infty} \frac{\# Y(\mathbb{F}_{q^n})}{n} t^n \right) \quad (5.0.3)$$

be the zeta function of $Y / \mathbb{F}_q$. We can reformulate (2.0.1) as the following formula:

$$Z(Y / \mathbb{F}_q, t) = Z([(E_p(Y / \mathbb{F}_q), F_q^*)])^{-1}. \quad (5.0.4)$$

Proposition 5.1. *Let Y be a proper SNC (not necessarily log) scheme over $\mathbb{F}_q$. Let $Y^{(i)}$ ($i \in \mathbb{Z}_{\geq 0}$) be the disjoint union of the $(i+1)$ -fold intersections of the irreducible components of Y . Then*

$$Z(Y / \mathbb{F}_q, t) = \prod_{i,j \geq 0} \det(1 - tF_q^* | H_{\text{rig}}^j(Y^{(i)} / K_0(\mathbb{F}_q)))^{(-1)^{i+j+1}}. \quad (5.1.1)$$

Proof. Let $Y_{\bullet}$ be the Čech diagram of an affine open covering of Y by finitely many affine open subschemes U_j 's of Y . Let $\mathring{Y}_1, \dots, \mathring{Y}_N$ be the irreducible components of $\mathring{Y}$. Set $U_j^{(i)} := \coprod_{0 \leq k_0 < k_1 < \dots < k_l \leq N} (\mathring{Y}_{k_0} \cap \dots \cap \mathring{Y}_{k_l} \cap U_j)$, $Y_0^{(i)} := \coprod_j U_j^{(i)}$ and $Y_n^{(i)} := \text{cosk}_0^{Y^{(i)}}(Y_0^{(i)})_n$ ($n \in \mathbb{N}$). Let $Y_{\bullet} \xrightarrow{\subseteq} \mathcal{P}_{\bullet}$ be a closed immersion into a formally smooth formal scheme over $\text{Spf}(\mathcal{W}(\mathbb{F}_q))$. Then we have a closed immersion $Y_0^{(i)} \xrightarrow{\subseteq} \coprod^{(i)} \mathcal{P}_0$, where $\coprod^{(i)} \mathcal{P}_0$ is a finite sum of $\mathcal{P}_0$ which depends on i . Let $\delta_j : Y_0^{(i+1)} \rightarrow Y_0^{(i)}$ ($0 \leq j \leq i$) be the standard face morphism. Then we have a natural morphism $\Delta_j : \coprod^{(i+1)} \mathcal{P}_0 \rightarrow \coprod^{(i)} \mathcal{P}_0$ fitting into the following commutative diagram:

$$\begin{array}{ccc}
Y_0^{(i+1)} & \xrightarrow{\delta_j} & Y_0^{(i)} \\
\cap \downarrow & & \downarrow \cap \\
\coprod^{(i+1)} \mathcal{P}_0 & \xrightarrow{\Delta_j} & \coprod^{(i)} \mathcal{P}_0
\end{array}$$

and satisfying the standard relations. Set $\mathcal{P}_\bullet^{(i)} := \text{cosk}_0^{\mathcal{W}(\mathbb{F}_q)}(\coprod^{(i)} \mathcal{P}_0)$.

Let $\text{sp}:]Y_\bullet^{(i)}[_{\mathcal{P}_\bullet^{(i)}} \rightarrow Y_\bullet^{(i)}$ be the specialization map. Then, as in [8, (2.3)], the following sequence:

$$0 \rightarrow \text{sp}_*(\Omega_{Y_\bullet[\mathcal{P}_\bullet]}^\bullet) \rightarrow \text{sp}_*(\Omega_{Y_\bullet^{(0)}[\mathcal{P}_\bullet^{(0)}]}^\bullet) \rightarrow \text{sp}_*(\Omega_{Y_\bullet^{(1)}[\mathcal{P}_\bullet^{(1)}]}^\bullet) \rightarrow \dots$$

is exact. Hence we have the following spectral sequence:

$$E_1^{ij} = H_{\text{rig}}^j(Y^{(i)}/K_0(\mathbb{F}_q)) \Rightarrow H_{\text{rig}}^{i+j}(Y/K_0(\mathbb{F}_q)). \quad (5.1.2)$$

By (5.0.4) and this spectral sequence, we obtain the following formula:

$$[(E_p(Y/\mathbb{F}_q), F_q^*)] = \sum_{i,j \geq 0} (-1)^{i+j} [H_{\text{rig}}^j(Y^{(i)}/K_0(\mathbb{F}_q)), F_q^*] \in \mathcal{K}(K_0(\mathbb{F}_q)). \quad (5.1.3)$$

This formula implies (5.1.1).

Corollary 5.2. *Let $X/\mathbb{F}_q$ be a non-smooth combinatorial K3 surface ([22, 36, 46]). (We do not assume that X has a log structure of simple normal crossing type.) Let m be the summation of the times of the processes of blowing downs making all irreducible components relatively minimal. Let M_1 (resp., M_2) be the cardinality of the irreducible components of $\overset{\circ}{X}$ whose relatively minimal models are $\mathbb{P}_{\mathbb{F}_q}^2$ (resp., Hirzebruch surfaces = relatively minimal rational ruled surfaces). Let M be the cardinality of the irreducible components of $\overset{\circ}{X}$. Then the following hold:*

(1) If X is of Type II with double elliptic curve $E / \mathbb{F}_q$, then

$$Z(\mathring{X} / \mathbb{F}_q, t) = \frac{\det(1 - qtF_q^* | H_{\text{rig}}^1(E/K_0(\mathbb{F}_q)))^{M-2}}{(1-t)\det(1 - tF_q^* | H_{\text{rig}}^1(E/K_0(\mathbb{F}_q)))(1-qt)^{M_1+2M_2+M-3+m}(1-q^2t)^M}. \quad (5.2.1)$$

(2) Assume that X is of Type III. Let d be the cardinality of the double curves of $\mathring{X}$. Then

$$Z(\mathring{X} / \mathbb{F}_q, t) = \frac{1}{(1-t)^2(1-qt)^{M_1+2M_2+m-d}(1-q^2t)^M}. \quad (5.2.2)$$

Proof. First we give a remark on the rigid cohomology of a smooth projective rational surface S over $\mathbb{F}_q$. Set $S_{\mathbb{F}_{q^n}} := S \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $S_{\overline{\mathbb{F}_q}} := S \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$.

Let $\overline{S}_{\min}$ be a relatively minimal model of $S_{\overline{\mathbb{F}_q}}$. If $\overline{S}_{\min} \simeq \mathbb{P}_{\overline{\mathbb{F}_q}}^2$, we see that the motive $H(S_{\overline{\mathbb{F}_q}})$ is as follows by ([12, (6.12)]):

$$H(S_{\overline{\mathbb{F}_q}}) \simeq H(\mathbb{P}_{\overline{\mathbb{F}_q}}^2) \oplus H(D)(-1),$$

where D is the disjoint sum of 0-dimensional points. Since $H^2(\mathbb{P}_{\overline{\mathbb{F}_q}}^2)$ is isomorphic to a Tate-twist, if a natural number n is big enough, then $(F_q^*)^n$ on $H_{\text{rig}}^2(S_{\mathbb{F}_{q^n}}/K_0(\mathbb{F}_{q^n}))$ is $\text{diag}(q^n, \dots, q^n)$. Hence the eigenvalues of $(F_q^*)^n$ are q^n ($n \gg 0$) and thus the eigenvalues of F_q^* are q .

If $\overline{S}_{\min}$ is isomorphic to a relatively minimal ruled surface over a smooth curve C over $\overline{\mathbb{F}_q}$, the motive $H(S_{\overline{\mathbb{F}_q}})$ is as follows by ([12, (6.10), (6.12)]):

$$H(S_{\overline{\mathbb{F}_q}}) \simeq H(C) \oplus H(C)(-1) \oplus H(D)(-1).$$

Hence we see that F_q^* on $H_{\text{rig}}^2(S/K_0(\mathbb{F}_q))$ is $\text{diag}(q, \dots, q)$ as above.

(1) It is easy to check that

$$H_{\text{rig}}^i(\mathring{X}^{(0)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^M & (i = 0), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))^{\oplus M-2} & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M_1+2M_2+2(M-2)+m} & (i = 2), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))(-1)^{\oplus M-2} & (i = 3), \\ K_0(\mathbb{F}_q)(-2)^M & (i = 4), \end{cases}$$

and

$$H_{\text{rig}}^i(\mathring{X}^{(1)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^{M-1} & (i = 0), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))^{\oplus M-1} & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M-1} & (i = 2). \end{cases}$$

Now (5.2.1) follows from (5.1.1).

(2) Let T be the cardinality of the triple points of $\mathring{X}$. It is easy to check that

$$H_{\text{rig}}^i(\mathring{X}^{(0)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^M & (i = 0), \\ 0 & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M_1+2M_2+m} & (i = 2), \\ 0 & (i = 3), \\ K_0(\mathbb{F}_q)(-2)^M & (i = 4), \end{cases}$$

$$H_{\text{rig}}^i(\mathring{X}^{(1)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^d & (i = 0), \\ 0 & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^d & (i = 2), \end{cases}$$

and

$$H_{\text{rig}}^0(\mathring{X}^{(2)}/K_0(\mathbb{F}_q)) = K_0(\mathbb{F}_q)^T.$$

Because the dual graph of $\mathring{X}$ is a circle, $M - d + T = \chi(\mathbb{S}^1) = 2$. Now (5.2.2) follows from (5.1.1). $\square$

Remark 5.3. If $p \neq 2$, we can prove that T is even (cf. [22]). However we do not use this fact in this article.

Corollary 5.4. *Assume that $p \neq 2$. Let $X/\mathbb{F}_q$ be a non-smooth combinatorial classical Enriques surface ([36, 46]). Let M_1, M_2, M, m and d be as in (5.2). Then the following hold:*

(1) *If X is of Type II with double elliptic curve $E/\mathbb{F}_q$, then*

$$Z(\mathring{X}/\mathbb{F}_q, t) = \frac{\det(1 - qtF_q^*|H_{\text{rig}}^1(E/K_0(\mathbb{F}_q)))^{M-1}}{(1-t)(1-qt)^{M_1+2M_2+M-1+m}(1-q^2t)^M}. \quad (5.4.1)$$

(2) *Assume that X is of Type III. Let d and T be the cardinalities of the double curves of $\mathring{X}$ and the triple points of $\mathring{X}$, respectively. Then*

$$Z(\mathring{X}/\mathbb{F}_q, t) = \frac{1}{(1-t)(1-qt)^{M_1+2M_2+m-d}(1-q^2t)^M}. \quad (5.4.2)$$

Proof. (1) It is easy to check that

$$H_{\text{rig}}^i(\mathring{X}^{(0)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^M & (i = 0), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))^{\oplus M-1} & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M_1+2M_2+2(M-1)+m} & (i = 2), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))(-1)^{\oplus M-1} & (i = 3), \\ K_0(\mathbb{F}_q)(-2)^M & (i = 4), \end{cases}$$

and

$$H_{\text{rig}}^i(\mathring{X}^{(1)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^{M-1} & (i = 0), \\ H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))^{\oplus M-1} & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M-1} & (i = 2). \end{cases}$$

(2) It is easy to check that

$$H_{\text{rig}}^i(\mathring{X}^{(0)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^M & (i = 0), \\ 0 & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^{M_1+2M_2+m} & (i = 2), \\ 0 & (i = 3), \\ K_0(\mathbb{F}_q)(-2)^M & (i = 4), \end{cases}$$

$$H_{\text{rig}}^i(\mathring{X}^{(1)}/K_0(\mathbb{F}_q)) = \begin{cases} K_0(\mathbb{F}_q)^d & (i = 0), \\ 0 & (i = 1), \\ K_0(\mathbb{F}_q)(-1)^d & (i = 2), \end{cases}$$

and

$$H_{\text{rig}}^0(\mathring{X}^{(2)}/K_0(\mathbb{F}_q)) = K_0(\mathbb{F}_q)^T.$$

Because the dual graph of $\mathring{X}$ is $\mathbb{P}^2(\mathbb{R})$, $M - d + T = \chi(\mathbb{P}^2(\mathbb{R})) = 1$. $\square$

Lastly we consider another type of local zeta functions.

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with finite residue field $\mathbb{F}_q$ and let K be the fraction field of $\mathcal{V}$. Let $\mathfrak{Y}$ be a proper smooth scheme over K of dimension d and let I be the inertia

group of the absolute Galois group $\text{Gal}(\overline{K}/K)$. Then the zeta function of $\mathfrak{Y}$ is defined as follows:

$$Z(\mathfrak{Y}, t) := \prod_{i=0}^{2d} \det(1 - t\sigma|H_{\text{et}}^i(\mathfrak{Y} \otimes_{\overline{K}} \mathbb{Q}_l)^I)^{(-1)^{h+1}},$$

where $\sigma \in \text{Gal}(\overline{K}/K)$ is a lift of the geometric Frobenius of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ and l is a prime which is prime to q . If $\mathfrak{Y}$ is the generic fiber of a proper semistable family $\mathcal{Y}$ over $\mathcal{V}$ with special fiber Y , then the following formula holds by [23] ([29]):

$$Z(\mathfrak{Y}, t) = \prod_{h=0}^{2d} \det(1 - t\sigma|H_{\text{ket}}^h(\overline{Y}, \mathbb{Q}_l)^I)^{(-1)^{i+1}}.$$

Let $\mathcal{X}$ be a proper strict semistable family of surfaces over $\mathcal{V}$ with log special fiber X over $s_{\mathbb{F}_q}$.

Then [41, (6.3.3)] tells us that $Z(\mathcal{X}_K, t)$ can be described by the log crystalline cohomologies by the coincidence of the monodromy filtration and the weight filtration. ([48, (8.3)], [41, (6.2.4)]; however see [47, (11.15)] and [48, (7.1)].)

$$Z(\mathcal{X}_K, t) = \prod_{i=0}^4 \det(1 - tF_q^*|(H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q})))_{K_0(\mathbb{F}_q)})^{N=0})^{(-1)^{i+1}},$$

where $\mathcal{W}(s_{\mathbb{F}_q})$ is the canonical lift of $s_{\mathbb{F}_q}$ over $\mathcal{W}(\mathbb{F}_q)$, $H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q}))$ is the i -th log crystalline cohomology of $X/\mathcal{W}(\mathbb{F}_q)$ and

$$N : H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q}))_{K_0(\mathbb{F}_q)} \rightarrow H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q}))_{K_0(\mathbb{F}_q)}(-1)$$

is the p -adic monodromy operator. More generally, for a proper SNCL scheme $Y/s_{\mathbb{F}_q}$ of pure dimension d , set

$$Z(H^i(Y/K_0(s_{\mathbb{F}_q})), t) := \det(1 - tF_q^*|(H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q})))_{K_0(\mathbb{F}_q)})^{N=0})^{(-1)^{i+1}}, \quad (5.4.3)$$

and

$$Z(Y/s_{\mathbb{F}_q}, t) := \prod_{i=0}^{2d} Z(H^i(Y/K_0(s_{\mathbb{F}_q})), t)^{(-1)^{i+1}}. \quad (5.4.4)$$

Let us recall the following result due to the author ([52, (8.3)], (cf. [40, (2.2)], [9, (6.4)]):

Theorem 5.5 ([52, (8.3)]). *Let κ be a perfect field of characteristic $p > 0$. Let X/s be an SNCL K3 surface. Let $H_{\log}^i(X)$ ($i \in \mathbb{N}$) be the i -th log crystalline cohomology or the i -th Kummer étale cohomology of X/s . Then the following hold:*

- (1) *The $\star$ -adic ($\star = p, l$) monodromy filtration and the weight one on $H_{\log}^i(X)$ coincide.*
- (2) *The following hold:*
 - (a) *X is of Type I if and only if $N = 0$ on $H_{\log}^2(X)$.*
 - (b) *X is of Type II if and only if $N \neq 0$ and $N^2 = 0$ on $H_{\log}^2(X)$.*
 - (c) *X is of Type III if and only if $N^2 \neq 0$ on $H_{\log}^2(X)$.*

Proof. For the completeness of this article, we give the proof of Theorem 5.5.

We give the proof of this theorem in the p -adic case because the proof in the l -adic case is the same as that in the p -adic case.

Recall the following weight spectral sequence ([41, 3.23], [47, (2.0.1)]):

$$E_1^{-k, i+k} = \bigoplus_{j \geq \max\{-k, 0\}} H_{\text{rig}}^{i-2j-k}(\overset{\circ}{X}^{(2j+k)}/K_0)(-j-k) \Rightarrow H_{\text{crys}}^i(X/\mathcal{W}(s))_{K_0}. \quad (5.5.1)$$

(See [47] for the mistakes in [41].) Here we have used Berthelot's comparison isomorphism $H_{\text{crys}}^i(Y/\mathcal{W})_{K_0} = H_{\text{rig}}^i(Y/K_0)$ ($i \in \mathbb{N}$) for a proper smooth scheme Y over κ . By [47, (3.6)] this spectral sequence degenerates at E_2 . (The l -adic analogue of this spectral sequence also degenerates at E_2 by Nakayama's theorem ([45, (2.1)]).)

(1) We may assume that κ is algebraically closed. If X/s is of Type I, there is nothing to prove.

If X is of Type III, the double curves and the irreducible components are rational, and hence $E_1^{0,1} = E_1^{1,1} = E_1^{0,3} = E_1^{-1,3} = 0$. By [46, (3.5) (3)], $H_{\text{log-crys}}^1(X/\mathcal{W}) = 0$ and hence we have $E_2^{-1,2} = 0$. (Note that we also have the similar vanishing for the first Kummer étale cohomology of X by the vanishing above and the existence of the $\mathbb{Q}$ -structure of $E_2^{-1,2}$ (cf. the proof of [48, (8.3)]).) By taking the duality in [47, (10.5)], $E_2^{1,2} = 0$. By [41, 6.2.1], the p -adic monodromy operator $N : H_{\text{crys}}^2(X/\mathcal{W}(s)) \rightarrow H_{\text{crys}}^2(X/\mathcal{W}(s))(-1)$ induces an isomorphism $N^2 : E_2^{-2,4} \xrightarrow{\sim} E_2^{2,0}(-2) = K_0$.

If X is of Type II, then $E_1^{-2,4} = E_1^{2,0} = 0$. By [46, (3.5) (3)] again, $H_{\text{log-crys}}^1(X/\mathcal{W}) = H_{\text{log-crys}}^3(X/\mathcal{W}) = 0$. Hence $E_2^{ij} = 0$ for $i + j = 1, 3$. Because $N : H_{\text{crys}}^2(X/\mathcal{W}(s)) \rightarrow H_{\text{crys}}^2(X/\mathcal{W}(s))(-1)$ induces an isomorphism $E_2^{-1,3} \xrightarrow{\sim} E_2^{1,1}(-1)$ by [41, 6.2.2], we have proved (1).

(2) (2) follows from (1) and the non-vanishings of $E_2^{1,1}$ in the Type II case and $E_2^{2,0}$ in the Type III case, respectively.

Remark 5.6. The author has found the Theorem 5.5 in December 1996 by using the p -adic weight spectral sequence (5.5.1). The key point of the proof is to notice to use the p -adic weight spectral sequence of X/s instead of the Clemens-Schmid exact sequence used in Kulikov's article [36]. (In fact, the complex analogue (5.7) below of (5.5) holds; this is a generalization of Kulikov's theorem in [loc. cit.] and the proof of (5.5) is simpler than that in [loc. cit.]. To my surprise, mathematicians who are working over $\mathbb{C}$ have not used the weight spectral sequence (5.7.1).) The author has finished writing the preprint [52] by 2000 at the latest (cf. [45, Remark 2.4 (3)]). However, after that, he has noticed that there are too many non-minor mistakes in theory of log de Rham-Witt complexes in Hyodo-Kato's article [26] and Mokrane's article [41] as pointed out in [47]. Because he has used Hyodo-Kato's and Mokrane's theory in [52] heavily, he has to use their results in correct ways. However he has used his too much time for correcting their results in [47], he has no will to publish [52] now (because [52] is quite long and because he has to use more time for adding comments about Hyodo-Kato's and Mokrane's articles in [52]). For example, ν in [41] is *not* a morphism of complexes, the left N in the diagram in [40, (2.2)] is incorrect.

In [40, (2.2)]. Matsumoto has proved (5.5) for semistable algebraic spaces of K3-surfaces after looking at the proof in [52]. (See "Proof of p -adic case" in the proof of [40, Proposition 2.2].)

Theorem 5.7 (cf. [36]). *Let s be the log point of $\mathbb{C}$. Let X/s be an analytic SNCL K3 surface. Let X_∞ be the base change of the Kato-Nakayama space $X^{\log}$ of X ([37]) with respect to the morphism $\mathbb{R} \ni x \mapsto \exp(2\pi\sqrt{-1}x) \in \mathbb{S}^1$. Let $N : H^i(X_\infty, \mathbb{Q}) \rightarrow H^i(X_\infty, \mathbb{Q})(-1)$ ($i \in \mathbb{N}$) be the monodromy operator constructed in [21]. Then the following hold:*

- (1) *The weight filtration on $H^i(X_\infty, \mathbb{Q})$ constructed in [21] coincide with the monodromy filtration on $H^i(X_\infty, \mathbb{Q})$*

(2) *The following hold:*

(a) *X is of Type I if and only if $N = 0$ on $H^2(X_\infty, \mathbb{Q})$.*

(b) *X is of Type II if and only if $N \neq 0$ and $N^2 = 0$ on $H^2(X_\infty, \mathbb{Q})$.*

(c) *X is of Type III if and only if $N^2 \neq 0$ on $H^2(X_\infty, \mathbb{Q})$.*

Proof. By [48, (2.1.10)] we have the following weight spectral sequence:

$$E_{1,\infty}^{-k,h+k} = \bigoplus_{j \geq \max\{-k, 0\}} H^{h-2j-k}(\overset{\circ}{X}^{(2j+k+1)}, \mathbb{Q})(-j-k) \Rightarrow H^h(X_\infty, \mathbb{Q}). \quad (5.7.1)$$

By [18, (5.9)], if X is a combinatorial Type II or Type III K3 surface over $\mathbb{C}$, then $H^0(X, \Omega_{X/\mathbb{C}}^1) = 0$. (Of course, if X is of Type I, then $H^0(X, \Omega_{X/\mathbb{C}}^1) = 0$ by Hodge symmetry.) Hence $H^1(X_\infty, \mathbb{C}) = H_{\text{dR}}^1(X/\mathbb{C}) = H^0(X, \Omega_{X/s}^1) \oplus H^1(X, \mathcal{O}_X) = 0$. Here we have used the isomorphism between Steenbrink complexes $A_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ and $A_{\mathbb{C}}$ of X and the isomorphism between $A_{\mathbb{C}}$ and $\Omega_{X/s}^\bullet$ ([21]). By the duality of the E_2 -terms of (5.7.1) ([48, (5.15) (2)]) and the degeneration at E_2 of (5.7.1) (by Hodge theory), we obtain the vanishing of $H^3(X_\infty, \mathbb{C})$. The rest of the proof is the same as that of (5.5).

Theorem 5.8 ([52, (15.1)]). *Let $X/s_{\mathbb{F}_q}$ be a projective SNCL K3 surface. Then the following hold:*

(1)

$$Z(H^i(X/K_0(s_{\mathbb{F}_q})), t) = \begin{cases} 1-t & (i=0), \\ 1 & (i=1, 3), \\ 1-q^2t & (i=4). \end{cases}$$

(2) If X is of Type II with double elliptic curve E , then

$$Z(H^2(X/K_0(s_{\mathbb{F}_q})), t) = \det(1 - tF_q^* | H_{\text{rig}}^1(E/K_0))(1 - qt)^{18}.$$

Consequently,

$$Z(X / s_{\mathbb{F}_q}, t) = \frac{1}{(1 - t) \det(1 - tF_q^* | H_{\text{rig}}^1(E/K_0))(1 - qt)^{18} (1 - q^2 t)}.$$

(3) If X is of Type III, then

$$Z(H^2(X / K_0(s_{\mathbb{F}_q})), t) = (1 - t)(1 - qt)^{19}.$$

Consequently

$$Z(X / s_{\mathbb{F}_q}, t) = \frac{1}{(1 - t)^2 (1 - qt)^{19} (1 - q^2 t)}.$$

Proof. (1) By [46, (3.5)], $H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q})) = 0$ ($i = 1, 3$). Thus $Z(H^i(X / K_0(s_{\mathbb{F}_q})), t) = 1$ ($i = 1, 3$). By [46, (6.9)], X is the log special fiber of a projective semistable family $\mathcal{X}$ over $\text{Spec}(W(\mathbb{F}_q))$. By [46, (6.10)], the generic fiber of $\mathcal{X}$ is a K3 surface. Hence, by Hyodo-Kato's isomorphism ([26, (5.1)] (however see [47, Section 7] for incompleteness of the proof of Hyodo-Kato isomorphism), $\dim_{K_0(\mathbb{F}_q)} H_{\text{crys}}^2(X / \mathcal{W}(s_{\mathbb{F}_q}))_{K_0(\mathbb{F}_q)} = 22$.

(2) In this case, by (5.5), $N \neq 0$, $N : E_2^{-1,2} \rightarrow E_2^{1,1}$ is an isomorphism, $N^2 = 0$ on $H_{\text{crys}}^2(X/\mathcal{W}(s_{\mathbb{F}_q}))_{K_0(\mathbb{F}_q)}$ and $E_2^{-2,4} = E_2^{2,0} = 0$. Hence we have the following exact sequence by (5.5):

$$0 \rightarrow E_2^{1,1} \rightarrow \text{Ker}(N) \rightarrow E_2^{0,2} \rightarrow 0.$$

Because $E_2^{1,1} \simeq H_{\text{rig}}^1(E/K_0(\mathbb{F}_q))$, $\det(1 - tF_q^*|E_2^{1,1}) = \det(1 - tF_q^*|H_{\text{rig}}^1(E/K_0(\mathbb{F}_q)))$. On the other hand, $E_2^{0,2}$ is a subquotient of

$$H_{\text{rig}}^0(X^{(2)}/K_0(\mathbb{F}_q))(-1) \oplus H_{\text{rig}}^2(X^{(1)}/K_0(\mathbb{F}_q)).$$

Hence F_q^* on $E_2^{0,2}$ is $\text{diag}(q, \dots, q)$ as shown in the proof of (5.2). Since $\text{Ker}(N)$ is 20-dimensional, we obtain (2).

(3) In this case, by (5.5), $N^2 \neq 0$, $N^3 = 0$ and $E_2^{-1,3} = E_2^{1,1} = 0$.

Because $N^2 : E_2^{-2,4} \rightarrow E_2^{2,0}(-2)$ is an isomorphism, $N : E_2^{0,2} \rightarrow E_2^{2,0}(-1)$ is surjective and hence the kernel of N is 20-dimensional. Obviously $F_q^* = \text{id}$ on $E_2^{2,0}$. As in (2), F_q^* on $E_2^{0,2}$ is $\text{diag}(q, \dots, q)$. Hence we obtain (3).

Theorem 5.9 ([52, (15.2)]). *Let $X/s_{\mathbb{F}_q}$ be a projective non-smooth SNCL classical Enriques surface. Then*

$$Z(H^i(X/K_0(s_{\mathbb{F}_q})), t) = \begin{cases} 1-t & (i=0), \\ 1 & (i=1, 3), \\ (1-qt)^{10} & (i=2), \\ 1-q^2t & (i=4). \end{cases}$$

Consequently

$$Z(X/s_{\mathbb{F}_q}, t) = \frac{1}{(1-t)(1-qt)^{10}(1-q^2t)}.$$

Proof. By [46, (7.1)], $H_{\text{crys}}^i(X/\mathcal{W}(s_{\mathbb{F}_q})) = 0$ ($i=1, 3$) and hence $Z(H^i(X/K_0(s_{\mathbb{F}_q})), t) = 1$ ($i=1, 3$). By [46, (7.1)] and the argument in [46, (6.8), (6.11)], X is the log special fiber of a projective semistable family $\mathcal{X}$ over $\mathcal{W}(\mathbb{F}_q)$ and the generic fiber of $\mathcal{X}$ is a classical Enriques

surface. Hence $\dim_{K_0(\mathbb{F}_q)} H_{\text{crys}}^2(X/\mathcal{W}(s_{\mathbb{F}_q}))_{K_0(\mathbb{F}_q)} = 10$. The rest of the proof is the same as that of (5.8) by noting that $0 = E_2^{-1,3} = E_2^{1,1} = E_2^{20} = E_2^{-2,4}$, where $E_2^{\bullet\bullet}$'s are E_2 -terms of the spectral sequence (5.5.1).

Appendix

6. A Remark on Katsura and Van der Geer's Result

In this section we generalize the argument in the proof of (1.7) (3).

First we recall the following theorem in [55]. This is a generalization of Katsura and Van der Geer's theorem ([67, (5.1), (5.2), (16.4)]).

Theorem 6.1 ([55, (2.3)]). *Let κ be a perfect field of characteristic $p > 0$. Let Y be a proper scheme over κ . (We do not assume that Y is smooth over κ .) Let q be a nonnegative integer. Assume that $H^q(Y, \mathcal{O}_Y) \simeq \kappa$, that $H^{q+1}(Y, \mathcal{O}_Y) = 0$ and that $\Phi_{Y/\kappa}^q$ is pro-representable. Assume also that the Bockstein operator*

$$\beta : H^{q-1}(Y, \mathcal{O}_Y) \rightarrow H^q(Y, \mathcal{W}_{n-1}(\mathcal{O}_Y))$$

arising from the following exact sequence:

$$0 \rightarrow \mathcal{W}_{n-1}(\mathcal{O}_Y) \xrightarrow{V} \mathcal{W}_n(\mathcal{O}_Y) \rightarrow \mathcal{O}_Y \rightarrow 0$$

is zero for any $n \in \mathbb{Z}_{\geq 2}$. Let $V : \mathcal{W}_{n-1}(\mathcal{O}_Y) \rightarrow \mathcal{W}_n(\mathcal{O}_Y)$ be the Verschiebung morphism and let $F : \mathcal{W}_n(\mathcal{O}_Y) \rightarrow \mathcal{W}_n(\mathcal{O}_Y)$ be the induced morphism by the Frobenius endomorphism of $\mathcal{W}_n(Y)$. Let $n^q(Y)$ be the minimum of positive integers n 's such that the induced morphism

$$F : H^q(Y, \mathcal{W}_n(\mathcal{O}_Y)) \rightarrow H^q(Y, \mathcal{W}_n(\mathcal{O}_Y))$$

by the $F : \mathcal{W}_n(\mathcal{O}_Y) \rightarrow \mathcal{W}_n(\mathcal{O}_Y)$ is not zero. (If $F = 0$ for all n , then set $n^q(Y) := \infty$.) Let $h^q(Y/\kappa)$ be the height of the Artin-Mazur formal group $\Phi_{Y/\kappa}^q$ of Y/κ . Then $h^q(Y/\kappa) = n^q(Y)$.

Proposition 6.2. *Let the notations be as in (6.1). Let $D(\kappa)$ be the Cartier-Dieudonné algebra over κ . Then the following hold:*

(1) *length $_{\mathcal{W}} H^q(Y, \mathcal{W}_n(\mathcal{O}_Y)) = n$ ($n \in \mathbb{Z}_{\geq 1}$).*

(2) *Set $h := h(\Phi_{Y/\kappa}^q)$. Assume that $h < \infty$. Let us consider the following natural surjective morphism $H^q(Y, \mathcal{W}(\mathcal{O}_Y)) \rightarrow H^q(Y, \mathcal{W}_h(\mathcal{O}_Y))$. Then this morphism induces the following isomorphism:*

$$H^q(Y, \mathcal{W}(\mathcal{O}_Y))/p \xrightarrow{\sim} H^q(Y, \mathcal{W}_h(\mathcal{O}_Y)) \quad (6.2.1)$$

of $D(\kappa)/p$ -modules.

Proof. (1) By the assumptions we have the following exact sequence:

$$0 \rightarrow H^q(Y, \mathcal{W}_{n-1}(\mathcal{O}_Y)) \xrightarrow{V} H^q(Y, \mathcal{W}_n(\mathcal{O}_Y)) \rightarrow H^q(Y, \mathcal{O}_Y) \rightarrow 0.$$

(1) immediately follows from this.

(2) First assume that $h = 1$. Then $H^q(Y, \mathcal{W}(\mathcal{O}_Y)) \simeq \mathcal{W}$. In this case, (2) is obvious.

Next assume that $1 < h < \infty$. Set $M_n := H^q(X, \mathcal{W}_n(\mathcal{O}_X))$ ($n \in \mathbb{Z}_{\geq 1}$) and $M := H^q(X, \mathcal{W}(\mathcal{O}_X))$. Consider the following exact sequence:

$$0 \rightarrow \mathcal{W}_{m-n}(\mathcal{O}_Y) \xrightarrow{V^n} \mathcal{W}_m(\mathcal{O}_Y) \rightarrow \mathcal{W}_n(\mathcal{O}_Y) \rightarrow 0,$$

for $m > n$. By the assumption and (3.2.1) we see that $H^{q+1}(Y, \mathcal{W}_m(\mathcal{O}_X)) = 0$ for any m . Hence the natural morphism $M_m \rightarrow M_n$ is surjective and

consequently the natural morphism $M \rightarrow M_n$ is surjective. In particular, the natural morphism $M \rightarrow M_h$ is surjective. Let η be an element of M_h . We claim that $p\eta = 0$.

We have to distinguish the operator $F : M_n \rightarrow M_n$ and the operator $F : M_n \rightarrow M_{n-1}$. The latter F is equal to $R_n F$, where $R_n : M_n \rightarrow M_{n-1}$ is the projection. We denote $R_n F$ by F_n to distinguish two F 's. Since the following diagram

$$\begin{array}{ccc} M_h & \xrightarrow{R_h} & M_{h-1} \\ F \downarrow & & \downarrow F \\ M_h & \xrightarrow{R_h} & M_{h-1} \end{array} \quad (6.2.2)$$

is commutative, we have the following:

$$p\eta = VF_h(\eta) = VR_h F(\eta) = VFR_h(\eta).$$

Since $F = 0$ on M_{h-1} by (6.1), the last term is equal to zero. Hence $p\eta = 0$. Consequently, the natural morphism $H^q(Y, \mathcal{W}(\mathcal{O}_Y)) \rightarrow H^q(Y, \mathcal{W}_h(\mathcal{O}_Y))$ factors through the projection $H^q(Y, \mathcal{W}(\mathcal{O}_Y)) \rightarrow H^q(Y, \mathcal{W}(\mathcal{O}_Y))/p$. Since the morphism (6.2.1) is surjective and $\dim_\kappa H^q(Y, \mathcal{W}(\mathcal{O}_Y))/p = h = \dim_\kappa H^q(Y, \mathcal{W}_h(\mathcal{O}_Y))$ by (1), the morphism (6.2.1) is an isomorphism. $\square$

Remark 6.3. Let the notations be as in (1.7) (2). By using only (6.2) for the case $q = d$, we can prove that

$$\# Y(\mathbb{F}_{q^k}) \equiv 1 \pmod{p^{\left[\frac{ke+1}{2}\right]}} \quad (k \in \mathbb{Z}_{\geq 1}), \quad (6.3.1)$$

where $[]$ is the Gauss symbol. However the congruence (6.3.1) is not sharper than (1.7.2); only in the case $h = 2$, (6.3.1) is equivalent to (1.7.2).

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