REES CONGRUENCES ON RESIDUATED SEMIGROUPS

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Abstract

Residuated semigroups form an important class of ordered algebra which can be regarded as a generalisation of quantales. We study the notion of quotient on a residuated semigroup extending some results of [27] to them. A particular attention is put on Rees congruences on residuated semigroups.

1. Introduction and Preliminaries

Residuation is a basic concept in mathematics [2]. It is strongly connected with Galois connections and closure operators. Residuated semigroups have been introduced in the 30s of the last century by Ward and Dilworth [26] to investigate ideal theory of commutative rings with unit. Nowadays, the investigation of residuated lattices (that is, residuated monoids on lattices) has become quite popular, and has been staying in the focus of strong international attention. An extensive

2010 Mathematics Subject Classification: Secondary 06F05, 06A15, 18A30.

Keywords and phrases: ordered monoids, congruences.

Received March 16, 2018; Revised April 5, 2018

monograph discussing residuated lattices went to print in 2007 [9]. This increasing and significant interest was initiated by the discovery of the strong connection between residuated lattices and substructural logics [18, 19]. Residuated semigroups constitute a vast class of ordered algebras comprising both residuated lattices and quantales. This paper deals with quotient of residuated semigroups; it extends some results of [27].

An **ordered semigroup** is an ordered set $(P, \leq)$ together with semigroup structure $(\otimes : P \times P \rightarrow P)$ such that the order $\leq$ that is compatible with the semigroup operation, meaning that $x \leq y$ implies $z \otimes x \leq z \otimes y$ and $x \otimes z \leq y \otimes z$ for all $x, y, z \in P$. **Ordered monoid** are defined analogously. If P is a group and it is ordered as a semigroup, one obtains the notion of **ordered group**. The terms **posemigroup**, **pogroup**, and **pomonoid** are also in use.

Definition 1.1. A partially ordered semigroup $(P, \leq, \otimes)$ is called **residuated** if all left translation $P \xrightarrow{a \otimes -} P$ and all right translations $P \xrightarrow{- \otimes a} P$ are residuated, i.e., have right adjoints $P \xrightarrow{a \setminus -} P$ and $P \xrightarrow{- / a} P$, respectively; this means that

$$a \otimes x \leq y \text{ and } x \leq a \setminus y \quad (1)$$

are equivalent condition; similarly for

$$x \otimes a \leq y \text{ and } x \leq y / a. \quad (2)$$

Residuated po-semigroups will be simply be called **residuated semigroups**.

Clearly,

$$a \leq b \text{ implies } b \setminus x \leq a \setminus x \text{ and } x \setminus a \leq x \setminus b. \quad (3)$$

It follows from (1), (2), and (3) that $a \leq b / (a \setminus b)$, with equality if and only if $a = b / x$, for some x in P .

The two binary operations $P \times P \xrightarrow{-\backslash-} P$ and $P \times P \xrightarrow{-/-} P$ are called the left and the right residuals of $\otimes$. More precisely c/b is called the right residual of c over b , and $b \backslash c$ the left residual of c over b .

Definition 1.2. If P and Q are residuated semigroups, then a mapping $F : P \rightarrow Q$ is a residuated semigroup homomorphism if and only if it preserves the multiplication $\otimes$ and the two residuals $-\backslash-$ and $-/-$.

A residuated po-monoid for which the order $\leq$ is the lattice order is called **residuated reticulated monoid** or simply a **residuated lattice**. Examples of residuated lattices include Boolean algebras, Heyting algebras [12], MV-algebras [4], basic logic algebras, [11] and lattice-ordered groups. Complete residuated semigroups are referred to as **quantales** and complete residuated monoid are referred to as **unital quantales**.

Example 1.3 (Powerset residuated semigroup). Let $(M, \cdot)$ be a semigroup. For two subsets X and Y of M , we define

$$X \otimes Y = \{x \cdot y : x \in X \text{ and } y \in Y\},$$

$$X \backslash Y = \{z \in M : X \otimes \{z\} \subseteq Y\},$$

and

$$Y / X = \{z \in M : \{z\} \otimes X \subseteq Y\}.$$

The structure $(2^M, \subseteq, \otimes, \backslash, /)$ is a residuated semigroup. It is called the powerset residuated semigroup over the semigroup $(M, \cdot)$. This example is of importance in the theory of languages where M is usually the free semigroup over an alphabet.

Example 1.4. Let E be a set and let's denote by **Rel**(E) the set $\mathcal{P}(E \times E)$ of relations on E . One gets the residuated monoid $(\text{Rel}(E); \subseteq, \otimes, \backslash, /, \Delta_E)$, where Δ_E denotes the equality relation on E and $R \otimes S$ the compositional relation of the relations R and S

(alternative notation is $S \circ R$). The right residuate S/R of S over R is $\{(x, y), \forall z[yRz \Rightarrow xSz]\}$; the left residuate R/S of S over R is $\{(y, z), \forall x[xRy \Rightarrow xSz]\}$.

Example 1.5. Let R be a ring with unit and let $\mathbf{Ideals}(R)$ be the set of all (two-sided) ideals of R equipped with set inclusion. For $I, J \in \mathbf{Ideals}(R)$, we write $I \otimes J = \{\sum_{\text{finite}} ij : i \in I, j \in J\}$, $I \setminus J = \{k : I \otimes \{k\} \subseteq J\}$, $J/I = \{k : \{k\} \otimes I \subseteq J\}$. One gets the residuated monoid $(\mathbf{Ideals}(R), \subseteq, \otimes, \setminus, /, R)$.

2. Regular Congruences on Residuated Ordered Semigroups

For an equivalence relation θ a set P we denote by $\theta^b(x)$ the equivalence class of x and therefore by $\theta^b : P \rightarrow P/\theta$ the canonical surjection. If θ be a congruence on a universal algebra $\mathbf{P} = (P, \otimes)$, the quotient semigroup (also known as the homomorphic image) $\mathbf{P}/\theta$ of $\mathbf{P}$ by θ has support the quotient set P/θ of θ -classes and its multiplication, again denoted by $\otimes$, is given by:

$$\theta^b(x) \otimes \theta^b(y) := \theta^b(x \otimes y).$$

It is known how to extend the concept of quotient to po-semigroup. Let us first quote the following definition:

Definition 2.1 ([23]). Let $(P, \leq, \otimes)$ be a po-semigroup, θ be a congruence on the semigroup $(P, \otimes)$. The congruence θ is called an **order congruence** if there is an order $\lesssim$ on the quotient semigroup $(P/\theta, \otimes)$ such that

(1) $(P/\theta, \lesssim, \otimes)$ is a po-semigroup;

(2) the surjective mapping $\theta^b : P \rightarrow P/\theta, x \mapsto \theta^b(x)$ is a monotone and multiplicative map.

Other now it is time to extend to residuated semigroups the notion of quotient. First we give the following definition:

Definition 2.2. Let $(P, \leq, \otimes, \backslash, /)$ be a residuated po-semigroup, θ be a congruence on the semigroup $(P, \otimes)$. The relation θ is called a **regular congruence** if there is an order $\lesssim$ and two binary operations $\backslash$ and $/$ on the quotient algebras P/θ such that

(1) $(P/\theta, \otimes, \lesssim, \backslash, /)$ is a residuated po-semigroup;

(2) the surjective mapping $\theta^b : P \rightarrow P/\theta, x \mapsto \theta^b(x)$ is monotone, multiplicative and preserves residuals.

2.1. Rees congruences on residuated semigroups

Definition 2.3 ([27]). A non empty subset I of a po-semigroup $\mathbf{P} = (P, \leq, \otimes)$ is called an ideal of $\mathbf{P}$ if (i) $P \otimes I \subseteq I$, (ii) $I \otimes P \subseteq I$. If moreover it satisfies the condition (iii) $x \leq y$ and $y \in I$ imply $x \in I$, then it is called an order ideal of $\mathbf{P}$.

Let I be an ideal of semigroup $(\mathbf{P}, \otimes)$. The Rees congruence θ_I induced by I is equal to $\Delta_P \cup (I \times I)$, where $\Delta_P = \{(x, x) : x \in P\}$. It is equivalently defined by $\theta_I = \Delta_{P \setminus I} \cup (I \times I)$. For simplicity, we denote P/θ_I by P/I . The factor semigroup $(P/I, \otimes)$ is called Rees factor semigroup. It is clear that $P/I = \{I\} \cup \{\{x\}, x \in P \setminus I\}$.

Proposition 2.4 ([27]). *Let I be a ideal of po-semigroup P . Then I is a congruence class of an order congruence on P if and only if I is convex, i.e., if $x, y \in I$, then for any c such that $x \leq c \leq y$, it is the case that $c \in I$.*

We extend this result to residuated semigroups by introducing the notion of residuated ideal.

Definition 2.5. A non empty subset of a residuated semigroup P is called a residuated ideal if it satisfies the following axioms (written in deduction form):

- (1) $\frac{x, x \backslash y \in I}{y \in I}$ and $\frac{x, y/x \in I}{y \in I}$ [left and right modus Ponens];
- (2) $\frac{x \in I, y \in P}{y \backslash x, x/y \in I}$ [I is residually closed].

Theorem 2.6. Let I be a residuated ideal of a residuated semigroup P . Then I is a congruence class of a regular congruence on P if and only if I is convex.

Proof. Let I be a congruence class for a regular congruence θ . From the proof for 2.4 in [27], we get that I is convex. For the converse, let θ_I be the Rees congruence induced by I . As clearly shown in [27], the order $\preccurlyeq$ on the factor algebra $(P/\theta_I, \otimes)$, defined by

$$\theta_I^b(x) \preccurlyeq \theta_I^b(y) \text{ iff } (x \leq y) \text{ or } (x \leq b, b' \leq y) \text{ for some } b, b' \in I,$$

gives the triple $(P/\theta_I, \preccurlyeq, \otimes)$ a structure of a po-semigroup and the canonical map $\theta_I^b : P \rightarrow P/\theta_I$ is a monotone multiplicative map. It remains to define the residuals on $(P/\theta_I, \preccurlyeq, \otimes)$. For the purpose, we first show that if $x\theta_I x'$ and $a\theta_I a'$, then $(a \backslash x)\theta_I (a \backslash x')$ and $(x/a)\theta_I (x'/a')$.

Now let's start with $x\theta_I x'$ and $a\theta_I a'$. Then $x, x' \in I$ or $x = x'$; and $a, a' \in I$ or $a = a'$. We consider four cases.

(i) $x, x' \in I$ and $a, a' \in I$. From condition (2) of 2.5, $a \backslash x, a \wedge x$ are elements of I . The right residual is treated in the same way.

(ii) $x, x' \in I$ and $a = a'$. Again from the same condition, $a \backslash x, a \wedge x$ are elements of I . The right residual is treated in the same way.

(iii) $x = x'$ and $a, a' \in I$. In this case, we show that x/a is in I iff x/a' is also in I and conversely. Now assume that $x/a \in I$; since a is in I , applying Left Modus Ponens, i.e., condition (1) of 2.5, we get that x is in I and so x/a' is in I . Similarly, if $x/a' \in I$, then $x/a \in I$.

(iv) $x = x'$ and $a = a'$. It is clear that $x/a = x'/a'$ and $a \setminus x = a' \setminus x'$.

Now that we have shown that the congruence θ_I is compatible with residuals, one can take as the definition of quotient of residuals

$$\theta^b(y) \setminus \theta^b(x) := \theta^b(y \setminus x) \text{ and } \theta^b(x) / \theta^b(y) := \theta^b(x / y),$$

and then get the residuated semigroup $(P/\theta, \preceq, \otimes, (-) \setminus (-), (-) / (-))$ equipped with the projection $\theta^b : P \rightarrow P/\theta$, which is monotone, multiplicative and preserves residuals.

2.2. Characterisation of regular congruences on residuated semigroups

It turns out that the notion of order congruence (Definition 2.1) can be expressed using a special property of θ .

Proposition 2.7 ([23]). *A binary relation on a po-semigroup is an order congruence if and only if for any non negative integer n and for any sequence $a_1, a_2, \dots, a_{2n}$ in P ,*

$$a_1 \theta a_2 \leq a_3 \theta a_4 \leq \dots a_{2n-1} \theta a_{2n} \leq a_1 \Rightarrow \theta^b(a_1) = \theta^b(a_2) = \dots = \theta^b(a_{2n}).$$

This proposition extends in extension to regular congruence as defined in Definition 2.2. One can observe that

Proposition 2.8. *A regular congruence of a residuated semigroup P is exactly a congruence θ on the underlying po-semigroup that respects the taking of residuals, i.e., for all x, y and a in P $x \theta y$ implies $(x/a) \theta (y/a)$ and $(x/a) \theta (y/a)$.*

We get the following characterization:

Proposition 2.9. *A binary relation on a residuated semigroup is a regular congruence if and only if for any non negative integer n and for any sequence $a_1, a_2, \dots, a_{2n}$ in P ,*

$$a_1 \theta a_2 \leq a_3 \theta a_4 \leq \dots a_{2n-1} \theta a_{2n} \leq a_1 \Rightarrow \theta^b(a_1) = \theta^b(a_2) = \dots = \theta^b(a_{2n}).$$

Proof. Assuming θ to be a regular congruence is just to say that we get the ordered algebra $(P, F, \leq)$ in the sense of [5] where $F = \{\otimes, a \setminus -, -/a, a \in P\}$. Then we can apply the Proposition 2.1 of [5]. $\square$

Let θ be a binary relation on a poset $(P, \leq)$. A θ -**chain** from an element $x \in P$ to an element $y \in P$ is a finite sequence $x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n$ with

$$x \leq x_1 \theta x'_1 \leq x_2 \theta x'_2 \leq \dots \leq x_n \theta x'_n \leq y.$$

We also need to refine the notion of congruence.

Theorem 2.10. *Let θ be a regular congruence on a residuated semigroup P .*

(i) *The relation $\preceq_\theta$ on P as follows*

$$x \preceq_\theta y \text{ iff there exists } \theta\text{-chain from } x \text{ to } y$$

is a compatible preorder, i.e., it a reflexive and transive relation compatible with the order.

(ii) *The relation $\vee(\theta) = \preceq_\theta \cap \preceq_\theta^0$ is a regular congruence on P . It is regular congruence generated by θ .*

Proof. That the relation $\preceq$ is a preorder is proved exactly as in [23] (see proof of Theorem 5). It remains to check the compatibility with taking the residuals. If $x \leq x_1 \theta x'_1 \leq x_2 \theta x'_2 \leq \dots \leq x_n \theta x'_n \leq y$ is a θ -chain from x to y , then the sequence $(a \otimes x) \leq (a \otimes x_1) \theta (a \otimes x'_1) \leq (a \otimes x_2) \theta (a \otimes x'_2) \leq \dots \leq (a \otimes x_n) \theta (a \otimes x'_n) \leq (a \otimes y)$ is a θ -chain from $a \otimes x$ to

$a \otimes y$ and the sequence $(x \otimes a) \leq (x_1 \otimes a)\theta(x'_1 \otimes a) \leq (x_2 \otimes a)\theta(x'_2 \otimes a) \leq \dots \leq \theta(x'_n \otimes a) \leq (y \otimes a)$ is a θ -chain from $x \otimes a$ to $y \otimes a$; this comes from the fact that $\leq$ and θ are compatible with the multiplication. Similarly, one gets a θ -chain from $a \setminus x$ to $a \setminus y$ and a θ -chain from x/a to y/a by dividing on the left or on the right by a ; this follows from the fact that θ is compatible with the taking of the residuals and that the residuals $a \setminus -$ and $-/a$ are monotone maps.

The rest of the proof is a direct rewriting of proof of Theorem 5 of [23] but now using Proposition 2.9. $\square$

The last proposition can be used to characterise the regular congruence on a residuated semigroup generated by a Rees congruence induced by a residuated lattice.

Theorem 2.11 (The regular congruence generated by a residuated ideal).

Let I be residuated ideal of P . The regular congruence R_I generated by I is described as follow xR_Iy iff $x = y$ or there exist $b, b', c, c' \in I$ such that $x \leq b, b' \leq y$ and $y \leq c, c' \leq x$. The order on $P/\nu(\theta_I)$ is given by

$$\theta_I^b(x) \leq \theta_I^b(x) \Leftrightarrow (x \leq y) \text{ or } (x \leq a, a' \leq y, \text{ for some } a, a' \in I).$$

Proof. If we follow the proof of Theorem 6 of [23], it is enough to check that the binary relation $\nu(\theta_I)$ is compatible with taking the residuals, which is actually true since the maps $a \setminus -$ and $-/a$ are increasing maps.

Acknowledgement

The authors would like to thank an anonymous referee for the valuable suggestions and corrections.

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