COMBINATORIAL RESULTS FOR CERTAIN SEMIGROUPS OF ORDER-DECREASING PARTIAL ISOMETRIES OF A FINITE CHAIN

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Abstract

Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \dots, n\}$ and let $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ be its subsemigroups of order-decreasing partial isometries and of order-preserving and order-decreasing partial isometries of X_n , respectively. In this paper, we investigate the cardinalities of some equivalences on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, which lead naturally to obtaining the order of the semigroups.

1. Introduction

Let $X_n = \{1, 2, \dots, n\}$ and $\mathcal{I}_n$ be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then $\mathcal{I}_n$ is an *inverse* semigroup (that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). The importance of $\mathcal{I}_n$ (more commonly known as the *symmetric inverse semigroup or monoid*) to inverse semigroup theory may be likened to that of the symmetric group $\mathcal{S}_n$ to group theory. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, the analogue of Cayley's theorem for finite groups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of $\mathcal{I}_n$, see, for example, [3, 5, 6, 10, 14, 19, 20].

A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving* (*order-reversing*) if $(\forall x, y \in \text{Dom } \alpha) x \leq y \Rightarrow x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$) and, an *isometry* (or *distance-preserving*) if $(\forall x, y \in \text{Dom } \alpha) |x - y| = |x\alpha - y\alpha|$. We shall denote by $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, the semigroups of partial isometries and of order-preserving partial isometries of an n -chain, respectively. Eventhough semigroups of partial isometries on more restrictive but richer mathematical structures have been studied by Wallen [21], and Bracci and Picasso [4] the study of the corresponding semigroups on chains was only initiated recently by Al-Kharousi et al. [1, 2]. A little while later, Kehinde et al. [13] studied $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, the order-decreasing analogues of $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, respectively.

Analogous to Al-Kharousi et al. [2], this paper investigates the combinatorial properties of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, thereby complementing the results in Kehinde et al. [13] which dealt mainly with the algebraic and rank properties of these semigroups. In this section we introduce basic definitions and terminology as well as quote some elementary results from Section 1 of Al-Kharousi et al. [1] and Kehinde et al. [13] that will be needed in this paper. In Section 2, we obtain the cardinalities of two equivalences defined on $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$. These equivalences lead to formulae for the orders of $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$ as well as new triangles of numbers that were as a result of this work recently recorded in [18].

For standard concepts in semigroup and symmetric inverse semigroup theory, see, for example, [12, 16]. In particular $E(S)$ denotes the set of idempotents of S . Let

$$\mathcal{DDP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x \in \text{Dom } \alpha) x\alpha \leq x\}. \quad (1)$$

be the subsemigroup of $\mathcal{I}_n$ consisting of all order-decreasing partial isometries of X_n . Also let

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{DDP}_n : (\forall x, y \in \text{Dom } \alpha) x \leq y \Rightarrow x\alpha \leq y\alpha\}, \quad (2)$$

be the subsemigroup of $\mathcal{DDP}_n$ consisting of all order-preserving and order-decreasing partial isometries of X_n . Then we have the following result.

Lemma 1.1. *$\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are subsemigroups of $\mathcal{I}_n$.*

Remark 1.2. $\mathcal{DDP}_n = \mathcal{DP}_n \cap \mathcal{I}_n^-$ and $\mathcal{ODDP}_n = \mathcal{ODP}_n \cap \mathcal{I}_n^-$, where $\mathcal{I}_n^-$ is the semigroup of partial one-to-one order-decreasing transformations of X_n [19].

Next, let α be an arbitrary element in $\mathcal{I}_n$. The *height or rank* of α is $h(\alpha) = |\text{Im } \alpha|$, the *right [left] waist* of α is $w^+(\alpha) = \max(\text{Im } \alpha)[w^-(\alpha)]$

$= \min(\text{Im } \alpha)$], the *right* [*left*] *shoulder* of α is $\varpi^+(\alpha) = \max(\text{Dom } \alpha)$ [$\varpi(\alpha) = \min(\text{Dom } \alpha)$], and *fix* of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |F(\alpha)|$, where

$$F(\alpha) = \{x \in X_n : x\alpha = x\}.$$

Next we quote some parts of ([1], Lemma 1.2) that will be needed as well as state some additional observations that will help us understand more the cycle structure of order-decreasing partial isometries.

Lemma 1.3. *Let $\alpha \in \mathcal{DP}_n$. Then we have the following:*

(a) *The map α is either order-preserving or order-reversing. Equivalently, α is either a translation or a reflection.*

(b) *If $f(\alpha) = p > 1$, then $f(\alpha) = h(\alpha)$. Equivalently, if $f(\alpha) > 1$, then α is a partial identity.*

(c) *If α is order-preserving and $f(\alpha) \geq 1$, then α is a partial identity.*

(d) *If α is order-preserving then it is either strictly order-decreasing ($x\alpha < x$ for all x in $\text{Dom } \alpha$) or strictly order-increasing ($x\alpha > x$ for all x in $\text{Dom } \alpha$) or a partial identity.*

(e) *If $F(\alpha) = \{i\}$ (for $1 \leq i \leq n$), then for all $x \in \text{Dom } \alpha$ we have that $x + x\alpha = 2i$.*

(f) *If α is order-decreasing and $i \in F(\alpha)$ ($1 \leq i \leq n$), then for all $x \in \text{Dom } \alpha$ such that $x < i$ we have $x\alpha = x$.*

(g) *If α is order-decreasing and $F(\alpha) = \{i\}$, then $\text{Dom } \alpha \subseteq \{i, i+1, \dots, n\}$.*

2. Combinatorial Results

For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups, we refer the reader to Umar [20].

As in Umar [20], for natural numbers $n \geq p \geq m \geq 0$ and $n \geq i \geq 0$, we define

$$F(n; p_i) = |\{\alpha \in S : h(\alpha) = |\text{Im } \alpha| = i\}|, \quad (3)$$

$$F(n; m_i) = |\{\alpha \in S : f(\alpha) = i\}|, \quad (4)$$

where S is any subsemigroup of $\mathcal{I}_n$. From ([2], Proposition 2.4), we have

Theorem 2.1. *Let $S = \mathcal{ODP}_n$. Then $F(n; p) = \frac{(2n-p+1)}{p+1} \binom{n}{p}$,*

where $n \geq p \geq 1$.

We now have

Proposition 2.2. *Let $S = \mathcal{ODDP}_n$. Then $F(n; p) = \binom{n+1}{p+1}$, where*

$n \geq p \geq 1$.

Proof. By virtue of Lemma 1.3 (d) and Theorem 2.1, we see that

$$\begin{aligned} F(n; p) &= \frac{1}{2} \left[\frac{2n-p+1}{p+1} \binom{n}{p} - \binom{n}{p} \right] + \binom{n}{p} \\ &= \frac{1}{2} \left[\frac{2(n-p)}{p+1} \binom{n}{p} \right] + \binom{n}{p} \\ &= \frac{n-p}{p+1} \binom{n}{p} + \binom{n}{p} = \binom{n}{p+1} + \binom{n}{p} = \binom{n+1}{p+1}. \end{aligned}$$

□

The proof of the next lemma is routine using Proposition 2.2.

Lemma 2.3. *Let $S = \mathcal{ODDP}_n$. Then $F(n; p) = F(n-1; p-1) + F(n-1; p)$, for all $n \geq p \geq 2$.*

Theorem 2.4. $|\mathcal{ODDP}_n| = 2^{n+1} - (n+1)$.

Proof. It is enough to observe that $|\mathcal{ODDP}_n| = \sum_{p=0}^n F(n; p)$.

Lemma 2.5. *Let $S = \mathcal{ODDP}_n$. Then $F(n; m) = \binom{n}{m}$, for all*

$n \geq m \geq 1$.

Proof. It follows directly from Lemma 1.3 (b), (c) and the fact that all idempotents are necessarily order-decreasing. $\square$

Proposition 2.6. *Let $U_n = \{\alpha \in \mathcal{ODDP}_n : f(\alpha) = 0\}$. Then $|U_n| = |\mathcal{ODDP}_{n-1}|$.*

Proof. The proof is similar to that of ([19], Theorem 4.3). $\square$

Remark 2.7. The triangles of numbers $F(n; p)$ and $F(n; m)$, have as a result of this work appeared in Sloane [18] as [A184049] and [A184050], respectively.

Now we turn our attention to counting order-reversing partial isometries. First recall from ([13], Section 3.2 (c)) that order-decreasing and order-reversing partial isometries exist only for heights less than or equal to $n/2$. We now have

Lemma 2.8. *Let $S = \mathcal{DDP}_n^*$ be the set of order-reversing partial isometries of X_n . Then $F(n; p_0) = 1$ and $F(n; p_1) = \binom{n+1}{2}$, for all $n \geq 1$.*

Proof. These follow from the simple observation that

$$\{\alpha \in \mathcal{ODDP}_n : h(\alpha) = 0 \text{ or } 1\} = \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = 0 \text{ or } 1\}$$

and Proposition 2.2. $\square$

Lemma 2.9. *Let $\alpha \in \mathcal{DDP}_n^*$. Then for all $p \geq 1$ we have*

$$F(2p+1, p+1) = 1 \text{ and } F(2p, p) = 3.$$

Proof. (i) By Lemma 1.3 (f), (g), we see that $i \in \{0, 1, \dots, p\}$, $\begin{pmatrix} p+1+i \\ p+1-i \end{pmatrix}$

is the unique order-reversing isometry of height $p+1$; and (ii) for

$i \in \{0, 1, \dots, p-1\}$, $\begin{pmatrix} p+i \\ p-i \end{pmatrix}$, $\begin{pmatrix} p+1+i \\ p-i \end{pmatrix}$, and $\begin{pmatrix} p+1+i \\ p+1-i \end{pmatrix}$ are the only

order-reversing isometries of height p . $\square$

The following technical lemma will be useful later.

Lemma 2.10. *Let $\alpha \in \mathcal{DDP}_n^*$. Suppose $\varpi^+(\alpha) - r \in \text{Dom } \alpha$ and $\varpi^+(\alpha) - s \notin \text{Dom } \alpha$ for all $1 \leq s < r$. Then $\varpi(\alpha) > r$.*

Proof. By order-reversing we see that $(\varpi^+(\alpha))\alpha = w^-(\alpha)$ and $(\varpi(\alpha))\alpha = w^+(\alpha)$. Thus $\varpi^+(\alpha) - r \geq \varpi(\alpha) \Rightarrow \varpi^+(\alpha) - \varpi(\alpha) \geq r$. So by isometry we have $w^+(\alpha) - w^-(\alpha) = \varpi^+(\alpha) - \varpi(\alpha) \geq r \Rightarrow w^+(\alpha) \geq w^-(\alpha) + r \Rightarrow w^+(\alpha) > r \Rightarrow \varpi(\alpha) > r$, as required. $\square$

Lemma 2.11. *Let $S = \mathcal{DDP}_n^*$. Then $F(n; p) = F(n-2; p-1) + F(n-2; p)$, for all $n \geq p \geq 2$.*

Proof. Let $\alpha \in \mathcal{DDP}_n^*$ and $h(\alpha) = p$. Define $A = \{\alpha \in \mathcal{DDP}_{n-2}^* : h(\alpha) = p\}$ and $B = \{\alpha \in \mathcal{DDP}_{n-2}^* : h(\alpha) = p-1\}$. Clearly, $A \cap B = \emptyset$. Define a map $\theta : \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = p\} \rightarrow A \cup B$ by $(\alpha)\theta = \alpha'$, where

(i) $x\alpha' = x\alpha(x \in \text{Dom } \alpha)$, if $\alpha \in A$. It is clear that α' is an order-decreasing isometry and $h(\alpha) = p$;

(ii) if $\{n-1, n\} \subseteq \text{Dom } \alpha$ and $\alpha \in B$, let $\text{Dom } \alpha' = \{x-1 : x \in \text{Dom } \alpha \text{ and } x < n\}$ and $(x-1)\alpha' = x\alpha - 1 \leq x-1$ and so α' is order-decreasing and $h(\alpha) = p-1$;

(iii) if $\{n-2, n-1\} \subseteq \text{Dom } \alpha$ and $\alpha \in B$, let $\text{Dom } \alpha' = \{x-1 : x \in \text{Dom } \alpha \text{ and } x < n-1\}$ and $(x-1)\alpha' = x\alpha - 1 \leq x-1$ and so α' is order-decreasing and $h(\alpha) = p-1$;

(iv) otherwise, if $\alpha \in B$, let $\text{Dom } \alpha' = \{x-r : x \in \text{Dom } \alpha \text{ and } x < \varpi^+(\alpha)\}$, where r is such that $\varpi^+(\alpha) - r \in \text{Dom } \alpha$ and $\varpi^+(\alpha) - s \notin \text{Dom } \alpha$ for all $1 \leq s < r$. Define $(x-r)\alpha' = x\alpha - r \leq x-r$ and so α' is order-decreasing and Lemma 2.10 ensures that $h(\alpha) = p-1$.

Moreover, in (ii) and (iii), we have $|(x-1)\alpha' - (y-1)\alpha'| = |(x\alpha - 1) - (y\alpha - 1)| = |x\alpha - y\alpha| = |x - y| = |(x-1) - (y-1)|$, and in (iv), we have $|(x-r)\alpha' - (y-r)\alpha'| = |(x\alpha - r) - (y\alpha - r)| = |x\alpha - y\alpha| = |x - y| = |(x-r) - (y-r)|$. Hence α' is an isometry.

Also observe that in (ii), we have $\varpi^+(\alpha') = n-2$; in (iii) we have $\varpi^+(\alpha') = n-3$; and in (iv) we have $\varpi^+(\alpha') < n-3$. These observations coupled with the definitions of α' ensures that θ is a bijection.

To show that θ is onto it is enough to note that we can in a symmetric manner define θ^{-1} from $A \cup B \rightarrow \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = p\}$. This establishes the statement of the lemma. $\square$

The next lemma which can be proved by induction, is necessary.

Lemma 2.12. *Let $S = \mathcal{DDP}_n^*$. Then we have the following:*

$$\sum_{i \geq 0} \binom{n-1-2i}{2} = \begin{cases} \frac{(n+1)(n-1)(2n-3)}{24}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(2n+1)}{24}, & \text{if } n \text{ is even.} \end{cases}$$

Lemma 2.13. *Let $S = \mathcal{DDP}_n^*$. Then we have the following:*

$$F(n; p_2) = \begin{cases} \frac{(n+1)(n-1)(2n-3)}{24}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(2n+1)}{24}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. By applying Lemmas 2.8 and 2.11 successively, we get

$$\begin{aligned}
 F(n; p_2) &= F(n-2; p_1) + F(n-2; p_2) = F(n-2; p_2) + \binom{n-1}{2} \\
 &= F(n-4; p_2) + \binom{n-3}{2} + \binom{n-1}{2} \\
 &= F(n-6; p_2) + \binom{n-5}{2} + \binom{n-3}{2} + \binom{n-1}{2}.
 \end{aligned}$$

By iteration the result follows from Lemma 2.12 and the facts that

$$F(2; p_2) = 0 \text{ and } F(3; p_2) = 1 = \binom{2}{2}. \quad \square$$

Proposition 2.14. *Let $S = \mathcal{DDP}_n^*$. Then for all $\lfloor (n+1)/2 \rfloor \geq p \geq 1$, we have*

$$F(n; p) = \begin{cases} \frac{(n+1)(n-1)(n-3)\cdots(n-2p+3)(2n-3p+3)}{2^p(p+1)!}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(n-4)\cdots(n-2p+2)(2n-p+3)}{2^p(p+1)!}, & \text{if } n \text{ is even.} \end{cases}$$

Proof (By Induction).

$$\text{Basis step: } F(n; p_1) = \binom{n+1}{1+1} = \binom{n+1}{2} \text{ is true by Lemma 2.8 and the}$$

observation made in its proof, while the formula for $F(n; p_2)$ is true by Lemma 2.13.

Inductive step: Suppose $F(m; p)$ is true for all $\lfloor (n+1)/2 \rfloor > m \geq p \geq 1$.

Case 1. If m is odd, consider (using the induction hypothesis)

$$\begin{aligned}
 F(m+2; p) &= F(m; p) + F(m; p-1) \\
 &= \frac{(m+1)(m-1)(m-3)\cdots(m-2p+3)(2m-3p+3)}{2^p(p+1)!} \\
 &\quad + \frac{(m+1)(m-1)(m-3)\cdots(m-2p+5)(2m-3p+6)}{2^{p-1}p!} \\
 &= \frac{(m+3)(m+1)(m-1)\cdots(m-2p+5)(2m-3p+7)}{2^p(p+1)!},
 \end{aligned}$$

which is the formula for $F(m+2; p)$ when m is odd.

Case 2. If m is even, consider (using the induction hypothesis)

$$\begin{aligned}
 F(m+2; p) &= F(m; p) + F(m; p-1) \\
 &= \frac{m(m-2)(m-4)\cdots(m-2p+2)(2m-p+3)}{2^p(p+1)!} \\
 &\quad + \frac{m(m-2)(m-4)\cdots(m-2p+4)(2m-p+4)}{2^{p-1}p!} \\
 &= \frac{(m+2)m(m-2)\cdots(m-2p+4)(2m-p+7)}{2^p(p+1)!},
 \end{aligned}$$

which is the formula for $F(m+2; p)$ when m is even. $\square$

Proposition 2.15. Let $S = \mathcal{DDP}_n^*$ and let $b_n = \sum_{p \geq 0} F(n; p)$. Then for $n \geq 0$, we have

$$(1) \quad b_{2n+1} = 5 \cdot 2^{n+1} - 4n - 8;$$

$$(2) \quad b_{2n} = 7 \cdot 2^n - 4n - 6.$$

Proof. Apply induction and use the fact that $|\mathcal{DDP}_n^*| = \sum_{p=0}^n F(n; p)$.

Proposition 2.16. *Let $S = \mathcal{DDP}_n$. Then*

(1) *if n is odd and $(n+1)/2 \geq p \geq 2$*

$$F(n; p) = \frac{(n+1)(n-1)(n-3)\cdots(n-2p+3)(2n-3p+3)}{2^p(p+1)!} + \binom{n+1}{p+1};$$

(2) *if n is even and $n/2 \geq p \geq 2$*

$$F(n; p) = \frac{n(n-2)(n-4)\cdots(n-2p+2)(2n-p+3)}{2^p(p+1)!} + \binom{n+1}{p+1};$$

(3) *if $\lfloor (n+1)/2 \rfloor < p$, $F(n; p) = \binom{n+1}{p+1}$.*

Proof. It follows from Propositions 2.2 & 2.14 and Lemmas 1.3 (c) & 2.8. $\square$

Combining Theorem 2.4, Lemmas 1.3 (a), (c) & 2.9, Proposition 2.15 and the observation made in the proof of Lemma 2.8 we get the order of $\mathcal{DDP}_n$ which we record as a theorem below.

Theorem 2.17. *Let $\mathcal{DDP}_n$. Then for all $n \geq 0$, we have*

$$(1) |\mathcal{DDP}_{2n+1}| = 2^{2n+2} + 5 \cdot 2^{n+1} - (2n^2 + 9n + 12);$$

$$(2) |\mathcal{DDP}_{2n}| = 2^{2n+1} + 7 \cdot 2^n - (2n^2 + 7n + 8).$$

Lemma 2.18. *Let $S = \mathcal{DDP}_n$. Then $F(n; m) = \binom{n}{m}$, for all $n \geq m \geq 2$.*

Proof. It follows directly from ([13], Lemma 3.18) and the fact that all idempotents are necessarily order-decreasing. $\square$

Proposition 2.19. *Let $S = \mathcal{DDP}_n$. Then $F(2n; m_1) = 2^{n+1} - 2$ and $F(2n-1; m_1) = 3 \cdot 2^{n-1} - 2$, for all $n \geq 1$.*

Proof. Let $F(\alpha) = \{i\}$. Then by Lemma 1.3 (e), for any $x \in \text{Dom } \alpha$ we have $x + x\alpha = 2i$. Thus, by Lemma 1.3 (g), there $2i - 2$ possible elements for $\text{Dom } \alpha : (x, x\alpha) \in \{(i, i), (i + 1, i - 1), (i + 2, i - 2), \dots, (2i - 1, 1)\}$. However, (excluding (i, i)) we see that there are

$$\sum_{j=0}^{i-1} \binom{i-1}{j} = 2^{i-1}, \text{ possible partial isometries with } F(\alpha) = \{i\}, \text{ where}$$

$2i - 1 \leq n \Leftrightarrow i \leq (n + 1)/2$. Moreover, by symmetry, we see that $F(\alpha) = \{i\}$ and $F(\alpha) = \{n - i + 1\}$ give rise to equal number of decreasing partial isometries. Note that if n is odd (even) the equation $i = n - i + 1$ has one (no) solution. Hence, if $n = 2a - 1$ we have

$$2 \sum_{i=1}^{a-1} 2^{i-1} + 2^{a-1} = 2(2^{a-1} - 1) + 2^{a-1} = 3 \cdot 2^{a-1} - 2$$

decreasing partial isometries with exactly one fixed point; if $n = 2a$ we have

$$2 \sum_{i=1}^a 2^{i-1} = 2(2^a - 1) = 2^{a+1} - 2$$

decreasing partial isometries with exactly one fixed point. $\square$

Theorem 2.20. *Let $\mathcal{DDP}_n$. Then*

$$a_n = |\mathcal{DDP}_n| = 3a_{n-1} - 2a_{n-2} - 2^{\lfloor \frac{n}{2} \rfloor} + n + 1,$$

with $a_0 = 1$ and $a_{-1} = 0$.

Proof. It follows from Propositions 2.6 & 2.19, Lemma 2.18 and the fact that $|\mathcal{DDP}_n| = \sum_{m=0}^n F(n; m)$. $\square$

Remark 2.21. The triangle of numbers $F(n; m)$ and sequence $|\mathcal{DDP}_n|$ have as a result of this work appeared in Sloane [18] as [A184051] and [A184052], respectively. However, the triangles of

numbers $F(n; p)$ for $\mathcal{DDP}_n$ and $\mathcal{DDP}_n^*$ and the sequence $|\mathcal{DDP}_n^*|$ are as at the time of submitting this paper not in Sloane [18]. For some computed values of $F(n; p)$, see Tables 3.1 and 3.2.

Table 3.1

$n \setminus p$	0	1	2	3	4	5	6	7	$\sum F(n; p) = \mathcal{DDP}_n^* $
0	1								1
1	1	1							2
2	1	3	0						4
3	1	6	1	0					8
4	1	10	3	0	0				14
5	1	15	7	1	0	0			24
6	1	21	13	3	0	0	0		38
7	1	28	22	8	1	0	0	0	60

Table 3.2

$n \setminus p$	0	1	2	3	4	5	6	7	$\sum F(n; p) = \mathcal{DDP}_n $
0	1								1
1	1	1							2
2	1	3	1						5
3	1	6	5	1					13
4	1	10	13	5	1				30
5	1	15	27	16	6	1			66
6	1	21	48	38	21	7	1		137
7	1	28	78	78	57	28	8	1	279

3. Number of $\mathcal{D}^*$ -classes

For the definitions of the Green's relations ($\mathcal{L}$, $\mathcal{R}$, and $\mathcal{D}$) and their starred analogues ($\mathcal{L}^*$, $\mathcal{R}^*$, and $\mathcal{D}^*$), we refer the readers to Howie [12] and Fountain [8], (respectively) or Ganyushkin and Mazorchuk [9].

First, notice that from ([1], Lemma 2.1), we deduce that number of $\mathcal{L}$ -classes in $K(n, p) = \{\alpha \in \mathcal{DP}_n : h(\alpha) = p\}$ (as well as the number of

$\mathcal{R}$ -classes there) is $\binom{n}{p}$. To describe the $\mathcal{D}$ -classes in $\mathcal{DP}_n$ and $\mathcal{ODP}_n$,

first we recall (from [1]) that the *gap* and *reverse gap of the image set of α* (with $h(\alpha) = p$) are ordered $(p-1)$ -tuples defined as follows:

$$g(\text{Im } \alpha) = (|a_2\alpha - a_1\alpha|, |a_3\alpha - a_2\alpha|, \dots, |a_p\alpha - a_{p-1}\alpha|),$$

and

$$g^R(\text{Im } \alpha) = (|a_p\alpha - a_{p-1}\alpha|, \dots, |a_3\alpha - a_2\alpha|, |a_2\alpha - a_1\alpha|),$$

where $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$ with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$.

Further, let $d_i = |a_{i+1}\alpha - a_i\alpha|$ for $i = 1, 2, \dots, p-1$. Then

$$g(\text{Im } \alpha) = (d_1, d_2, \dots, d_{p-1}) \text{ and } g^R(\text{Im } \alpha) = (d_{p-1}, d_{p-2}, \dots, d_1).$$

For example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 7 & 8 \\ 3 & 4 & 6 & 9 & 10 \end{pmatrix}, \beta = \begin{pmatrix} 2 & 4 & 7 & 8 \\ 10 & 8 & 5 & 4 \end{pmatrix} \in \mathcal{DP}_{10},$$

then $g(\text{Im } \alpha) = (1, 2, 3, 1)$, $g(\text{Im } \beta) = (2, 3, 1)$, $g^R(\text{Im } \alpha) = (1, 3, 2, 1)$ and $g^R(\text{Im } \beta) = (1, 3, 2)$. Next, let $d(n, p)$ be the number of distinct ordered p -tuples: $(d_1, d_2, \dots, d_p)$ with $\sum_{i=1}^p d_i = n$. This is clearly the number of *compositions* of n into p parts. Thus, we have

Lemma 3.1 ([17, p.151]). $d(n, p) = \binom{n-1}{p-1}$.

We shall henceforth use the following well-known binomial identity when needed:

$$\sum_{m=p}^n \binom{m}{p} = \binom{n+1}{p+1}.$$

We take this opportunity to state and prove a result which was omitted in [2].

Theorem 3.2. *Let $S = \mathcal{ODP}_n$. Then*

(1) *the number of $\mathcal{D}$ -classes in $K(n, p) (p \geq 1)$ is $\binom{n-1}{p-1}$;*

(2) *the number of $\mathcal{D}$ -classes in S is $1 + 2^{n-1}$.*

Proof.

(1) It follows from ([1], Theorem 2.5): $(\alpha, \beta) \in \mathcal{D}$ if and only if $g(\text{Im } \alpha) = g(\text{Im } \beta)$; ([1], Lemma 3.3): $p-1 \leq \sum_{i=1}^{p-1} d_i \leq n-1$; Lemma 3.1; and so the number of $\mathcal{D}$ -classes is $\sum_{i=p-1}^{n-1} d(i, p-1) = \sum_{i=p-1}^{n-1} \binom{i-1}{p-2} = \binom{n-1}{p-1}$.

(2) The number of $\mathcal{D}$ -classes in S is $1 + \sum_{p=1}^n \binom{n-1}{p-1} = 1 + 2^{n-1}$. $\square$

The following results from [13] will be needed:

Lemma 3.3 ([13], Lemma 2.3). *Let $\alpha, \beta \in \mathcal{DDP}_n$ or $\mathcal{ODDP}_n$. Then*

(1) $\alpha \leq_{\mathcal{R}^*} \beta$ *if and only if* $\text{Dom } \alpha \subseteq \text{Dom } \beta$;

(2) $\alpha \leq_{\mathcal{L}^*} \beta$ *if and only if* $\text{Im } \alpha \subseteq \text{Im } \beta$;

(3) $\alpha \leq_{\mathcal{H}^*} \beta$ *if and only if* $\text{Dom } \alpha \subseteq \text{Dom } \beta$ *and* $\text{Im } \alpha \subseteq \text{Im } \beta$.

From ([13], (3)), for $\alpha, \beta \in \mathcal{DDP}_n$, we have $(\alpha, \beta) \in \mathcal{D}^*$ if and only if

$$g(\text{Im } \alpha) = \begin{cases} g(\text{Im } \beta); & \text{or} \\ g^R(\text{Im } \beta), & \text{if } p \leq a_p - a_1 \leq (n-1)/2. \end{cases} \quad (5)$$

Similarly, from ([13], (4)), for $\alpha, \beta \in \mathcal{ODDP}_n$, we have

$$(\alpha, \beta) \in \mathcal{D}^* \text{ if and only if } g(\text{Im } \alpha) = g(\text{Im } \beta). \quad (6)$$

Now a corollary of Theorem 3.2 follows:

Corollary 3.4. *Let $S = \mathcal{ODDP}_n$. Then*

$$(1) \text{ the number of } \mathcal{D}^* \text{-classes in } K(n, p) (p \geq 1) \text{ is } \binom{n-1}{p-1};$$

$$(2) \text{ the number of } \mathcal{D}^* \text{-classes in } S \text{ is } 1 + 2^{n-1}.$$

Observe that for all $\alpha \in \mathcal{DP}_n$ with $h(\alpha) = p$,

$$a_p - a_1 = \sum_{i=1}^{p-1} (a_{i+1} - a_i) = \sum_{i=1}^{p-1} d_i, \quad (7)$$

where $g(\text{Dom } \alpha) = (d_1, d_2, \dots, d_{p-1})$. Moreover, an ordered p -tuple: $(d_1, d_2, \dots, d_p)$ is said to be *symmetric* if

$$(d_1, d_2, \dots, d_p) = (d_1, d_2, \dots, d_p)^R = (d_p, d_{p-1}, \dots, d_1).$$

Now, let $d_s(n, p)$ be the number of distinct symmetric ordered p -tuples:

$(d_1, d_2, \dots, d_p)$ with $\sum_{i=1}^p d_i = n$. Then we have

Lemma 3.5 ([2, Lemma 3.5]).

$$d_s(n; p) = \begin{cases} 0, & \text{if } n \text{ is odd and } p \text{ is even;} \\ \left\lfloor \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor, & \text{otherwise.} \end{cases}$$

Now by virtue of (5) and ([1], Theorem 2.5), it is not difficult to see that the number of $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$ is the same as the number of $\mathcal{D}$ -classes in $\mathcal{ODP}_n$ less those pairs that are merged into single $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$. Thus, we have

Lemma 3.6. *Let $g(m, p)$ be the number of $\mathcal{D}$ -classes in $\mathcal{ODP}_n$ (consisting of maps of height p and $\sum d_i = m$) that are merged into single $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$. Then $m \leq (n-1)/2$, and*

$$g(m, p) = \begin{cases} \frac{1}{2} \binom{m-1}{p-2}, & \text{if } n \text{ is odd and } p \text{ is odd;} \\ \frac{1}{2} \left[\binom{m-1}{p-2} - \left\lfloor \frac{m-1}{2} \right\rfloor \right], & \text{otherwise.} \end{cases}$$

Proof. The result follows from (5), Lemmas 3.1 & 3.5 and the observation that

$$g(n, p) = \frac{d(n-1, p-1) - d_s(n-1, p-1)}{2}.$$

□

Now we have the main result of this section.

Theorem 3.7. *Let $B(n, p)$ be the number of $\mathcal{D}$ -classes in $\mathcal{ODP}_n$ (consisting of maps of height p) that are merged into single $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$. Then for $n \geq p \geq 1$, we have*

$$B(n, p) = \begin{cases} \frac{1}{2} \left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - \binom{\lfloor \frac{n-1}{4} \rfloor}{\frac{p-1}{2}} \right], & \text{if } p \text{ is odd;} \\ \frac{1}{2} \left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - 2 \binom{\lfloor \frac{n-1}{4} \rfloor}{\frac{p}{2}} \right], & \text{if } n \equiv 1, 2(\text{mod } 4), \text{ \& } p \text{ is even;} \\ \frac{1}{2} \left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - 2 \binom{\lfloor \frac{n-3}{4} \rfloor}{\frac{p}{2}} - \binom{\lfloor \frac{n-3}{4} \rfloor}{\frac{p-2}{2}} \right], & \text{if } n \equiv -1, 0(\text{mod } 4), \text{ \& } p \text{ is even.} \end{cases}$$

Proof. The result follows from (5), (7) and Lemma 3.6. To see this, let $n \equiv 0 \pmod{4}$ and p be even. Then $n = 4k$ for some integer k , and

$$\begin{aligned}
B(n, p) &= \sum_{m=p}^{\lfloor \frac{n-1}{2} \rfloor} g(m, p) = \sum_{m=p}^{2k-1} g(m, p) \\
&= g(p, p) + g(p+2, p) + \cdots + g(2k-2, p) \\
&\quad + g(p+1, p) + g(p+3, p) + \cdots + g(2k-1, p) \\
&= \frac{1}{2} \left[\binom{p-1}{p-2} - \binom{\frac{p-2}{2}}{\frac{p-2}{2}} + \binom{p+1}{p-2} - \binom{\frac{p}{2}}{\frac{p-2}{2}} + \cdots + \binom{2k-3}{p-2} - \binom{k-2}{\frac{p-2}{2}} \right] \\
&\quad + \frac{1}{2} \left[\binom{p}{p-2} - \binom{\frac{p}{2}}{\frac{p-2}{2}} + \binom{p+2}{p-2} - \binom{\frac{p+2}{2}}{\frac{p-2}{2}} + \cdots + \binom{2k-2}{p-2} - \binom{k-1}{\frac{p-2}{2}} \right] \\
&= \frac{1}{2} \left[\binom{2k-1}{p-1} - 2 \binom{k-1}{\frac{p}{2}} - \binom{k-1}{\frac{p-2}{2}} \right] \\
&= \frac{1}{2} \left[\binom{\frac{n-2}{2}}{p-1} - 2 \binom{\frac{n-4}{4}}{\frac{p}{2}} - \binom{\frac{n-4}{4}}{\frac{p-2}{2}} \right].
\end{aligned}$$

All the other cases are handled similarly. $\square$

Now we have the main result of this section.

Corollary 3.8. *The number of $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$ (consisting of maps of height $p \geq 1$) is $\binom{n-1}{p-1} - B(n, p)$.*

Proof. The result follows from Theorem 3.7 and the remarks preceding Lemma 3.6. $\square$

Corollary 3.9. *The number of $\mathcal{D}^*$ -classes in $\mathcal{DDP}_n$ denoted by d_n is*

$$d_n = \begin{cases} 2^{n-1} - 2^{\lfloor \frac{n-3}{2} \rfloor} + 2^{\lfloor \frac{n+1}{4} \rfloor}, & \text{if } n \equiv -1, 0(\text{mod } 4); \\ 2^{n-1} - 2^{\lfloor \frac{n-3}{2} \rfloor} + 3 \cdot 2^{\lfloor \frac{n-3}{4} \rfloor}, & \text{if } n \equiv 1, 2(\text{mod } 4). \end{cases}$$

Proof. The result follows from Theorem 3.7 and Corollary 3.8. To see this, let $n \equiv 1, 2(\text{mod } 4)$. Then $n = 4k + 1, 4k + 2$ for some integer k , and

$$\begin{aligned} d_n &= 1 + \sum_{p=1}^n \binom{n-1}{p-1} - \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} B(n, p) = 1 + 2^{n-1} - \sum_{p=1}^{2k} B(n, p) \\ &= 1 + 2^{n-1} - [B(n, 1) + B(n, 3) + \cdots + B(n, 2k-1)] \\ &\quad - [B(n, 2) + B(n, 4) + \cdots + B(n, 2k)] \\ &= 1 + 2^{n-1} - \frac{1}{2} \left[\binom{2k}{0} - \binom{k}{0} + \binom{2k}{2} - \binom{k}{1} + \cdots + \binom{2k}{2k-2} - \binom{k}{k-1} \right] \\ &\quad - \frac{1}{2} \left[\binom{2k}{1} - 2 \binom{k}{1} + \binom{2k}{3} - 2 \binom{k}{2} + \cdots + \binom{2k}{2k-1} - 2 \binom{k}{k} \right] \\ &= 1 + 2^{n-1} - \frac{1}{2} \left[(2^{2k} - 1) - (3 \cdot 2^k + 1) + 2 \right] \\ &= 2^{n-1} - 2^{\lfloor \frac{n-3}{2} \rfloor} + 3 \cdot 2^{\lfloor \frac{n-3}{4} \rfloor}. \end{aligned}$$

The case $n \equiv -1, 0(\text{mod } 4)$ is handled similarly. $\square$

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