ON ROW-FACTORIZATION MATRICES

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Abstract

In this paper, we study the determinant of RF-matrices for pseudo-Frobenius numbers in a numerical semigroup.

1. Introduction

It is very basic and significant to investigate semigroups in the set of non negative integers. To describe how a given semigroup is, there are lots of definitions for semigroups such as Frobenius number, genus, type, and so on. To compute them, there are tools such as Apéry sets (see [7]). In 2016, we get one more tool, a row-factorization matrix, in short an RF-matrix, given by Moscariello [6]. He use it to prove that the upper bound of the type of almost symmetric semigroup of embedding dimension four is three. However, properties of RF-matrices are not sufficiently revealed. So we post a question whether there exists an RF-matrix for the Frobenius number whose determinant is coincident with it

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up to sign. We show the reason why this question is important and a partial answer for it in the latter half of this section. In the next section, we prove it.

Now we start with definitions. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{N}$ be the set of non negative integers.

1.1. Numerical semigroups. We consider a semigroup S in $\mathbb{N}$, that is, S is a subset of $\mathbb{N}$, containing 0, and closed under usual addition. If there are $s > 0$ and $n_1, \dots, n_s \in S$ satisfying

$$S = \mathbb{N}n_1 + \dots + \mathbb{N}n_s = \{a_1n_1 + \dots + a_sn_s : a_1, \dots, a_s \in \mathbb{N}\},$$

we say that S is generated by $n_1, \dots, n_s$. We also say that S is minimally generated by $n_1, \dots, n_s$, if any proper subset of $\{n_1, \dots, n_s\}$ does not generate S . In this case, we say that s is the embedding dimension of S and denote S by $\langle n_1, \dots, n_s \rangle$. If $\mathbb{N} - S$ is finite, we say that S is numerical. Then $\langle n_1, \dots, n_s \rangle$ is numerical if and only if the general common divisor of $n_1, \dots, n_s$ is one.

For $d \in S = \langle n_1, \dots, n_s \rangle$, if there is the unique expression $d = \sum_i a_i n_i$, where $a_i \in \mathbb{N}$ for each i , then we say that d has UF (a unique factorization). For each i , we denote the minimal positive number d such that dn_i does not have UF, by α_i .

1.2. Frobenius number, pseudo-Frobenius numbers. Let S be a numerical semigroup. The number

$$F(S) = \max\{a \in \mathbb{Z} : a \notin S\},$$

is called the Frobenius number of S . We also define

$$PF(S) = \{a \in \mathbb{Z} \setminus S : a + x \in S \text{ if } x \in S \text{ and } x \neq 0\},$$

and an element in $\text{PF}(S)$ is called a pseudo-Frobenius number. Obviously, the Frobenius number is a pseudo-Frobenius number. We say that the number of $\text{PF}(S)$ is the type of S , denoted by $t(S)$. If $t(S) = 1$, then S is called symmetric. Note that S is symmetric if and only if the following condition is satisfied:

$$x \in S \quad \Leftrightarrow \quad F(S) - x \notin S.$$

1.3. RF-Matrices. Let $S = \langle n_1, \dots, n_s \rangle$ be a numerical semigroup and $d \in S$. We define the Apéry set $\text{Ap}(S, d)$ as follows:

$$\text{Ap}(S, d) = \{x \in S : x - d \notin S\}.$$

Note $|\text{Ap}(S, d)| = d$ and, for any $a \in \mathbb{Z}$, there is $x \in \text{Ap}(S, d)$ with $a \equiv x \pmod{d}$. Let $f \in \mathbb{Z} - S$. For each i , there is $a_{ii} < 0$ with $f - a_{ii}n_i \in \text{Ap}(S, n_i)$. Then there are $a_{ij} \geq 0$ for $j \neq i$ satisfying $f - a_{ii}n_i = \sum_{j \neq i} a_{ij}n_j$. We say that the matrix $\text{RF}(f) = (a_{ij})$ is an RF-matrix (row-factorization matrix) for f in S . We denote the ij -entry of $\text{RF}(f)$ by $\text{RF}_{ij}(f)$, and $\text{RF}(F(S))$ by $\text{RF}(S)$. Note that an RF-matrix is not unique.

1.4. Associated semigroup ring. Let $S = \langle n_1, \dots, n_s \rangle$ be a numerical semigroup. Put $\varphi_S : \mathbb{Z}^s = \bigoplus_{i=1}^s \mathbb{Z}e_i \rightarrow \mathbb{Z}$ be a homomorphism which sends $\sum_{i=1}^s a_i e_i$ to $\sum_{i=1}^s a_i n_i$. Since S is numerical, φ_S is surjective. And $V(S) = \text{Ker} \varphi_S$ is a torsion-free submodule of $\mathbb{Z}^s$ of rank $s - 1$.

Let k be a field. We define the associated semigroup ring $k[S] = k[t^d]_{d \in S}$. And there exists a surjective ring homomorphism $\tilde{\varphi}_S : k[X_1, \dots, X_s] \rightarrow k[S]$ with $\tilde{\varphi}_S(X_i) = t^{n_i}$. Put $I(S) = \text{Ker} \tilde{\varphi}_S$, where $I(S)$ is called the defining ideal of $k[S]$. It is known that $I(S)$ is generated

by binomials $F(v) = X^{v^-} - X^{v^+}$, where $v \in V(S)$. Note $v^+ = \sum_i \max\{\sigma_i(v), 0\}e_i$ and $v^- = \sum_i \max\{-\sigma_i(v), 0\}e_i$, where $\sigma_i(v)$ is the i -th entry of v for each i .

1.5. The determinants of RF-matrices. Let $S = \langle n_1, \dots, n_s \rangle$ be a numerical semigroup. For $d_1, d_2 \in \mathbb{Z}$, we write $d_1 \leq_S d_2$ if $d_2 - d_1 \in S$. Since

$$\text{Ap}(S, n_1) = \{x \in S : \exists f \in \text{PF}(S); x \leq_S f + n_1\},$$

by computing the number of the Apéry set from all expressions of $f + n_1$, where $f \in \text{PF}(S)$, we find the relation between the number n_1 and RF-matrices, whose first rows induce expressions of $f + n_1$. On the other hand, let $\text{RF}(S) = (a_{ij})$ be an RF-matrix for $F(S)$ and put $v_{ij} = \sum_{l=1}^s (a_{il} - a_{jl})e_l$ for each i, j . Then v_{ij} is contained in $V(S)$ and $W = \sum_{i,j} \mathbb{Z}v_{ij}$ is a submodule of $V(S)$. Further, $I(W) = I(S)$ if and only if $W = V(S)$, where $I(W) = (F(v))_{v \in W}$ is a binomial ideal. We note that $W = V(S)$ if and only if $|\det \text{RF}(S)| = F(S)$. Therefore, it is necessary to compute $\det \text{RF}(S)$ to determine whether $I(W) = I(S)$ holds or not. We also have

$$k[X_1, \dots, X_s] / I(S) + (X_1) \cong k[S] / (t^{n_1}) \cong \bigoplus_{d \in \text{Ap}(S, n_1)} k \cdot t^d.$$

Hence, the equality $I(S) = (F(v_{ij}))_{i,j} + (X_1)$ holds, if

$$(1) \text{ ht}(F(v_{ij}))_{i,j} = s,$$

$$(2) \dim_k k[X_1, \dots, X_s] / (F(v_{ij}))_{i,j} + (X_1) = n_1.$$

We can easily check the condition (2) from the RF-matrix $\text{RF}(S)$, because the left hand side is computed from v_{1j}^+ and the right hand side is the number of Apéry set, obtained by previous way from the RF-matrix. Note that $I(S) = (F(v_{ij}))_{i,j}$ implies $I(S) = I(W)$, thus $|\det \text{RF}(S)| = F(S)$. By taking account of the signature, we put a conjecture:

Conjecture 1.1. Let S be a numerical semigroup of embedding dimension s . Then there exists an RF-matrix $\text{RF}(S)$ for $F(S)$ in S satisfying $\det \text{RF}(S) = (-1)^{s+1} F(S)$.

Note 1.2. In general, the determinant of an RF-matrix for f in S is a multiple of f . Consider a matrix $M = (a_{ij})$ satisfying $a_{1j} = \text{RF}_{1j}(f)$ and $a_{ij} = \text{RF}_{ij}(f) - \text{RF}_{1j}(f)$ if $i > 1$. Then $\det M = \det \text{RF}(f)$. If we put $M' = (a_{ij})_{i>1}$, then the row vectors of M is contained in $V(S)$. If the rank of the submodule W generated by them is $s-1$, $V(S)/W$ is a torsion module. Put $d = |V(S)/W|$. Then we have $|\det M| = df$. If $\text{rank } W < s-1$, then $\det M = 0$. In each case, $\det M = \det \text{RF}(f)$ is a multiple of f . Further, the determinant of the matrix by deleting j -th column from M' is dn_j up to sign (note $d = 0$ if $\text{rank } W < s-1$) for each j , thus, by calculating one of $(s-1)$ -minors of M' , we can judge whether the conjecture is true or not.

Theorem 1.3. *The above conjecture is true in the following cases:*

- (1) S is completely glued,
- (2) $s = 3$,
- (3) $s = 4$ and the defining ideal of associated semigroup is an almost complete intersection,
- (4) $s = 4$, S is symmetric,
- (5) $s = 4$, S is almost symmetric.

2. Proof of Theorem 1.3

2.1. Gluing.

Definition 2.1. Let S_1, S_2 be numerical semigroups, $d_1 \in S_2$ and $d_2 \in S_1$. If d_1 and d_2 are coprime, then

$$S = d_1 S_1 + d_2 S_2 = \{d_1 x + d_2 y : x \in S_1, y \in S_2\},$$

is a numerical semigroup. We say that S is glued by S_1 and S_2 .

Definition 2.2. We say that S is completely glued if one of the following is satisfied:

- (1) $S = \langle 1 \rangle$,
- (2) S is glued by completely glued semigroups.

If the embedding dimension of S is two, then S is completely glued.

Assume that S is glued by S_1 and S_2 . Then

$$F(S) = d_1 F(S_1) + d_2 F(S_2) + d_1 d_2$$

holds. Note that a completely glued numerical semigroup is symmetric ([7], Corollary 9.12).

Theorem 2.3. Let $S_1 = \langle n_1, \dots, n_{s_1} \rangle$, $S_2 = \langle n'_1, \dots, n'_{s_2} \rangle$ be numerical semigroups and put $S = d_1 S_1 + d_2 S_2$, where $d_1 \in S_2$, $d_2 \in S_1$. Assume $f_1 \notin S_1$ (resp., $f_2 \notin S_2$) and $f_1 + d_2 \in S_1$ (resp., $f_2 + d_1 \in S_2$). Then $f = d_1 f_1 + d_2 f_2 + d_1 d_2 \notin S$. Further, if we write $f_1 + d_2 = \sum_i a_i n_i$ and $f_2 + d_1 = \sum_i a'_i n'_i$, then we have

$$\text{RF}(f) = \begin{pmatrix} \text{RF}(f_1) & N_2 \\ N_1 & \text{RF}(f_2) \end{pmatrix},$$

where N_1 (resp., N_2) is an $s_2 \times s_1$ -matrix (resp., $s_1 \times s_2$ -matrix) whose ij -entry is a_i (resp., a'_i) for each i, j . Further,

$$\det \text{RF}(f) = -f(\det \text{RF}(f_1))(\det \text{RF}(f_2)) / f_1 f_2.$$

Therefore, if $\text{RF}(f_1) = (-1)^{s_1+1} f_1$ and $\text{RF}(f_2) = (-1)^{s_2+1} f_2$, then $\text{RF}(f) = (-1)^{s_1+s_2+1} f$.

Proof. Suppose $f \in S$. Then there are $x_1 \in S_1$ and $x_2 \in S_2$ satisfying $f = d_1 x_1 + d_2 x_2$. Since d_1 and d_2 are coprime, there is $t \in \mathbb{Z}$ with $f_1 + d_2 = x_1 + t d_2$ and $f_2 = x_2 - t d_2$. Since $f_1 \notin S_1$ (resp., $f_2 \notin S_2$), we have $t \leq 0$ (resp., $t > 0$), a contradiction. Hence $f \notin S$.

Put $M_1 = (a_{ij})$ (resp., $M_2 = (b_{ij})$) be an $s_1 \times s_1$ -matrix (resp., $s_2 \times s_2$ -matrix) with $a_{1j} = n_j \text{RF}_{1j}(f_1)$ (resp., $b_{1j} = n'_j \text{RF}_{1j}(f_2)$) and $a_{ij} = n_j (\text{RF}_{ij}(f_1) - \text{RF}_{1j}(f_1))$ (resp., $b_{ij} = n'_j (\text{RF}_{ij}(f_2) - \text{RF}_{1j}(f_2))$) if $i > 1$. Then $\det M_1 = \prod_{j=1}^{s_1} n_j \cdot \det \text{RF}(f_1)$ (resp., $\det M_2 = \prod_{j=1}^{s_2} n'_j \cdot \det \text{RF}(f_2)$). Put $M'_1 = (a_{ij})_{i,j>1}$ (resp., $M'_2 = (b_{ij})_{i,j>1}$) be an $(s_1 - 1) \times (s_1 - 1)$ -matrix (resp., $(s_2 - 1) \times (s_2 - 1)$ -matrix). Then $\det M_1 = f_1 \det M'_1$ (resp., $\det M_2 = f_2 \det M'_2$). Let D be a diagonal matrix whose diagonal entries are $n_1, \dots, n_{s_1}, n'_1, \dots, n'_{s_2}$. Then $\det(\text{RF}(f)D)$ is transformed into

$$\begin{vmatrix} f_1 & * & f_2 + d_1 & * \\ 0 & M'_1 & 0 & 0 \\ f_1 + d_2 & * & f_2 & * \\ 0 & 0 & 0 & M'_2 \end{vmatrix} = (f_1 f_2 - (f_1 + d_2)(f_2 + d_1)) \det M'_1 \cdot \det M'_2.$$

Hence

$$\det \text{RF}(f) = -f(\det \text{RF}(f_1))(\det \text{RF}(f_2)) / f_1 f_2.$$

□

Corollary 2.4. *If S is completely glued, then $\text{RF}(S) = (-1)^{s+1} F(S)$ holds where s is the embedding dimension of S .*

Proof. If $S = \langle 1 \rangle$, then $\text{RF}(S) = \text{RF}(-1) = -1$ and the assertion holds. Then we inductively prove it by Theorem 2.3. $\square$

2.2. The case of embedding dimension three. Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup. If S is completely glued, the conjecture is true by Corollary 2.4. If S is not complete glued, by well known result, there are

$$v_1 = \begin{pmatrix} \alpha_1 \\ -a_{21} \\ -a_{31} \end{pmatrix}, v_2 = \begin{pmatrix} -a_{12} \\ \alpha_2 \\ -a_{32} \end{pmatrix}, v_3 = \begin{pmatrix} -a_{13} \\ -a_{23} \\ \alpha_3 \end{pmatrix},$$

where $v_1 + v_2 + v_3 = 0$ and $I(S) = (F(v_1), F(v_2), F(v_3))$. In this case,

$$\text{PF}(S) = \begin{cases} f_1 = (\alpha_2 - 1)n_2 + (\alpha_{31} - 1)n_3 - n_1 \\ f_2 = (a_{21} - 1)n_2 + (\alpha_3 - 1)n_3 - n_1 \end{cases}.$$

Then

$$\text{RF}(f_1) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_{31} - 1 \\ \alpha_{12} - 1 & -1 & \alpha_3 - 1 \\ \alpha_1 - 1 & a_{23} - 1 & -1 \end{pmatrix}, \text{RF}(f_2) = \begin{pmatrix} -1 & a_{21} - 1 & \alpha_3 - 1 \\ \alpha_1 - 1 & -1 & a_{32} - 1 \\ \alpha_{13} - 1 & \alpha_2 - 1 & -1 \end{pmatrix}.$$

Since $n_1 = \alpha_2\alpha_3 - a_{23}a_{32}$, we have $\det \text{RF}(f_l) = f_l$ for each l .

2.3. Almost complete intersection of embedding dimension four.

Let $S = \langle n_1, n_2, n_3, n_4 \rangle$ be a numerical semigroup such that the defining ideal of associated semigroup ring is an almost complete intersection, that is, generated by four polynomials. According to [3],

there exist three cases. Let $v_1, v_2, v_3, v_4 \in V(S)$ with $I(S) = (F(v_1), F(v_2), F(v_3), F(v_4))$.

Case 1.

$$v_1 = \begin{pmatrix} \alpha_1 \\ -a_{21} \\ -a_{31} \\ -a_{41} \end{pmatrix}, v_2 = \begin{pmatrix} -a_{12} \\ \alpha_2 \\ 0 \\ -a_{42} \end{pmatrix}, v_3 = \begin{pmatrix} -a_{13} \\ -a_{23} \\ \alpha_3 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} -a_{14} \\ 0 \\ -a_{34} \\ \alpha_4 \end{pmatrix},$$

where $v_1 + v_2 + v_3 + v_4 = 0$. In this case,

$$\text{PF}(S) = \begin{cases} f_1 = (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (a_{41} - 1)n_4 - n_1 \\ f_2 = (\alpha_2 - 1)n_2 + (a_{31} - 1)n_3 + (\alpha_4 - 1)n_4 - n_1 \\ f_3 = (a_{21} - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4 - n_1 \end{cases}.$$

Then

$$\text{RF}(f_1) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{41} - 1 \\ a_{12} - 1 & -1 & \alpha_3 - 1 & \alpha_4 - 1 \\ a_{12} + a_{13} - 1 & a_{23} - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{23} - 1 & a_{34} - 1 & -1 \end{pmatrix},$$

$$\text{RF}(f_2) = \begin{pmatrix} -1 & \alpha_2 - 1 & a_{31} - 1 & \alpha_4 - 1 \\ a_{12} + a_{14} - 1 & -1 & \alpha_3 - 1 & a_{42} - 1 \\ \alpha_1 - 1 & a_{23} - 1 & -1 & a_{42} - 1 \\ a_{14} - 1 & \alpha_2 - 1 & \alpha_3 - 1 & -1 \end{pmatrix},$$

$$\text{RF}(f_3) = \begin{pmatrix} -1 & a_{21} - 1 & \alpha_3 - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & -1 & a_{34} - 1 & a_{42} - 1 \\ a_{13} - 1 & \alpha_2 - 1 & -1 & \alpha_4 - 1 \\ a_{13} + a_{14} - 1 & \alpha_2 - 1 & a_{34} - 1 & -1 \end{pmatrix}.$$

Since $n_1 = \alpha_2 \alpha_3 \alpha_4 - a_{23} a_{34} a_{42}$, we have $\det \text{RF}(f_l) = -f_l$ for each l .

Case 2.

$$v_1 = \begin{pmatrix} \alpha_1 \\ -\alpha_2 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ -a_{32} \\ -a_{42} \end{pmatrix}, v_3 = \begin{pmatrix} -a_{13} \\ -a_{23} \\ \alpha_3 \\ -a_{43} \end{pmatrix}, v_4 = \begin{pmatrix} -a_{14} \\ -a_{24} \\ -a_{34} \\ \alpha_4 \end{pmatrix},$$

where $v_2 + v_3 + v_4 = 0$. In this case,

$$\text{PF}(S) = \begin{cases} f_1 = (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (a_{42} - 1)n_4 - n_1 \\ f_2 = (\alpha_2 - 1)n_2 + (a_{32} - 1)n_3 + (\alpha_4 - 1)n_4 - n_1 \end{cases}.$$

Then

$$\text{RF}(f_1) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{42} - 1 \\ \alpha_1 - 1 & -1 & \alpha_3 - 1 & a_{42} - 1 \\ \alpha_1 + a_{13} - 1 & a_{23} - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 + a_{12} - 1 & a_{22} - 1 & a_{34} - 1 & -1 \end{pmatrix},$$

$$\text{RF}(f_2) = \begin{pmatrix} -1 & \alpha_2 - 1 & a_{32} - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & -1 & a_{32} - 1 & \alpha_4 - 1 \\ \alpha_1 + a_{12} - 1 & a_{22} - 1 & -1 & a_{43} - 1 \\ \alpha_1 + a_{14} - 1 & a_{24} - 1 & \alpha_3 - 1 & -1 \end{pmatrix}.$$

Since $n_1 = \alpha_2(\alpha_3\alpha_4 - a_{34}a_{43})$, we have $\det \text{RF}(f_l) = -f_l$ for each l .

Case 3. The case in which S is glued by $\langle 1 \rangle$ and a semigroup S_1 of embedding dimension 3 which is not completely glued. Since we proved $\det \text{RF}(S_1) = F(S_1)$ in Subsection 2.3, the conjecture is true by Theorem 2.3.

2.4. Symmetric case of embedding dimension four. We assume that $S = \langle n_1, n_2, n_3, n_4 \rangle$ is symmetric. If S is not completely glued, we have

$$\text{RF}(S) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{14} \\ a_{21} & -1 & \alpha_3 - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{32} & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 & a_{43} & -1 \end{pmatrix},$$

where $a_{(i+1)i} < \alpha_i - 1$ for $i = 1, 2, 3$ and $a_{14} < \alpha_4 - 1$. Then

$$\begin{aligned} F(S) + n_4 &= (\alpha_1 - 1)n_1 + (\alpha_2 - 1)n_2 + a_{43}n_3 \\ &= a_{21}n_1 + (\alpha_2 - a_{32} - 2)n_2 + (\alpha_3 + a_{43})n_3, \end{aligned}$$

and

$$\begin{aligned} &\begin{vmatrix} a_{21} + 1 & -\alpha_2 & 0 \\ \alpha_1 & a_{32} - \alpha_2 + 1 & -\alpha_3 \\ \alpha_1 & 0 & a_{43} - \alpha_3 + 1 \end{vmatrix} \\ &= \alpha_1\alpha_2(a_{43} + 1) + (a_{21} + 1)(\alpha_2 - a_{32} - 1)(\alpha_3 - a_{43} - 1) = n_4, \end{aligned}$$

and $\det \text{RF}(S) = -F(S)$.

2.5. Almost symmetric case of embedding dimension four. In this subsection, we refer to ([4], Section 5). We first study pseudo-symmetric case. Let $S_1 = \langle n_1, n_2, n_3, n_4 \rangle$ be pseudo-symmetric. By ([4], Subsection 5.1),

$$\text{RF}(S_1) = \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_4 - 1 & \alpha_3 - 1 & -1 \end{pmatrix},$$

where $0 < \alpha_4 < \alpha_2 - 1$. Then

$$F(S_1) + n_2 = (\alpha_1 - 1)n_1 + (\alpha_3 - 2)n_3 + (\alpha_4 - 1)n_4,$$

and

$$\begin{vmatrix} \alpha_1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & -\alpha_3 & \alpha_4 - 1 \\ \alpha_1 & 0 & -1 \end{vmatrix} = \alpha_1(\alpha_3 - 1)\alpha_4 = n_2,$$

thus we have $\det \text{RF}(S_1) = -F(S_1)$. Note that we also have $\det \text{RF}(F(S_1)/2) = -F(S_1)/2$ by the similar calculation.

Let $S_2 = \langle n_1, n_2, n_3, n_4 \rangle$ and $S_3 = \langle n'_1, n'_2, n'_3, n'_4 \rangle$ be almost symmetric with

$$\text{RF}(S_2) = \begin{pmatrix} -1 & \alpha_2 - 2 & 1 & \alpha_4 - 2 \\ \alpha_1 - 2 & -1 & 0 & \alpha_4 - 1 \\ \alpha_1 + b_{41} - 2 & b_{32} - 1 & -1 & 0 \\ b_{41} - 1 & \alpha_2 + b_{32} - 2 & 0 & -1 \end{pmatrix},$$

$$\text{RF}(S_3) = \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 1 & \alpha_4 - 2 \\ \alpha_{21} - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & 0 \\ \alpha_1 - 1 & \alpha_2 - 2 & \alpha_3 - 2 & -1 \end{pmatrix},$$

where $b_{41} > 0$, $b_{42} > 0$, $\alpha_{21} > 0$ and $\alpha_2 = 2$ or $\alpha_4 = 2$ holds (cf. [4], Theorem 3.6). Note that, for each l , $F(S_l) + n_1$ has UF (a unique factorization), that is, there are unique non-negative integers $\alpha_2, \alpha_3, \alpha_4$ satisfying $F(S_l) + n_1 = \sum_i \alpha_i n_i$. And

$$F(S_2) + n_1 = (\alpha_2 - 2)n_2 + n_3 + (\alpha_4 - 2)n_4,$$

$$F(S_3) + n_1 = (\alpha_2 - 2)n'_2 + (\alpha_3 - 1)n'_3 + (\alpha_4 - 2)n'_4,$$

and $\det \text{RF}(S_2) = 0$. Note $b_{41} > 1$ or $b_{32} > 1$. Otherwise,

$$\begin{pmatrix} -1 & 0 & 0 & \alpha_4 - 1 \\ 0 & -1 & 1 & \alpha_4 - 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

is an RF-matrix for one of a pseudo-Frobenius number in S_2 by ([4], Theorem 3.6), and we have $2n_1 = \alpha_4 n_4$, $2n_2 = 2n_3 + (\alpha_4 - 2)n_4$, thus $\alpha_1 = \alpha_2 = 2$, this contradicts to $\alpha_1 n_1 = \alpha_2 n_2$. If $b_{41} > 1$ (resp., $b_{32} > 1$), then

$$\begin{pmatrix} -1 & \alpha_2 - 2 & 1 & \alpha_4 - 2 \\ \alpha_1 - 2 & -1 & 0 & \alpha_4 - 1 \\ b_{41} - 2 & \alpha_2 + b_{32} - 1 & -1 & 0 \\ b_{41} - 1 & \alpha_2 + b_{32} - 2 & 0 & -1 \end{pmatrix} \text{ (resp., } \begin{pmatrix} -1 & \alpha_2 - 2 & 1 & \alpha_4 - 2 \\ \alpha_1 - 2 & -1 & 0 & \alpha_4 - 1 \\ \alpha_1 + b_{41} - 2 & b_{32} - 1 & -1 & 0 \\ \alpha_1 + b_{41} - 1 & b_{32} - 2 & 0 & -1 \end{pmatrix})$$

is an RF-matrix for $F(S_2)$ whose determinant is $-F(S_2)$. Concerning S_3 , since

$$\begin{vmatrix} -\alpha_2 + 1 & -1 & 1 \\ 1 & -\alpha_3 & -\alpha_4 + 2 \\ 0 & -1 & -\alpha_4 + 1 \end{vmatrix} = -\alpha_4 + (\alpha_2 - 1)(-2 + \alpha_3 + \alpha_4 - \alpha_3\alpha_4),$$

if $\alpha_2 = 2$ or $\alpha_4 = 2$, we have

$$-\alpha_4 + (\alpha_2 - 1)(-2 + \alpha_3 + \alpha_4 - \alpha_3\alpha_4) = -(\alpha_2 - 1)\alpha_3(\alpha_4 - 1) - 2 = -n_1,$$

and $\det \text{RF}(S_3) = -F(S_3)$.

Let $S_4 = \langle n_1, n_2, n_3, n_4 \rangle$ and $S_5 = \langle n'_1, n'_2, n'_3, n'_4 \rangle$ be almost symmetric with

$$\begin{aligned} \text{RF}(S_4) &= \begin{pmatrix} -1 & 0 & \alpha_3 - 1 & \alpha_4 - 2 \\ \alpha_1 - 2 & -1 & 0 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 2 & -1 & 0 \\ 0 & \alpha_2 - 1 & \alpha_3 - 2 & -1 \end{pmatrix}, \\ \text{RF}(S_5) &= \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 2 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & \alpha_4 - 2 \\ \alpha_1 - 2 & \alpha_2 - 2 & \alpha_3 - 1 & -1 \end{pmatrix} \end{aligned}$$

(cf. [4], Theorem 4.8). Note $\alpha_i > 2$ for each i in S_4 , otherwise $F(S_4) + n_{i'}$ has UF for some i' and we discussed before. Recall RF-matrices of pseudo-Frobenius number, not equal to $F(S_4)$, are

$$\begin{pmatrix} -1 & 0 & 0 & \alpha_4 - 1 \\ 0 & -1 & 1 & \alpha_4 - 2 \\ 0 & \alpha_2 - 1 & -1 & 0 \\ 1 & \alpha_2 - 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & \alpha_3 - 2 & 0 \\ \alpha_1 - 1 & -1 & 0 & 0 \\ \alpha_1 - 2 & 0 & -1 & 1 \\ 0 & 0 & \alpha_3 - 1 & -1 \end{pmatrix}.$$

Then

$$F(S_4) + n_1 = (\alpha_3 - 1)n_3 + (\alpha_4 - 2)n_4$$

$$= \alpha_2 n_2 + (\alpha_3 - 3)n_3,$$

$$F(S_5) + n_1 = (\alpha_2 - 2)n'_2 + (\alpha_3 - 2)n'_3 + (\alpha_4 - 1)n'_4$$

$$= (\alpha_2 - 3)n'_2 + (2\alpha_3 - 2)n'_3,$$

and $\det \text{RF}(S_4) = 0$. Since $\alpha_3 > 2$,

$$\begin{pmatrix} -1 & \alpha_2 & \alpha_3 - 3 & 0 \\ \alpha_1 - 2 & -1 & 0 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 2 & -1 & 0 \\ 0 & \alpha_2 - 1 & \alpha_3 - 2 & -1 \end{pmatrix}$$

is an RF-matrix for $F(S_4)$ in S_4 , whose determinant is $-F(S_4)$. We have

$$\begin{vmatrix} -\alpha_2 + 1 & 0 & -1 \\ 1 & -\alpha_3 + 1 & -1 \\ 0 & 1 & -\alpha_4 \end{vmatrix} = -\alpha_2 - \alpha_4 + \alpha_2\alpha_4 + \alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4,$$

and

$$n'_1 = (\alpha_2 - 1)(\alpha_3 - 1)\alpha_4 + (\alpha_2 - 2) + 2,$$

thus $\det \text{RF}(S_5) = -F(S_5)$.

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