MEET-REDUCIBLE SUBMAXIMAL CLONES DETERMINED BY NONTRIVIAL EQUIVALENCE RELATIONS

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Abstract

The structure of the lattice of clones on a finite set has been proven to be very complex. To better understand the top of this lattice, it is important to provide a characterization of submaximal clones in the lattice of clones. It is known that the clones $\text{Pol}(\theta)$ and $\text{Pol}(\rho)$ (where θ is a nontrivial equivalence relation on $E_k = \{0, \dots, k-1\}$, and ρ belongs to one of the six types of relations which characterize maximal clones) are maximal clones. In this paper, we provide a classification of relations (of Rosenberg's list) ρ on E_k for which the clone $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$.

1. Introduction

The structure of the lattice of clones on a finite set of more than two elements is quite complex. For instance, its cardinality is $2^{\aleph_0}$. For a better comprehension of some intervals in this lattice, it is important to provide a characterization of maximal and submaximal clones. Maximal clones have been investigated extensively by I. G. Rosenberg, and a complete characterization of these can be found in [14]. More precisely, it is proved that for a given nontrivial equivalence relation θ on a finite set and a central relation ρ on the same set, the clones $\text{Pol}(\theta)$ and $\text{Pol}(\rho)$ are maximal. For a unary central relation on an arbitrary finite set, Rosenberg and Szendrei [16, 17] investigated the submaximal clones of their polymorphisms and obtain new results on those of prime permutations on a finite set. Submaximal clones for a set with two and three elements were completely described and classified in [7, 11, 12]. However, for sets with more than three elements, only partial results on their submaximal clones are found in the literature (see, e.g., [7]). Recently, Temgoua and Rosenberg obtained a characterization of all binary central relations for which the clone $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$, given any nontrivial equivalence relation θ and a binary central relation ρ [18].

In this paper, we characterize the relations ρ for which the clones of the form $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$, where θ is a nontrivial equivalence relation on a given finite set.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions and notations needed for the clarity of our presentation. In Section 3, we present the non-submaximality when ρ is a partial order or a prime affine relation. Section 4 is devoted to the characterization of type of equivalence relations or prime permutation relations which give submaximality. In Section 5, we characterize the relations ρ (resp., central relations or h -regularly generated relations) for which $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$. Section 6 concludes the paper.

2. Preliminaries

In this section, we recall some of the main definitions and notations. Readers needing more background on the topic may refer to [7]. Let $E_k = \{0, 1, \dots, k-1\}$ be a finite set of k elements with $k \geq 1$. Let $n, s \in \mathbb{N}^*$. An n -ary operation on E_k is a function from E_k^n to E_k . The set of all n -ary operations on E_k is denoted by $\mathcal{O}^n(E_k)$ and we set $\mathcal{O}(E_k) = \bigcup_{0 < n < \omega} \mathcal{O}^n(E_k)$. For $1 \leq i \leq s$, the s -ary i -th projection e_i^s is defined as $e_i^s(x_1, \dots, x_s) = x_i$ for all $x_1, \dots, x_s$ in E_k . For $f \in \mathcal{O}^n(E_k)$ and $g_1, \dots, g_n \in \mathcal{O}^m(E_k)$, we define their *composition* to be the m -ary operation $f[g_1, \dots, g_n]$ defined by:

$$f[g_1, \dots, g_n](x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)).$$

A *clone* on E_k is a subset F of $\mathcal{O}(E_k)$ which contains all the projections and is closed under composition. It is known that the intersection of an

arbitrary set of clones on E_k is a clone on E_k . Thus for $F \subseteq \mathcal{O}(E_k)$, there exists a smallest clone containing F , called the *clone generated by F* and denoted by $\langle F \rangle$. $\langle F \rangle$ is also the set of term operations of the non-indexed algebra $\mathcal{A} = (A; F)$, with $A = E_k$. The clones on E_k , ordered by inclusion, form a complete lattice denoted by $\mathcal{L}(E_k)$. A clone $C \in \mathcal{L}(E_k)$ is called *maximal* if it is covered only by $\mathcal{O}(E_k)$. A clone $C \in \mathcal{L}(E_k)$ is called *submaximal* if it is covered only by a maximal clone.

Let h be a positive integer. An h -ary relation ρ is a subset of E_k^h . For $\rho \subseteq E_k^2$, we write apb for $(a, b) \in \rho$. An h -ary relation ρ is called *totally symmetric* if for every permutation σ of $\{1, \dots, h\}$ and each h -tuple $(a_1, \dots, a_h) \in E_k^h$,

$$(a_1, \dots, a_h) \in \rho \text{ if } (a_{\sigma(1)}, \dots, a_{\sigma(h)}) \in \rho.$$

$\tau_h^{E_k}$ is the h -ary relation defined by $(a_1, \dots, a_h) \in \tau_h^{E_k}$ if there exist $i, j \in \{1, \dots, h\}$ such that $i \neq j$ and $a_i = a_j$. An h -ary relation ρ is called *totally reflexive* if $\tau_h^{E_k} \subseteq \rho$. For $h = 2$, the concepts totally reflexive and totally symmetric coincide with the usual notions of reflexive and symmetric. If ρ is totally reflexive and totally symmetric, we define the *center* of ρ denoted by C_ρ as follows:

$$C_\rho = \{a \in E_k : (a, a_2, \dots, a_h) \in \rho \text{ for all } a_2, \dots, a_h \in E_k\}.$$

Let θ be a binary relation and $m \in \mathbb{N}^*$; for $\mathbf{a} = (a_1, \dots, a_m) \in E_k^m$ and $\mathbf{b} = (b_1, \dots, b_m) \in E_k^m$, we write $\mathbf{a}\theta\mathbf{b}$ if $(a_i, b_i) \in \theta$ for $1 \leq i \leq m$. Assume that θ is an equivalence relation on E_k . The θ -class of $a \in E_k$ will be denoted by $[a]_\theta$.

A permutation π on E_k is *prime* if all cycles of π have the same prime length.

A subset $\sigma \subseteq E_k^4$ is called *affine* if there is a binary operation $+$ on E_k such that $(E_k, +)$ is an abelian group and $(a, b, c, d) \in \sigma \Leftrightarrow a + b = c + d$. An *affine relation* σ is *prime* if $(E_k, +)$ is an abelian p -group for some prime p , that is, all elements of the group have the same prime order p .

An h -ary relation ρ on E_k is called *central*, if ρ is a nonempty proper subset of E_k or ρ has the following three properties: ρ is totally reflexive; ρ is totally symmetric and C_ρ is a nonempty proper subset of E_k .

For $h \geq 3$, a family $T = \{\mathcal{V}_1; \dots; \mathcal{V}_m\}$ of equivalence relations on E_k is called *h -regular* if each $\mathcal{V}_i$ has exactly h equivalence classes and $\bigcap \{B_i | 1 \leq i \leq m\}$ is nonempty for arbitrary equivalence classes B_i of $\mathcal{V}_i$, $1 \leq i \leq m$. For $3 \leq h \leq k$, an *h -regular* (or *h -regularly generated*) relation on E_k determined by the h -regular family T (often denoted by λ_T), consists of all h -tuples whose set of components meets at most $h - 1$ classes of each $\mathcal{V}_i$ ($1 \leq i \leq m$).

Let $f \in \mathcal{O}^n(E_k)$ and ρ be an h -ary relation on E_k . The operation f preserves ρ if for all $(a_{1,i}, \dots, a_{h,i}) \in \rho$ ($i = 1, \dots, n$), we have

$$(f(a_{1,1}, \dots, a_{1,n}), f(a_{2,1}, \dots, a_{2,n}), \dots, f(a_{h,1}, \dots, a_{h,n})) \in \rho.$$

The set of operations on E_k preserving ρ is a clone denoted by $\text{Pol}(\rho)$. The maximal clones have been described for $k = 2$ (respectively, $k = 3$ and $k \geq 4$) in Post [11] (respectively, Jablonskij [5] and Rosenberg [14, 15]). They are of the form $\text{Pol}(\rho)$ where ρ belongs to one of six families of relations which include some familiar and easily defined

relations. For clones C and D on E_k , we say that C is *maximal in D* if D covers C in $\mathcal{L}(E_k)$; submaximal clones have already been define on page 2. All submaximal clones are known for $k = 2$ (see [11]) and $k = 3$ (see [7]).

For $n \geq 3$, an n -ary operation f is called a *near-unanimity operation* provided that $f(y, x, \dots, x) \approx f(x, y, \dots, x) \approx \dots \approx f(x, x, \dots, y) \approx x$ for all $x, y \in E_k$. We recall the following Baker-Pixley Theorem which will be useful in the proofs:

Theorem 2.1 ([1]). *Let $\mathcal{A} = (A, F)$ be a finite algebra which contains a “near unanimity function” of arity $d + 1$ ($(d + 1)$ -ary near-unanimity term or nu-term). Then, an operation $f : A^n \rightarrow A$ is term function for $\mathcal{A}$ iff each subuniverse of $\mathcal{A}^d$ is preserved by f .*

3. Partial Order, Prime Affine Relations

In this section, we prove that the clones $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$ in case θ is a nontrivial equivalence relation on E_k and ρ is either a partial order with least and greatest elements or a prime affine relation on E_k .

Theorem 3.1. *If θ is a nontrivial equivalence relation and ρ is a partial order with least and greatest elements, on a finite set E_k , then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not submaximal in $\text{Pol}(\theta)$.*

Proof. θ and ρ are incomparable. In fact, $\theta \not\subseteq \rho$ because θ is a nontrivial symmetric relation and ρ is an antisymmetric relation. We also have $\rho \not\subseteq \theta$, otherwise since ρ is a partial order with least and greatest element and θ is a transitive relation, θ will be trivial (equal to E_k^2); this is a contradiction. Without lost of generality, we can consider that the least element of ρ is 0 and the greatest element of ρ is 1.

Let ρ' and r be the relations defined by: $\rho' := \theta \circ \rho \circ \theta$ and $r := \rho \cap \theta$.

If $\rho' \neq E_k^2$, we have $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\rho') \subsetneq \text{Pol}(\theta)$.

Suppose that $\rho' = E_k^2$, then $(1, 0) \in E_k^2 = \rho'$ and it follows that $(1, 0) \in \theta$ or there is $b \in E_k \setminus \{0\}$ such that $(b, 0) \in \theta$. Therefore $r \neq \Delta_{E_k} = \{(a, a) : a \in E_k\}$.

Since r and r^{-1} are subset of θ , $r \circ r^{-1}$ is a subset of θ .

If $r \circ r^{-1} \neq \theta$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(r \circ r^{-1}) \subsetneq \text{Pol}(\theta)$.

If $r \circ r^{-1} = \theta$, then $\text{Pol}(r) \subsetneq \text{Pol}(\theta)$ and it can be proved that each equivalence class of θ contains a least and a greatest element. It follows from Theorem 3.3 of [19] that $\text{Pol}(r)$ is a meet-irreducible maximal subclone of $\text{Pol}(\theta)$. Using the fact that $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(r)$, we conclude that $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$. $\square$

For a prime affine relation, there is no meet-reducible submaximality in the set of polymorphisms of a nontrivial equivalence relation. This is proved by the following theorem.

Theorem 3.2. *Let α be a prime affine relation and θ be a nontrivial equivalence relation. We have*

$$\text{Pol}(\theta) \cap \text{Pol}(\alpha) \subsetneq \text{Pol}(\alpha_1) \subsetneq \text{Pol}(\theta),$$

where

$$\alpha_1 = \{(a, b, c, d) \in \alpha : (a, b), (a, c), (a, d) \in \theta\}.$$

Proof. Let $(a, b) \in \theta$ such that $a \neq b$, then the binary operation g_1 defined on E_k by:

$$g_1(x, y) = \begin{cases} a & \text{if } (x, y) \in \{(a, a); (a, b); (b, a)\}, \\ b & \text{otherwise,} \end{cases}$$

preserves θ and does not preserve α_1 . Therefore $\text{Pol}(\alpha_1) \subsetneq \text{Pol}(\theta)$.

Also it is easy to see that $\text{Pol}(\theta) \cap \text{Pol}(\alpha) \subseteq \text{Pol}(\alpha_1)$. Now, let $(a, b) \notin \theta$ and let g_2 be a ternary operation on E_k defined by:

$$g_2(x, y, z) = \begin{cases} b & \text{if } (x, y, z) \in \theta(a, a, b), \\ a & \text{otherwise,} \end{cases}$$

g_2 preserves α_1 and does not preserve α ; therefore $\text{Pol}(\theta) \cap \text{Pol}(\alpha) \subsetneq \text{Pol}(\alpha_1)$. □

4. Equivalence Relations, Prime Permutation Relations

Let $k > 1$, θ a nontrivial equivalence relation on E_k with blocks (equivalence classes) $B_0, \dots, B_{t_1-1}$ (where $2 \leq t_1 \leq k$) and ρ a nontrivial equivalence relation distinct from θ with blocks $C_0, \dots, C_{t_2-1}$ (where $2 \leq t_2 \leq k$). In this section, we determine the meet-reducible clones of the form $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ maximal in $\text{Pol}(\theta)$, where θ and ρ are two distinct nontrivial equivalence relations on E_k .

We define $\mu : E_k \rightarrow E_{t_1}$ by $\mu(x) = i$ if $x \in B_i$ and $\nu : E_k \rightarrow E_{t_2}$ by $\nu(x) = i$ if $x \in C_i$. Set $D = \text{Pol}(\theta) \cap \text{Pol}(\rho)$, $\nabla_{E_k} = E_k^2$ and $\gamma = \theta \cap \rho$. Clearly γ is an equivalence relation on E_k and $D \subseteq \text{Pol}(\theta) \cap \text{Pol}(\gamma)$.

Theorem 4.1. *Let θ and ρ be two distinct nontrivial equivalence relations on a finite set E_k . $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is submaximal in $\text{Pol}(\theta)$ if and only if θ and ρ satisfy one of the following statement:*

- (a) $\theta \subsetneq \rho$ or $\rho \subsetneq \theta$;
- (b) $\rho \cap \theta = \Delta_{E_k}$ and $\rho \circ \theta = \nabla_{E_k}$.

To prove this result, we need the following remark and lemmas.

Remark 4.2. In the condition (b), it follows from $\rho \circ \theta = \nabla_{E_k}$ that there exist $u_i \in B_i (i = 0, \dots, t_1 - 1)$ such that $(u_p, u_q) \in \rho$ for all $0 \leq p, q \leq t_1 - 1$. Also $\rho \circ \theta = \nabla_{E_k}$ is equivalent to $\theta \circ \rho = \nabla_{E_k}$.

Lemma 4.3. *Let β be a binary relation on E_k , ρ and θ satisfying the condition (a) or (b) of Theorem 4.1.*

If $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subseteq \text{Pol}(\beta)$, then $\beta \in \{\Delta_{E_k}; \rho; \theta; \nabla_{E_k}\}$.

Proof.

- Assume that ρ and θ satisfy condition (a). Moreover, suppose that $\rho \subsetneq \theta$, $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subseteq \text{Pol}(\beta)$ and $\beta \neq \Delta_{E_k}$. For any distinct $a_1, a_2 \in E_k$ and for any $b_i \in [a_i]_\rho (i = 1, 2)$, the function $f : E_k \rightarrow E_k$ defined by $f(a_i) = b_i, (i = 1, 2)$ and $f(x) = x$ for all $x \in E_k \setminus \{a_1, a_2\}$ satisfies $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$. Since $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subseteq \text{Pol}(\beta)$, $f \in \text{Pol}(\beta)$. It follows that $[a_1]_\rho \times [a_2]_\rho \subseteq \beta$ whenever $(a_1, a_2) \in \beta$. In the following, we denote by β / ρ the relation on $E_k / \rho := \{[a]_\rho | a \in E_k\}$ defined by: $([a_1]_\rho, [a_2]_\rho) \in \beta / \rho \Leftrightarrow (a_1, a_2) \in \beta$. Similarly, we define ρ / ρ and θ / ρ and we have $\text{Pol}(\theta / \rho) \cap \text{Pol}(\rho / \rho) \subseteq \text{Pol}(\beta / \rho)$. Since $\rho / \rho = \Delta_{E_k / \rho}$, $\text{Pol}(\theta / \rho) \cap \text{Pol}(\rho / \rho) = \text{Pol}(\theta / \rho)$ is the maximal clone on E_k / ρ determined by the nontrivial equivalence relation θ / ρ and it follows that $\beta / \rho \in \{\rho / \rho = \Delta_{E_k / \rho}; \theta / \rho; \nabla_{E_k / \rho} = \nabla_{E_k} / \rho\}$. Hence $\beta \in \{\rho; \theta; \nabla_{E_k}\}$.

• Assume that ρ and θ satisfy condition (b). In this case the function $\varphi : E_k \rightarrow E_k / \rho \times E_k / \theta$, defined by $a \mapsto ([a]_\rho, [a]_\theta)$ is a bijection. This bijection gives a decomposition of E_k into a Cartesian product of E_k / ρ and E_k / θ and one deduces that the operations in $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ correspond to the operations that act coordinatewise on $E_k / \rho \times E_k / \theta$.

□

Lemma 4.4. *Let β be a binary relation on E_k . If ρ and θ satisfy condition (a) or (b) in Theorem 4.1, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ contains a majority operation.*

Proof.

• Assume that ρ and θ satisfy condition (a). In addition, assume that $\rho \subsetneq \theta$. In each set B_i , we fix an element v_{B_i} ($i = 0, \dots, t_1 - 1$). We consider the majority operation m defined on E_k by:

$$m(x_1, x_2, x_3) = \begin{cases} x_i & \text{if } (x_i, x_j) \in \rho \text{ and } x_l \notin [x_i]_\rho \\ & \text{for some } 1 \leq i < j \leq 3, \text{ such that } l \notin \{i, j\}, \\ x_1 & \text{if } \{x_1, x_2\} \subseteq [x_3]_\rho, \\ v_{[x_l]_\theta} & \text{if } x_i \notin [x_j]_\rho \text{ for } 1 \leq i < j \leq 3 \text{ and} \\ & (x_l, x_m) \in \theta \text{ for some } 1 \leq l < m \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $m \in \text{Pol}(\rho) \cap \text{Pol}(\theta)$.

Let $(a_i, b_i) \in \rho$, $1 \leq i \leq 3$;

– if $\{a_1, a_2\} \subseteq [a_3]_\rho$, then $\{b_1, b_2\} \subseteq [b_3]_\rho$ and $m(a_1, a_2, a_3) = a_1 \rho b_1 = m(b_1, b_2, b_3)$,

– otherwise, if there exist $1 \leq i < j \leq 3$ such that $(a_i, a_j) \in \rho$, $l \notin \{i, j\}$ and $a_l \notin [a_i]_\rho$, then $(b_i, b_j) \in \rho$, $l \notin \{i, j\}$ and $b_l \notin [b_i]_\rho$, hence $m(a_1, a_2, a_3) = a_i \rho b_i = m(b_1, b_2, b_3)$,

– otherwise, if there exist $1 \leq l < m \leq 3$ such that $(a_l, a_m) \in \theta$, then with transitivity of θ and the fact that $\rho \subsetneq \theta$ we have $(b_l, b_m) \in \theta$ and $m(a_1, a_2, a_3) = v_{[a_l]_\theta} = v_{[b_l]_\theta} = m(b_1, b_2, b_3)$,

– otherwise, we have $m(a_1, a_2, a_3) = 0 = m(b_1, b_2, b_3)$. Therefore $m \in \text{Pol}(\rho)$.

Let $(a_i, b_i) \in \theta$, $1 \leq i \leq 3$;

– if $\{a_1, a_2\} \subseteq [a_3]_\rho$ then the transitivity of θ and the fact that $\rho \subsetneq \theta$, we have $\{a_1, a_2\} \subseteq [a_3]_\theta$ and $\{b_1, b_2\} \subseteq [b_3]_\theta$; therefore

$$(m(a_1, a_2, a_3), m(b_1, b_2, b_3)) \subseteq \{a_i, b_i; v_{[a_i]_\theta}; v_{[b_i]_\theta}\}^2 \subseteq \theta,$$

– otherwise, if there exist $1 \leq i < j \leq 3$ such that $(a_i, a_j) \in \rho$, $l \notin \{i, j\}$ and $a_l \notin [a_i]_\rho$, then $(b_i, b_j) \in \theta$; it follows that $m(b_1, b_2, b_3) \neq 0$ and $m(b_1, b_2, b_3) \in [m(a_1, a_2, a_3)]_\theta$. Since $l \notin \{i, j\}$ and $b_l \notin [b_i]_\rho$,

$$m(a_1, a_2, a_3) = a_i \rho b_i = m(b_1, b_2, b_3),$$

– otherwise, if there exist $1 \leq l < m \leq 3$ such that $(a_l, a_m) \in \theta$, then with transitivity of θ , we have $(b_l, b_m) \in \theta$, and

$$m(a_1, a_2, a_3) = v_{[a_l]_\theta} = v_{[b_l]_\theta} = m(b_1, b_2, b_3),$$

– otherwise, we have $m(a_1, a_2, a_3) = 0 = m(b_1, b_2, b_3)$. Therefore $m \in \text{Pol}(\theta)$.

• Assume that ρ and θ satisfy condition (b). With the decomposition of E_k into a Cartesian product of E_k / ρ and E_k / θ , we can say that, if m_1 is a majority operation on E_k / ρ and m_2 is a majority operation on

E_k / θ , then the operation m on $E_k / \rho \times E_k / \theta$ that acts like m_i in the i -th coordinate ($i = 1, 2$) is a majority operation on $E_k / \rho \times E_k / \theta$ that preserves ρ and θ . $\square$

Lemmas 4.3 and 4.4 together with Theorem 2.1 prove the if-direction of Theorem 4.1 and the only if-direction follows from the following lemmas.

Lemma 4.5. *If $\Delta_{E_k} \subsetneq \gamma \subsetneq \theta$ and $\gamma \subsetneq \rho$, then $D \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\gamma) \subsetneq \text{Pol}(\theta)$.*

Proof. γ is a nontrivial equivalence relation on E_k distinct from θ and ρ . Thus $D \subseteq \text{Pol}(\theta) \cap \text{Pol}(\gamma) \subsetneq \text{Pol}(\theta)$. Let us prove that $D \neq \text{Pol}(\theta) \cap \text{Pol}(\gamma)$; let $(a, b) \in \theta \setminus \gamma$ and $(c, d) \in \rho \setminus \gamma$. Then $(a, b) \notin \rho$ and $(c, d) \notin \theta$. Define $f \in \mathcal{O}^1(E_k)$ by:

$$f(x) = \begin{cases} a & \text{if } x \in B_{\mu(c)}, \\ b & \text{otherwise.} \end{cases}$$

Since $a\theta b$, $f \in \text{Pol}(\theta)$. In addition $\gamma \subseteq \theta$ and f is constant on each block of θ , hence $f \in \text{Pol}(\gamma)$. Therefore $f \in \text{Pol}(\theta) \cap \text{Pol}(\gamma)$ while $f \notin D$ since cpd and $(f(c), f(d)) = (a, b) \notin \rho$. $\square$

Lemma 4.6. *Let ρ and θ be two nontrivial equivalence relations which are incomparable.*

If $\rho \cap \theta \neq \Delta_{E_k}$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\rho \cap \theta) \subsetneq \text{Pol}(\theta)$ and $\text{Pol}(\theta) \cap \text{Pol}(\rho \cap \theta)$ is maximal in $\text{Pol}(\theta)$.

Proof. It follows from the assumptions and Lemma 4.5 that $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\rho \cap \theta) \subsetneq \text{Pol}(\theta)$. As $\Delta_{E_k} \neq \rho \cap \theta \subsetneq \theta \neq \nabla_{E_k}$, the sufficiency part yields $\text{Pol}(\theta) \cap \text{Pol}(\rho \cap \theta)$ is maximal in $\text{Pol}(\theta)$. $\square$

Lemma 4.7. *Let ρ and θ be two nontrivial equivalence relations which are incomparable.*

If $\rho \cap \theta = \Delta_{E_k}$ and $\rho \circ \theta \neq \nabla_{E_k}$, then for $\sigma = \rho \circ \theta \cap \theta \circ \rho$, we have $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\sigma) \subsetneq \text{Pol}(\theta)$.

Proof. By assumptions we have $\sigma \neq \nabla_{E_k}$, hence $\text{Pol}(\theta) \cap \text{Pol}(\sigma) \subsetneq \text{Pol}(\theta)$. From the definition of σ , we get $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subseteq \text{Pol}(\theta) \cap \text{Pol}(\sigma)$. Let us prove that $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\sigma)$; choose $a\theta b$ with $a \neq b$ and $u\rho v$ with $u \neq v$ and define the following unary operation g on E_k by:

$$g(x) = \begin{cases} a & \text{if } x = u, \\ b & \text{otherwise.} \end{cases}$$

As $(a, b) \in \theta$, we have $g \in \text{Pol}(\theta)$ and $g \in \text{Pol}(\sigma)$; ($\theta \subseteq \sigma$). Since $(g(u), g(v)) = (a, b)$, $(a, b) \in \theta \setminus \Delta_{E_k}$ and $\rho \cap \theta = \Delta_{E_k}$, $g \notin \text{Pol}(\rho)$. Hence

$$g \notin \text{Pol}(\theta) \cap \text{Pol}(\rho) \text{ while } g \in \text{Pol}(\theta) \cap \text{Pol}(\sigma).$$

□

The two previous lemmas proved that if $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is a submaximal clone of $\text{Pol}(\theta)$ and ρ and θ are incomparable, then $\rho \cap \theta = \Delta_{E_k}$ and $\rho \circ \theta = \nabla_{E_k}$, so the condition (b) holds.

Proof (of Theorem 4.1). It follows from the previous lemmas. □

We end this section with the following theorem due to Lau and Rosenberg, that deals with prime permutation relations.

Theorem 4.8 ([4]). *If θ is a nontrivial equivalence relation and ρ is a graph of prime permutation π , on a finite set E_k , then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is submaximal in $\text{Pol}(\theta)$ if and only if θ and ρ satisfy one of the following statements:*

(a) $\rho \subsetneq \theta$

(b) *The image of an equivalence class of θ is include in another class of θ surjectively.*

5. Central Relations and h -Regular Relations

In this section, θ is a nontrivial equivalence relation on E_k , whose equivalence classes are: $C_0, C_1, \dots, C_{t-1}$. Our aim is to characterize the central relations or h -regular relations ρ such that $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$.

Firstly, we give some definitions to be used. Let ρ be an h -ary relation ($h > 1$) on E_k . For $i \in \{0; 1; 2; \dots; h-1\}$, we define the relation $\rho_{i,\theta}$ by

$$\rho_{0,\theta} = \{(a_1, \dots, a_h) \in E_k^h / \exists u_i \in [a_i]_\theta, i \in \{1; \dots; h\} \text{ with } (u_1, u_2, \dots, u_h) \in \rho\}$$

and for $i \geq 1$,

$$\begin{aligned} \rho_{i,\theta} = \{ & (a_1, \dots, a_h) \in E_k^h / \exists u_j \in [a_j]_\theta, j \in \{i+1; \dots; h\} \\ & \text{with } (a_1, a_2, \dots, a_i, u_{i+1}, \dots, u_h) \in \rho \}. \end{aligned}$$

For $\sigma \in \mathcal{S}_h$ and γ an h -ary relation, we set

$$\gamma_\sigma = \{(a_{\sigma(1)}, \dots, a_{\sigma(h)}) / (a_1, \dots, a_h) \in \gamma\}.$$

For $J = \{j_1; \dots; j_n\} \subseteq \{1; \dots; h\}$ with $j_1 < \dots < j_n$, we define the h -ary relation ρ_J or $\rho_{j_1 \dots j_n}$ on E_k as follow:

$$\rho_J = \{(a_1, a_2, \dots, a_h) / \exists u_{i,J} \in [a_i]_\theta, i \in \{1; \dots; h\} \setminus J = \{i_1, \dots, i_{h-n}\}\}$$

such that $(a_{j_1}, \dots, a_{j_n}, u_{i_1,J}, \dots, u_{i_{h-n},J}) \in \rho\}$.

The next remark gives some properties of those relations.

Remark 5.1.

- (1) For $i \in \{0; 1; 2; \dots; h-1\}$, $\rho \subseteq \rho_{i,\theta}$ and $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\rho_{i,\theta})$.
- (2) $\rho_{0,\theta}$ is totally symmetric.
- (3) For $\sigma \in \mathcal{S}_h$, $\text{Pol}(\rho) = \text{Pol}(\rho_\sigma)$.
- (4) If $\{j_1, \dots, j_n\} \subseteq \{r_1, \dots, r_m\}$, then $\rho_{\{r_1, \dots, r_m\}} \subseteq \rho_{\{j_1, \dots, j_n\}}$.
- (5) For $J = \{1; \dots; n\}$ we have $\rho_J = \rho_{n,\theta}$.
- (6) For all $1 \leq j_1 < \dots < j_n \leq h$, there exists a permutation $\sigma \in \mathcal{S}_h$ such that $\rho_{j_1 \dots j_n} = (\rho_{n,\theta})_\sigma$.

Definition 5.2. Let ρ be an h -ary relation and θ be a nontrivial equivalence relation on E_k with t classes.

- (1) There is a transversal T for the θ -classes means that there exist $u_1, \dots, u_t \in E_k$ such that $(u_i, u_j) \notin \theta$ for all $1 \leq i < j \leq t$, $(u_{i_1}, u_{i_2}, \dots, u_{i_h}) \in \rho$ for all $1 \leq i_1, \dots, i_h \leq t$ and $T = \{u_1, \dots, u_t\}$.
- (2) There is a transversal T of order l ($1 \leq l \leq h-1$) for the θ -classes means that there exist $u_1, \dots, u_t \in E_k$ such that $(u_i, u_j) \notin \theta$ for all $1 \leq i < j \leq t$, $(a_1, a_2, \dots, a_l, v_{l+1}, v_{l+2}, \dots, v_h) \in \rho$, for all $a_1, a_2, \dots, a_l \in E_k$ and $v_{l+1}, v_{l+2}, \dots, v_h \in \{u_1; \dots; u_t\}$; and $T = \{u_1, \dots, u_t\}$.

Definition 5.3. A transversal of order 0 for the θ -classes means a transversal for the θ -classes.

Definition 5.4. Let ρ be an h -ary relation and θ be a nontrivial equivalence relation on E_k with t classes.

- (1) ρ is θ -closed means that $\rho = \rho_{0,\theta}$.
- (2) ρ is weakly θ -closed of order l ($1 \leq l \leq h-1$) means that there is a transversal $T = \{u_1; \dots; u_t\}$ of order $l-1$ for the θ -classes and $\rho = \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{l,\theta})_\sigma$.

Secondly, we characterize some particular relations. We consider the surjective map

$$\begin{aligned}\varphi : E_k &\rightarrow E_t \\ x &\mapsto \varphi(x) = i \text{ if } x \in C_i.\end{aligned}$$

For an n -ary relation α on E_t , we set

$$\varphi^{-1}(\alpha) = \{(a_1, \dots, a_n) \in E_k^n : (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)) \in \alpha\};$$

for an n -ary relation β on E_k , we set

$$\varphi(\beta) = \{(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)) : (a_1, \dots, a_n) \in \beta\}.$$

Remark 5.5. With the previous considerations and for a central relation ρ , we have:

- ρ is θ -closed if and only if there exists an h -ary central relation γ on E_t such that $\rho = \varphi^{-1}(\gamma)$.
- If ρ is weakly θ -closed of order l ($1 \leq l \leq h-1$), then $\text{Pol}((\rho_{l,\theta})_{\sigma_1} \cap \dots \cap (\rho_{l,\theta})_{\sigma_n}) \subseteq \text{Pol}(\rho)$ for $\{\sigma_1; \dots; \sigma_n\} \subseteq \mathcal{S}_h$.

Remark 5.6. Let ρ be a binary relation and θ be a nontrivial equivalence relation on E_k with t classes.

- (1) ρ is θ -closed if and only if $\rho = \theta \circ \rho \circ \theta$,
- (2) ρ is weakly θ -closed of order 1 (or simply weakly θ -closed) if and only if $\rho = (\theta \circ \rho) \cap (\rho \circ \theta)$, and there is a transversal T for the θ -classes.

Thirdly, we characterize the central relations ρ generating the submaximality.

5.1. Central relations. We recall that for $k = 2$, we have the result in the Post's description. If $k \geq 3$ and $h \in \{1; 2\}$ the following results give the characterization of existing submaximal classes.

Lemma 5.7 ([7]). *If $h = 1$, then $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$ if and only if the following condition is valid:*

$$(\exists I \subseteq \{0; \dots; t-1\} : \rho = \bigcup_{i \in I} C_i) \vee \forall j \in \{0; \dots; t-1\}, \rho \cap C_j \neq \emptyset.$$

Theorem 5.8 ([18]). *If $h = 2$, then $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$ if and only if one of the following conditions is valid:*

- (i) $\theta \subseteq \rho$ and every θ -class contains a central element of ρ ;
- (ii) ρ is θ -closed;
- (iii) ρ is weakly θ -closed of order 1.

In the remaining of this subsection we suppose that $h \geq 3$ and $k \geq 3$.

For a nontrivial equivalence relation θ , we define the h -ary relation η by

$$\eta = \{(u_1, u_2, \dots, u_h) \in E_k^h / u_1 \theta u_2\}.$$

Here we state the main theorem of this subsection:

Theorem 5.9. *Let $k \geq 3$ and let θ be a nontrivial equivalence relation on E_k with t equivalence classes. For an h -ary central relation ρ on E_k , the clone $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is a submaximal clone of $\text{Pol}(\theta)$ if and only if $h \leq t$ and ρ satisfies one of the following three conditions:*

- I. $\eta \subseteq \rho$ and every θ -class contains a central element of ρ .
- II. ρ is θ -closed.
- III. ρ is weakly θ -closed of order l and $\eta \subseteq \rho$.

The proof of Theorem 5.9 will follow from the results obtained below. It will be given at the end of this subsection.

Definition 5.10. Let $l \in \{\text{I; II; III}\}$, ρ is of type l if ρ satisfies the condition l of Theorem 5.9.

The following examples clarify the type of relations defined above.

Example 5.11. Let $k \geq 3$ be an integer and $0 \leq i < j < r < n \leq k-1$, we denote by $A_{i,j,r}$ and $A_{i,j,r,n}$ the sets

$$A_{i,j,r} := \{(\sigma(i), \sigma(j), \sigma(r)); \sigma \in \mathcal{S}_{\{i,j,r\}}\},$$

and

$$A_{i,j,r,n} := \{(\sigma(i), \sigma(j), \sigma(r), \sigma(n)); \sigma \in \mathcal{S}_{\{i,j,r,n\}}\}.$$

We consider the following equivalence relations θ_i defined by their equivalence classes denoted by C_m^i :

$$\theta_1 \text{ is defined on } E_6 \text{ by } C_0^1 = \{0; 1\}, C_1^1 = \{2; 3\}, C_2^1 = \{4; 5\};$$

$$\theta_2 \text{ is defined on } E_5 \text{ by } C_0^2 = \{0; 1\}, C_1^2 = \{2\}, C_2^2 = \{3\}, C_3^2 = \{4\};$$

$$\theta_3 \text{ is defined on } E_4 \text{ by } C_0^3 = \{0; 1\}, C_1^3 = \{2\}, C_2^3 = \{3\};$$

$$\theta_4 \text{ is defined on } E_8 \text{ by } C_0^4 = \{0; 1; 2\}, C_1^4 = \{3; 4; 5\}, C_2^4 = \{6; 7\};$$

$$\theta_5 \text{ is defined on } E_8 \text{ by } C_0^5 = \{0; 1; 2\}, C_1^5 = \{3; 4\}, C_2^5 = \{5; 6\}, C_3^5 = \{7\};$$

and the relations

$$\Upsilon_1 = E_6^3 \setminus A_{1,2,5}, \Upsilon_2 = E_5^3 \setminus A_{2,3,4}, \Upsilon_3 = E_4^3 \setminus A_{1,2,3},$$

$$\begin{aligned} \Upsilon_4 = E_8^3 \setminus (A_{1,4,6} \cup A_{1,4,7} \cup A_{1,5,7} \cup A_{1,5,6} \cup A_{2,3,7} \cup A_{2,4,6} \cup A_{2,4,7} \\ \cup A_{2,5,6} \cup A_{2,5,7}), \end{aligned}$$

$$\Upsilon_5 = E_8^4 \setminus (A_{1,4,6,7} \cup A_{2,4,6,7}).$$

It is easy to see that: Υ_1 is a central relation of type I with θ_1 ; Υ_2 is a central relation of type II but not of type I with θ_2 ; with θ_3 , Υ_3 is a central relation whose center is $\{0\}$ but it is neither of type, I, II or III.

Υ_4 is weakly θ_4 -closed of order 2 with a transversal of order 1, $T_1 = \{0; 3; 6\}$;

Υ_5 is weakly θ_5 -closed of order 3 with a transversal of order 2, $T_2 = \{0; 3; 5; 7\}$.

Definition 5.12. Let θ be an equivalence relation on E_k . An h -ary relation τ on E_k is said to be diagonal through θ if there exists an equivalence relation ε_1 on $\{1; 2; \dots; h\}$ with equivalence classes $A_1, A_2, \dots, A_l$ and an equivalence relation ε_2 on $\{\min(A_m); 1 \leq m \leq l\}$ such that

$$\tau = \{(a_1, a_2, \dots, a_h) \in E_k^h / ((i, j) \in \varepsilon_1 \Rightarrow a_i = a_j) \text{ and } ((i, j) \in \varepsilon_2 \Rightarrow a_i \theta a_j)\}.$$

Given two equivalence relations θ_1 and θ_2 satisfying Definition 5.12, we denote by $D_{\theta_1\theta_2}$ the corresponding diagonal relation through θ .

5.1.1. Proof of the necessity criterion in Theorem 5.9

Proposition 5.13. *If $k \geq 3$, θ is a nontrivial equivalence relation on E_k , and ρ is an h -ary central relation on E_k such that one of conditions I-III is satisfied, then the clone $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is a submaximal clone of $\text{Pol}(\theta)$.*

Before the proof of Proposition 5.13, we give some results characterizing relations containing $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ and we show that $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ contains an h -near-unanimity operation.

For the proof of this proposition we choose a fixed central element c of ρ and denote by C_ρ the set of all central elements of ρ . If ρ is of type I, choose a central element c_B from the θ -class B which can be $\min$

$(B \cap C_\rho)$, and if ρ is of type III and of order l , choose a transversal T of order $l-1$ of the θ -classes and denote by T_B the element of T representing the θ -class B .

We begin with the following lemma characterizing the diagonal relations through θ .

Lemma 5.14. *For an equivalence relation θ on E_k and a diagonal relation τ through θ , with arity h on E_k , we have $\text{Pol}(\tau) = \text{Pol}(\theta)$ or $\text{Pol}(\tau) = \mathcal{O}(E_k)$.*

Proof. Let θ be an equivalence relation on E_k and $\tau = D_{\varepsilon_1 \varepsilon_2}$ be a diagonal relation through θ . Let $T = \{\min A_m; 1 \leq m \leq l\}$, where $A_m, 1 \leq m \leq l$ are as in Definition 5.12. We distinguish two cases: (a) $\varepsilon_2 \neq \Delta_T$ and (b) $\varepsilon_2 = \Delta_T$.

(a) Assume that $\varepsilon_2 \neq \Delta_T$.

There exist $u, v \in T$ with $u < v$ such that for all $(a_1, a_2, \dots, a_h) \in \tau$, we have $(a_u, a_v) \in \theta$. Using the definition of τ , we have $\text{Pol}(\theta) \subseteq \text{Pol}(\tau)$. By setting

$$pr_{uv}(\tau) := \{(e_u^h(\mathbf{a}), e_v^h(\mathbf{a})); \mathbf{a} = (a_1, a_2, \dots, a_h) \in \tau\},$$

it follows that $pr_{uv}(\tau) = \theta$, therefore $\text{Pol}(\tau) \subseteq \text{Pol}(\theta)$, and it appears that

$$\text{Pol}(\tau) = \text{Pol}(\theta).$$

(b) Assume that $\varepsilon_2 = \Delta_T$.

It is clear that

$$\tau = D_{\varepsilon_1 \Delta_T} = \{(a_1, a_2, \dots, a_h) \in E_k^h / i\varepsilon_1 j \Rightarrow a_i = a_j\}.$$

Hence $\text{Pol}(\tau) = \mathcal{O}(E_k)$.

□

Lemma 5.15. *Under the assumptions of Proposition 5.13, we have:*

(a) $\eta \subseteq \rho$;

(b) *If ρ is of type I or II, then an h -ary relation τ on E_k is preserved by every operation in $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ if and only if τ is either the empty relation, a diagonal relation through θ , or the relation ρ ;*

(c) *If ρ is of type III and of order l , then an h -ary relation τ on E_k is preserved by every operation in $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ if and only if τ is either the empty relation, a diagonal relation through θ , or an intersection of relations of the form $(\rho_{l,\theta})_\sigma$ with $\sigma \in \mathcal{S}_h$.*

Proof. (a) The proof of $\eta \subseteq \rho$ is straightforward.

(b) Assume τ is either the empty relation, or a diagonal relation through θ . Then $\text{Pol}(\tau) \in \{\text{Pol}(\theta); \mathcal{O}(E_k)\}$ (see Lemma 5.14); hence $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\tau)$. Assume τ is of the form $(\rho_{l,\theta})_\sigma$. Then it is easy to see that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}((\rho_{l,\theta})_\sigma)$. Hence, if τ is an intersection of relations of the form $(\rho_{l,\theta})_\sigma$, then $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\tau)$.

Conversely, assume τ is preserved by all operations in $\text{Pol}(\rho) \cap \text{Pol}(\theta)$.

Let us suppose that τ is not the empty relation. We have to prove that τ is either a diagonal relation through θ , or an intersection of relations of the form $(\rho_{l,\theta})_\sigma$ with l the order of ρ and $\sigma \in \mathcal{S}_h$.

For this purpose, we define two equivalence relations. The first one denoted by ϵ_1 is defined on $\{1; 2; \dots; h\}$ by:

$$i \epsilon_1 j \text{ iff } \forall (a_1, a_2, \dots, a_h) \in \tau, a_i = a_j.$$

ϵ_1 is an equivalence relation with classes $A_0, A_1, \dots, A_m$. The second one, denoted by ϵ_2 is defined on $T = \{\min(A_i); 0 \leq i \leq m\}$ by:

$$i \epsilon_2 j \text{ iff } \forall (a_1, a_2, \dots, a_h) \in \tau, (a_i, a_j) \in \theta.$$

It follows that $D_{\epsilon_1 \epsilon_2}$ is a diagonal relation through θ . In order to complete the proof of this lemma, we distinguish two cases: $(\epsilon_1 \neq \Delta_{\{1;2;\dots;h\}} \text{ or } \epsilon_2 \neq \Delta_T)$ and $(\epsilon_1 = \Delta_{\{1;2;\dots;h\}} \text{ and } \epsilon_2 = \Delta_T)$.

(i) We suppose that $\epsilon_1 \neq \Delta_{\{1;2;\dots;h\}} \text{ or } \epsilon_2 \neq \Delta_T$. We prove that $D_{\epsilon_1 \epsilon_2} = \tau$. It suffices to show that $D_{\epsilon_1 \epsilon_2} \subseteq \tau$; $\tau \subseteq D_{\epsilon_1 \epsilon_2}$ by definitions of ϵ_1 and ϵ_2 . We need only consider three subcases: $(\epsilon_1 = \nabla_{\{1;2;\dots;h\}})$, $(\epsilon_1 \neq \nabla_{\{1;2;\dots;h\}} \text{ and } \epsilon_2 = \nabla_T)$, and $(\epsilon_1 \neq \nabla_{\{1;2;\dots;h\}} \text{ and } \epsilon_2 \neq \nabla_T)$.

(a) If $\epsilon_1 = \nabla_{\{1;2;\dots;h\}}$, then $D_{\epsilon_1 \epsilon_2} = \{(x, \dots, x); x \in E_k\}$. Since $\tau \subseteq D_{\epsilon_1 \epsilon_2}$ and τ is not the empty relation, for each $b \in E_k$ the constant function of value b preserves θ and ρ ; hence $(b, \dots, b) \in \tau$ and $\tau = D_{\epsilon_1 \epsilon_2}$.

(b) If $\epsilon_1 \neq \nabla_{\{1;\dots;h\}}$ and $\epsilon_2 = \nabla_T$, then there exists $(i, j) \in \nabla_{\{1;\dots;h\}}$ such that $(i, j) \notin \epsilon_1$. For all $1 \leq i, j \leq h$ such that $(i, j) \notin \epsilon_1$, we choose $\mathbf{a}^{ij} = (a_1^{ij}, \dots, a_h^{ij}) \in \tau$ such that $a_i^{ij} \neq a_j^{ij}$ and we set $B = \{\mathbf{a}^{ij}; (i, j) \notin \epsilon_1\}$. Let $q = |B|$; to simplify our notation, we suppose that $B = \{\mathbf{b}^1; \mathbf{b}^2; \dots; \mathbf{b}^q\}$ and we define the sequence $(\mathbf{x}_i)_{1 \leq i \leq h}$ by $\mathbf{x}_i = (b_i^1; b_i^2; \dots; b_i^q)$. It is easy to see that: $(i, j) \notin \epsilon_1 \Rightarrow \mathbf{x}_i \neq \mathbf{x}_j$. Let $(u_1, \dots, u_h) \in D_{\epsilon_1 \epsilon_2}$, we consider the q -ary operation defined on E_k by:

$$f(\mathbf{y}) = \begin{cases} u_l & \text{if } \mathbf{y} = \mathbf{x}_l \in \{\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_h\}, \\ u_1 & \text{otherwise.} \end{cases}$$

Since $\{u_1, u_2, \dots, u_h\}^2 \subseteq \theta$ and $\eta \subseteq \rho$, $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$. Therefore $f \in \text{Pol}(\tau)$ and $(u_1, \dots, u_h) \in \tau$.

(c) If $\epsilon_1 \neq \nabla_{\{1;2;\dots;h\}}$ and $\epsilon_2 \neq \nabla_T$, then there exists $(i, j) \in \nabla_T$ such that $(i, j) \notin \epsilon_2$. For all $1 \leq i, j \leq h$ such that $(i, j) \notin \epsilon_2$ we choose $\mathbf{a}^{ij} = (a_1^{ij}, a_2^{ij}, \dots, a_h^{ij}) \in \tau$ such that $(a_i^{ij}, a_j^{ij}) \notin \theta$ and we consider the set $B = \{\mathbf{a}^{ij}, (i, j) \notin \epsilon_2\}$. We set $q = |B|$. To simplify our notation we set $B = \{\mathbf{b}^1; \mathbf{b}^2; \dots; \mathbf{b}^q\}$ which allows us to define $\mathbf{x}_1 = (b_1^1, b_1^2, \dots, b_1^q)$, $\mathbf{x}_2 = (b_2^1, b_2^2, \dots, b_2^q)$, $\dots$, $\mathbf{x}_s = (b_s^1, b_s^2, \dots, b_s^q)$. We remark that $(i, j) \notin \epsilon_2 \Rightarrow (\mathbf{x}_i, \mathbf{x}_j) \notin \theta$.

Let $(u_1, u_2, \dots, u_h) \in D_{\epsilon_1 \epsilon_2}$ and consider the q -ary operation defined on E_k by:

$$f(\mathbf{y}) = \begin{cases} u_l & \text{if } \mathbf{y} = \mathbf{x}_l, \\ u_{\sigma(\mathbf{y})} & \text{if } \mathbf{y} \notin \{\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_h\}, \exists l \in \{1, 2, \dots, h\} / \mathbf{y} \theta \mathbf{x}_l \text{ and} \\ & \sigma(\mathbf{y}) = \min\{t / \mathbf{y} \theta \mathbf{x}_t\}, \\ c & \text{otherwise,} \end{cases}$$

where c is a central element. Since ρ is totally reflexive and $D_{\epsilon_1 \epsilon_2} \subseteq \rho$, it follows that $f \in \text{Pol}(\rho)$. By the definition of f , we have $f \in \text{Pol}(\theta)$, thus $f \in \text{Pol}(\tau)$. Since $\mathbf{b}^1, \dots, \mathbf{b}^q \in \tau$, it follows that $f(\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^q) = (f(\mathbf{x}_1), f(\mathbf{x}_2), \dots, f(\mathbf{x}_h)) = (u_1, u_2, \dots, u_h) \in \tau$, therefore $D_{\epsilon_1 \epsilon_2} \subseteq \tau$.

(ii) We suppose that $\epsilon_1 = \Delta_{\{1;2;\dots;h\}}$ and $\epsilon_2 = \Delta_T$. We show that τ is an intersection of relations of the form $(\rho_{l,\theta})_\sigma$ or the relation E_k^h .

For all $1 \leq i, j \leq h$ such that $i \neq j$, choose $\mathbf{a}^{ij} = (a_1^{ij}, \dots, a_s^{ij}) \in \tau$ such that $(a_i^{ij}, a_j^{ij}) \notin \theta$ and consider the set $B = \{\mathbf{a}^{ij}, (i, j) \notin \epsilon_2\}$. Using similar notation as in part (c) of (i) and the same q -ary operation f on E_k for a given $(u_1, u_2, \dots, u_h) \in \rho$, we have

$$f(\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^q) = (f(\mathbf{x}_1), f(\mathbf{x}_2), \dots, f(\mathbf{x}_h)) = (u_1, u_2, \dots, u_h) \in \tau,$$

and then $\rho \subseteq \tau$. By the definition of $(\rho_{l,\theta})_\sigma$ with l the order of ρ and $\sigma \in \mathcal{S}_h$, we have: $\rho \subseteq (\rho_{l,\theta})_\sigma$ for all $\sigma \in \mathcal{S}_h$. We distinguish once more two subcases: $\rho = \tau$ or $\rho \neq \tau$.

(a) If $\rho = \tau$, then it is finished.

(b) Otherwise $\rho \subsetneq \tau \subseteq E_k^h$.

If ρ is of type I or II, then we show that $\tau = E_k^h$.

As $\rho \subsetneq \tau$ then $\tau \setminus \rho \neq \emptyset$. Let us consider $(a_1, a_2, \dots, a_h) \in \tau \setminus \rho$. Let $(u_1, u_2, \dots, u_h) \in E_k^h$; assume ρ is of type I and consider the unary operation f defined on E_k by:

$$f(x) = \begin{cases} u_i & \text{if } x = a_i, \\ c_{[a_i]_\emptyset} & \text{if } x\theta a_i \text{ and } x \neq a_i, \\ c & \text{otherwise,} \end{cases}$$

where c is a central element. $f \in \text{Pol}(\rho)$ because $\eta \subseteq \rho$ and ρ is totally symmetric. Moreover $f \in \text{Pol}(\theta)$, then $f \in \text{Pol}(\tau)$ and $(u_1, u_2, \dots, u_h) = (f(a_1), f(a_2), \dots, f(a_h)) \in \tau$.

Assume ρ is of type II and consider the unary operation f defined on E_k by:

$$f(x) = \begin{cases} u_i & \text{if } x\theta a_i, \\ c & \text{otherwise,} \end{cases}$$

where c is a central element. $f \in \text{Pol}(\rho)$ because ρ is θ -closed. Moreover $f \in \text{Pol}(\theta)$, then $f \in \text{Pol}(\tau)$ and $(u_1, u_2, \dots, u_h) = (f(a_1),$

$f(a_2), \dots, f(a_h)) \in \tau$. Thus $\tau = E_k^h = D_{\Delta_{\{1, \dots, h\}} \Delta_{\{1, \dots, h\}}}$. Let's suppose that ρ is of type III and of order l , i.e., ρ is weakly θ -closed of order l and $\eta \subseteq \rho$.

Since $\tau \setminus \rho$ is not empty, there exists $(a_1, a_2, \dots, a_h) \in (\tau \setminus \rho)$. Therefore $(a_1, a_2, \dots, a_h) \notin \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{l, \theta})_\sigma$. We suppose that there is $\sigma' \in \mathcal{S}_h$ such that $(a_1, a_2, \dots, a_h) \in (\rho_{l, \theta})_{\sigma'}$. We consider the set $R_1 := \{\sigma' \in \mathcal{S}_h / (a_1, a_2, \dots, a_h) \in (\rho_{l, \theta})_{\sigma'}\}$ and define the relation φ by

$$\varphi = \bigcap_{\sigma' \in R_1} (\rho_{l, \theta})_{\sigma'}.$$

We have $\rho \subsetneq \varphi$.

We show that $\varphi \subseteq \tau$. Let $(u_1, \dots, u_h) \in \varphi$ and set

$$D = \{(b_1, \dots, b_h) \in E_k^h; b_i \in \{u_1, \dots, u_h\} \cup \{c, T_{[u_1]_\theta}, \dots, T_{[u_h]_\theta}\}, 1 \leq i \leq h, \\ \text{and } \{b_1, \dots, b_h\} \neq \{u_1, \dots, u_h\}\}.$$

We define the unary operation h on E_k by:

$$h(x) = \begin{cases} u_i & \text{if } x = a_i, \\ T_{[u_i]_\theta} & \text{if } x \in [a_i]_\theta \setminus \{a_i\}, \\ c & \text{otherwise.} \end{cases}$$

For $(y_1, \dots, y_h) \in \rho$, we have $(h(y_1), \dots, h(y_h)) \in D \cap \rho \subseteq \rho$. Therefore $h \in \text{Pol}(\rho) \cap \text{Pol}(\theta)$. Hence $h \in \text{Pol}(\tau)$ and $(u_1, \dots, u_h) = (h(a_1), \dots, h(a_h)) \in \tau$.

If $\tau = \varphi$, then it is finished. Otherwise, we have $\varphi \subsetneq \tau$.

There exists $(a_1, a_2, \dots, a_h) \in \tau \setminus \varphi$. If there exists $s \in \mathcal{S}_h$ such that $(a_1, a_2, \dots, a_h) \in (\rho_{l,\theta})_s$, then we use the same argument to construct φ' such that $\varphi \subsetneq \varphi'$ and $\varphi' \subseteq \tau$. Therefore $\tau = \varphi'$ or $\varphi' \subsetneq \tau$. So we have the same conclusion as above. We continue this process until obtained a h -tuple $(a_1, a_2, \dots, a_h) \in \tau$ such that, for each $\sigma \in \mathcal{S}_h$, $(a_1, a_2, \dots, a_h) \notin (\rho_{l,\theta})_\sigma$. Let $(u_1, \dots, u_h) \in E_k^h$, using the unary operation h defined above and the fact that ρ is weakly θ -closed of order l and there is a transversal T of order $l-1$ for the θ -classes, we show that $(u_1, \dots, u_h) \in \tau$. Therefore $\tau = E_k^h$. $\square$

Lemma 5.16. *Under the assumptions of Proposition 5.13, $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ contains an h -near-unanimity operation.*

Proof. If ρ is of type I, let us consider the $(s+1)$ -ary operation m defined on E_k by:

$$m(x_1, \dots, x_h, x_{h+1}) = \begin{cases} x_{i_1} & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } x_{i_1} = x_{i_2} = \dots = x_{i_s}, \\ c_{[x_{i_1}]_\theta} & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } [x_{i_1}]_\theta = [x_{i_2}]_\theta = \dots = [x_{i_s}]_\theta, \\ c & \text{otherwise;} \end{cases}$$

if ρ is of type II, let us consider the $(s+1)$ -ary operation m defined on E_k by:

$$m(x_1, \dots, x_h, x_{h+1}) = \begin{cases} x_{i_1} & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } x_{i_1} = x_{i_2} = \dots = x_{i_h}, \\ x_{i_1} & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } [x_{i_1}]_\theta = [x_{i_2}]_\theta = \dots = [x_{i_h}]_\theta, \\ c & \text{otherwise;} \end{cases}$$

if ρ is of type III, let us consider the $(s+1)$ -ary operation m defined on E_k by:

$$m(x_1, \dots, x_h, x_{h+1}) = \begin{cases} x_{i_1} & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } x_{i_1} = x_{i_2} = \dots = x_{i_h}, \\ T[x_{i_1}]_\theta & \text{if there exist } 1 \leq i_1 < \dots < i_h \leq h+1 \\ & \text{such that } [x_{i_1}]_\theta = [x_{i_2}]_\theta = \dots = [x_{i_h}]_\theta, \\ c & \text{otherwise;} \end{cases}$$

where c is a central element. By definition, m is a near unanimity function of order $h+1$. We show that $m \in \text{Pol}(\rho) \cap \text{Pol}(\theta)$. To do this, we prove firstly that $m \in \text{Pol}(\theta)$ and secondly that $m \in \text{Pol}(\rho)$.

(a) To see that m preserves θ , assume that $(u_i, v_i) \in \theta$ for $1 \leq i \leq s+1$. Since

$$|\{[u_1]_\theta; [u_2]_\theta; \dots; [u_{h+1}]_\theta\}| = |\{[v_1]_\theta; [v_2]_\theta; \dots; [v_{h+1}]_\theta\}|,$$

if there exist $1 \leq i_1 < i_2 < \dots < i_h \leq h+1$ such that $[u_{i_1}]_\theta = [u_{i_2}]_\theta = \dots = [u_{i_h}]_\theta$, then $m(u_1, u_2, \dots, u_{h+1}) \in [u_{i_1}]_\theta = [v_{i_1}]_\theta$ because $[u_i]_\theta = [v_i]_\theta$ for $i = 1, 2, \dots, h+1$.

Therefore, $(m(u_1, u_2, \dots, u_{h+1}), m(v_1, v_2, \dots, v_{h+1})) \in [u_{i_1}]_\theta^2 \subseteq \theta$. In the other case, we have $(m(u_1, u_2, \dots, u_{h+1}), m(v_1, v_2, \dots, v_{h+1})) = (c, c) \in \theta$; then $m \in \text{Pol}(\theta)$.

(b) Let us prove now that m preserves ρ .

Let $(u_{11}, u_{21}, \dots, u_{h1}), (u_{12}, u_{22}, \dots, u_{h2}), \dots, (u_{1h+1}, u_{2h+1}, \dots, u_{hh+1}) \in \rho$.
If $\{m(u_{i1}, u_{i2}, \dots, u_{ih+1}); 1 \leq i \leq h\}$ contains a central element of ρ , then

$$(m(u_{11}, u_{12}, \dots, u_{1h+1}), \dots, m(u_{h1}, u_{h2}, \dots, u_{hh+1})) \in \rho;$$

else we distinguish the three type of ρ .

Suppose ρ be of type I, there exist $1 \leq i_r^1 < i_r^2 < \dots < i_r^h \leq h+1$, $1 \leq r \leq h$ such that $u_{ri_r^1} = u_{ri_r^2} = \dots = u_{ri_r^h}$ for $r \in \{1; \dots; h\}$; as $\{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\} \neq \emptyset$, let us consider a fixed element $i \in \{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\}$. We have

$$(m(u_{11}, u_{12}, \dots, u_{1h+1}), \dots, m(u_{h1}, u_{h2}, \dots, u_{hh+1})) = (u_{1i}, \dots, u_{hi}) \in \rho.$$

Therefore $m \in \text{Pol}(\rho)$.

If ρ is of type II there exist $1 \leq i_r^1 < i_r^2 < \dots < i_r^h \leq h+1$, $1 \leq r \leq h$ such that $[u_{ri_r^1}]_\theta = [u_{ri_r^2}]_\theta = \dots = [u_{ri_r^h}]_\theta$ for $r \in \{1; \dots; h\}$; as $\{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\} \neq \emptyset$, then with $i \in \{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\}$ we have

$$(m(u_{11}, u_{12}, \dots, u_{1h+1}), \dots, m(u_{h1}, u_{h2}, \dots, u_{hh+1})) \in [u_{1i}]_\theta \times \dots \times [u_{hi}]_\theta \subseteq \rho$$

because ρ is θ -closed. Thus $m \in \text{Pol}(\rho)$.

Finally, if ρ is of type III, there exist $1 \leq i_r^1 < i_r^2 < \dots < i_r^h \leq h+1$, $1 \leq r \leq h$ such that $[u_{ri_r^1}]_\theta = [u_{ri_r^2}]_\theta = \dots = [u_{ri_r^h}]_\theta$ for $r \in \{1; \dots; h\}$; as

$$\{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\} \neq \emptyset,$$

then with $i \in \{i_1^1; i_1^2; \dots; i_1^h\} \cap \{i_2^1; i_2^2; \dots; i_2^h\} \cap \dots \cap \{i_h^1; i_h^2; \dots; i_h^h\}$ we have (with $Z := (m(u_{11}, \dots, u_{1h+1}), \dots, m(u_{h1}, \dots, u_{hh+1}))$)

$$Z \in \{u_{1i}, \dots, u_{hi}, T_{[u_{1i}]_0}, \dots, T_{[u_{hi}]_0}\}^h \subseteq \rho.$$

Hence $m \in \text{Pol}(\rho)$. $\square$

Proof (of Proposition 5.13). Let $f \in \text{Pol}(\theta) \setminus (\text{Pol}(\theta) \cap \text{Pol}(\rho))$. We prove that $G = \langle \text{Pol}(\theta) \cap \text{Pol}(\rho) \cup \{f \rangle$ is equal to $\text{Pol}(\theta)$. From Lemma 5.16, it follows that $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ contains an h -near-unanimity function m . According to Theorem 2.1 and the fact that $m \in G$, we have $G = \text{PolInv}^{(h)}G$. If $\tau \in \text{Inv}^{(h)}G$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho) \subseteq \text{Pol}(\theta) \cap \text{Pol}(\rho) \cup \{f\} = G = \text{PolInv}^{(h)}G \subseteq \text{Pol}(\tau)$. By Lemma 5.15, if ρ is of type I or II, then τ is either the empty relation, either a diagonal relation through θ or ρ ; if ρ is of type III and of order l , then τ is either the empty relation, either a diagonal relation through θ or an intersection of relations of the form $(\rho_{l,\theta})_\sigma$ with $\sigma \in \mathcal{S}_h$. Since $f \notin \text{Pol}(\rho)$, therefore f can not preserve an intersection of relations of the form $(\rho_{l,\theta})_\sigma$ with $\sigma \in \mathcal{S}_h$; therefore τ is the empty relation or a diagonal relation through θ . In the light of Lemma 5.14, we have $\text{Pol}(\theta) \subseteq G$. $\square$

Remark 5.17. Let us mention that if ρ is of type I or II, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is also maximal in $\text{Pol}(\rho)$, whereas if ρ is of type III, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\rho)$.

5.1.2. Proof of the completeness criterion

In this subsection, we show that the relations of type I, II and III are the only central relations ρ such that $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$. We assume that $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$ and show that apart from these types, we don't have maximality.

Proposition 5.18. *If $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$, then ρ is either of type I, II, or III.*

Before the proof of Proposition 5.18, we give some results on the properties of the relation ρ .

Lemma 5.19. *If $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$, then $\eta \subseteq \rho$.*

Proof. By contraposition, suppose that $\eta \not\subseteq \rho$, therefore there exists

$$(u_1, u_2, \dots, u_h) \in \eta \quad \text{such that} \quad (u_1, u_2, \dots, u_h) \notin \rho.$$

We denote by η^1 the relation

$$\eta^1 = \{(u_1, u_2, \dots, u_h) \in \rho / u_1 \theta u_2\}.$$

The following inclusions hold:

$$\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\eta^1) \subsetneq \text{Pol}(\theta).$$

(i) Let $f \in \text{Pol}(\rho) \cap \text{Pol}(\theta)$ with arity n , we prove that $f \in \text{Pol}(\eta^1)$.

Let $(u_{11}, u_{21}, \dots, u_{h1}), \dots, (u_{1n}, u_{2n}, \dots, u_{hn}) \in \eta^1$.

We have $f(u_{11}, \dots, u_{1n}) \theta f(u_{21}, \dots, u_{2n})$ since $f \in \text{Pol}(\theta)$ and

$$(f(u_{11}, \dots, u_{1n}), f(u_{21}, \dots, u_{2n}), \dots, f(u_{h1}, \dots, u_{hn})) \in \rho$$

since $f \in \text{Pol}(\rho)$. Therefore $f \in \text{Pol}(\eta^1)$.

Let $(u_1, u_2, \dots, u_h) \in \eta \setminus \rho$ and $(a, b) \notin \theta$ fixed, we define the h -ary operation f on E_k by:

$$f(x_1, x_2, \dots, x_h) = \begin{cases} u_1 & \text{if } (x_1, x_2, \dots, x_h) \theta (a, a, \dots, a), \\ u_2 & \text{if } (x_1, x_2, \dots, x_h) \theta (a, b, a, \dots, a), \\ \vdots & \\ u_h & \text{if } (x_1, x_2, \dots, x_h) \theta (a, a, \dots, a, b), \\ u_1 & \text{otherwise.} \end{cases} \quad (5.1)$$

With $(u_{11}, u_{21}, \dots, u_{h1}), \dots, (u_{1s}, u_{2s}, \dots, u_{ss}) \in \eta^1$, according to definition of η^1 , we have $f(u_{11}, \dots, u_{1s}) = f(u_{21}, \dots, u_{2s})$ since $f \in \text{Pol}(\theta)$ and $(u_{11}, \dots, u_{1n})\theta(u_{21}, \dots, u_{2n})$. From the fact that ρ is totally reflexive we have

$$(f(u_{11}, \dots, u_{1n}), f(u_{21}, \dots, u_{2n}), \dots, f(u_{h1}, \dots, u_{hn})) \in \eta^1.$$

Hence $f \in \text{Pol}(\eta^1)$.

$$\text{Furthermore, } \begin{pmatrix} a & a & a & \cdots & a & a \\ a & b & a & \cdots & a & a \\ \vdots & & & & & \\ a & a & a & \cdots & a & b \end{pmatrix} \subseteq \rho \text{ and by the definition of } f \text{ we}$$

have

$$(u_1, u_2, \dots, u_h) = (f(a, a, a, \dots, a), f(a, b, a, \dots, a), \dots, f(a, a, \dots, a, b)) \notin \rho.$$

Therefore, $f \notin \text{Pol}(\rho)$ and $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\eta^1)$.

(ii) From the equality $pr_{12}(\eta^1) = \theta$, it follows that $\text{Pol}(\eta^1) \subseteq \text{Pol}(\theta)$.

Let $(u_1, u_2, \dots, u_h) \in \eta \setminus \rho$ be a fixed element and c be a central element of ρ . It is obvious that $(u_1, u_2, \dots, u_h) \notin \eta^1$. Let $a, b, c \in E_k$ such that $(a, b) \in \theta$, $a \neq b$ and $(a, c) \notin \theta$.

For $1 \leq i \leq h$, we define the $(h-1)$ -tuple $\mathbf{W}_i = (w_{i1}, \dots, w_{ih-1})$ by:

$$w_{1l} = a \text{ for } 1 \leq l \leq h-1,$$

$$w_{2l} = \begin{cases} b & \text{if } l = h-1, \\ a & \text{elsewhere,} \end{cases}$$

$$\text{for } m \geq 3, w_{ml} = \begin{cases} c & \text{if } l = m - 2, \\ a & \text{elsewhere.} \end{cases}$$

We have $\mathbf{W}_1 \neq \mathbf{W}_2$ and for every $1 \leq i < j \leq h$, $\mathbf{W}_i \theta \mathbf{W}_j$ if and only if $i = 1$ and $j = 2$. We define the $(h - 1)$ -ary operation g on E_k by:

$$g(x_1, x_2, \dots, x_{h-1}) = \begin{cases} u_1 & \text{if } (x_1, x_2, \dots, x_{h-1}) = \mathbf{W}_1, \\ u_2 & \text{if } (x_1, x_2, \dots, x_{h-1}) = \mathbf{W}_2, \\ u_i & \text{if } (x_1, x_2, \dots, x_{h-1}) \theta \mathbf{W}_i \text{ for some } 3 \leq i \leq h - 1, \\ u_h & \text{elsewhere.} \end{cases}$$

Since $g(\theta) \subseteq \{(u_1, u_2); (u_2, u_1)\} \cup \{(u_i, u_i) : i \in \{1; \dots; h\}\}$, $g \in \text{Pol}(\theta)$.

The matrix $(w_{ij})_{\substack{1 \leq i \leq h \\ 1 \leq j \leq h-1}}$ is a subset of η^1 and

$$g\left((w_{ij})_{\substack{1 \leq i \leq h \\ 1 \leq j \leq h-1}}\right) = g((\mathbf{W}_1), \dots, g(\mathbf{W}_h)) = (u_1, \dots, u_h) \notin \eta^1.$$

Hence $g \notin \text{Pol}(\eta^1)$. Therefore $\text{Pol}(\eta^1) \subsetneq \text{Pol}(\theta)$.

□

Remark 5.20. If $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$, then the arity of ρ is less than or equal to t , where t is the number of equivalence classes of θ . Indeed, if the arity of ρ is greater than t , then $\eta \not\subseteq \rho$, because $\rho \neq E_k^{\text{arity}(\rho)}$ and $(\eta \subseteq \rho \Rightarrow \rho = E_k^{\text{arity}(\rho)})$.

From now on we suppose that $\eta \subseteq \rho$.

According to the definition of $\rho_{0,\theta}$ we have $\rho \subseteq \rho_{0,\theta}$.

Lemma 5.21. *If $\rho = \rho_{0,\theta}$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$.*

Proof. If $\rho = \rho_{0,\theta}$, then ρ is θ -closed. Hence ρ is of type II and $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$ from Proposition 5.13. $\square$

Now we suppose that $\rho \subsetneq \rho_{0,\theta}$ and we have two cases given by the following lemmas.

Lemma 5.22. *If $\rho \subsetneq \rho_{0,\theta} \subsetneq E_k^h$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$*

Proof. It is easy to see that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\theta) \cap \text{Pol}(\rho_{0,\theta}) \subseteq \text{Pol}(\theta)$.

Let $a, b \in E_k$ such that $(a, b) \notin \theta$ and $(u_1, u_2, \dots, u_h) \in \rho_{0,\theta} \setminus \rho$ (respectively, $E_k^h \setminus \rho$). Using the h -ary operation f defined in the proof of Lemma 5.19, we show that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\rho_{0,\theta})$ (respectively, $\text{Pol}(\theta) \cap \text{Pol}(\rho_{0,\theta}) \subsetneq \text{Pol}(\theta)$.) Hence $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\rho_{0,\theta}) \subsetneq \text{Pol}(\theta)$. $\square$

Lemma 5.23. *If $\rho \subsetneq \rho_{0,\theta} = E_k^h$ and there exists an integer $l > h$ such that $\rho_{0,\theta}^l \neq E_k^l$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$, where $\rho_{0,\theta}^l$ is the relation*

$$\rho_{0,\theta}^l = \{\mathbf{u} \in E_k^l / \exists (e_1, \dots, e_l) \in [u_1]_\theta \times \dots \times [u_l]_\theta; \{e_1, \dots, e_l\}^h \subseteq \rho\}.$$

Proof. We show that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$, where $m = \min\{l \in \mathbb{N} \setminus \{0; 1; 2; \dots; h\}; \rho_{0,\theta}^l \neq E_k^l\}$. We distinguish two cases; one for each inclusion.

(i) Let us prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta)$.

Let $f \in \mathcal{O}^n(E_k)$ be an n -ary operation on E_k such that $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$. According to the definition of $\rho_{0,\theta}^m$, for

$$(u_{11}, \dots, u_{m1}), \dots, (u_{1n}, \dots, u_{mn}) \in \rho_{0,\theta}^m,$$

there exists $a_{ij} \in [u_{ij}]_\theta$ such that for all $j \in \{1; \dots; n\}$ we have $\{a_{1j}, \dots, a_{mj}\}^h \subseteq \rho$.

Then, it follows that there exists $a_{ij} \in [u_{ij}]_\theta$ such that for all $i_1, i_2, \dots, i_h \in \{1; \dots; m\}$, we have

$$(a_{i_1 1}, a_{i_2 1}, \dots, a_{i_h 1}), \dots, (a_{i_1 n}, a_{i_2 n}, \dots, a_{i_h n}) \in \rho;$$

which implies that there exist $a_{ij} \in [u_{ij}]_\theta$ such that for all $i_1, i_2, \dots, i_h \in \{1; \dots; m\}$, we have

$$(f(a_{i_1 1}, \dots, a_{i_1 n}), f(a_{i_2 1}, \dots, a_{i_2 n}), \dots, f(a_{i_h 1}, \dots, a_{i_h n})) \in \rho.$$

Since $f \in \text{Pol}(\theta)$ and for all $i \in \{1; \dots; m\}$, $(u_{i1}, \dots, u_{in})_\theta(a_{i1}, \dots, a_{in})$, it appears that

$$(f(u_{11}, \dots, u_{1n}), \dots, f(u_{m1}, \dots, u_{mn})) \in \rho_{0,\theta}^m \quad \text{and} \quad f \in \rho_{0,\theta}^m.$$

Hence $f \in \text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta)$.

Let $(v_1, \dots, v_h) \in E_k^h \setminus \rho$ and $(a, b) \notin \theta$, using the h -ary operation f on E_k , specified by:

$$f(x_1, x_2, \dots, x_h) = \begin{cases} v_1 & \text{if } (x_1, x_2, \dots, x_h)\theta(a, a, \dots, a), \\ v_2 & \text{if } (x_1, x_2, \dots, x_h)\theta(a, b, a, \dots, a), \\ \vdots & \\ v_h & \text{if } (x_1, x_2, \dots, x_h)\theta(a, a, \dots, a, b, a), \\ v_h & \text{if } (x_1, x_2, \dots, x_h)\theta(a, a, \dots, a, a, b), \\ v_1 & \text{otherwise,} \end{cases}$$

and the fact that $m > h$ and $\rho_{0,\theta}^m$ is totally reflexive, we obtain $f \notin \text{Pol}(\rho)$ and $f \in \text{Pol}(\rho_{0,\theta}^m)$.

(ii) We prove that $\text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$.

Let $(u_1, \dots, u_m) \in E_k^m \setminus \rho_{0,\theta}^m(E_k^m \setminus \rho_{0,\theta}^m)$ is not empty because $\rho_{0,\theta}^m \neq E_k^m$ and $(a, b) \notin \theta$. Let us consider the m -ary operation h on E_k defined by:

$$h(x_1, \dots, x_m) = \begin{cases} u_1 & \text{if } (x_1, \dots, x_m)\theta(a, \dots, a, a) = a_1, \\ u_2 & \text{if } (x_1, \dots, x_m)\theta(a, b, a, \dots, a) = a_2, \\ u_3 & \text{if } (x_1, \dots, x_m)\theta(a, a, b, a, \dots, a) = a_3, \\ \vdots & \\ u_m & \text{if } (x_1, \dots, x_m)\theta(a, a, a, \dots, a, b) = a_m, \\ u_m & \text{otherwise.} \end{cases} \quad (5.2)$$

The implication $a_i\theta a_j \Rightarrow i = j$ allows us to say that $h \in \text{Pol}(\theta)$. It is clear that

$$\begin{pmatrix} a \\ \vdots \\ a \\ a \\ a \\ a \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ a \\ \vdots \\ a \\ a \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ a \\ b \\ a \\ \vdots \\ a \\ a \\ a \end{pmatrix}, \dots, \begin{pmatrix} a \\ a \\ a \\ \vdots \\ a \\ a \\ a \\ b \end{pmatrix} \in \rho_{0,\theta}^m,$$

but following the definition of h , it appears that

$$(u_1, u_2, u_3, \dots, u_m) = (h(a_1), h(a_2), h(a_3), \dots, h(a_m)) \notin \rho_{0,\theta}^m.$$

Therefore, $h \notin \text{Pol}(\rho_{0,\theta}^m)$ and $\text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$.

Hence $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\rho_{0,\theta}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

In the light of Lemma 5.23, it is natural to suppose $\rho_{0,\theta} = E_k^h$ and $\rho_{0,\theta}^l = E_k^l$ for all $l > h$. In particular, for $l = t$, we have $\rho_{0,\theta}^t = E_k^t$ and there exists $(x_1, x_2, \dots, x_t) \in C_0 \times C_1 \times \dots \times C_{t-1}$ such that $\{x_1, x_2, \dots, x_t\}^h \subseteq \rho$. Let ς be the relation

$$\varsigma = \{\mathbf{a} \in E_k^h \mid \exists u_{ij} \in [a_i]_0, 1 \leq i, j \leq h \text{ with } i \neq j \text{ such that } \forall j \in \{1; \dots; h\},$$

$$(a_j, u_{j_1j}, u_{j_2j}, \dots, u_{j_{h-1}j}) \in \rho, \text{ with } \{j_1; \dots; j_{h-1}\} = \{1; \dots; h\} \setminus \{j\}\}.$$

Since ρ is totally symmetric, $\varsigma = \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{1,\theta})_\sigma$ and it is obvious that

$$\rho \subseteq \varsigma.$$

Lemma 5.24. *If $\rho = \varsigma$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$.*

Proof. If $\rho = \varsigma$, then ρ is weakly θ -closed of order 1. Hence ρ is a relation of type III. By Proposition 5.13, $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$. $\square$

Lemma 5.25. *If $\rho \subsetneq \varsigma \subsetneq E_k^h$ then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$*

Proof. We prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\varsigma) \subsetneq \text{Pol}(\theta)$. Let $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$, n -ary.

Let $(a_{11}, a_{21}, \dots, a_{h1}), \dots, (a_{1n}, a_{2n}, \dots, a_{sn}) \in \varsigma$, we show that

$$(f(a_{11}, \dots, a_{1n}), f(a_{21}, \dots, a_{2n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \varsigma.$$

Firstly, we show that $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \rho_{1,\theta}$. Since $(a_{1i}, \dots, a_{hi}) \in \varsigma$ for $1 \leq i \leq n$, there exist $u_{ji} \in [a_{ji}]_\theta$, $2 \leq j \leq h$ such that $(a_{1i}, u_{2i}, \dots, u_{hi}) \in \rho$ for $1 \leq i \leq n$. Therefore

$$(f(a_{11}, \dots, a_{1n}), f(u_{21}, \dots, u_{2n}), \dots, f(u_{h1}, \dots, u_{hn})) \in \rho,$$

and $f(u_{j1}, \dots, u_{jn}) \theta f(a_{j1}, \dots, a_{jn})$ for $2 \leq j \leq h$. Hence

$$(f(a_{11}, \dots, a_{1n}), f(a_{21}, \dots, a_{2n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \rho_{1,\theta}.$$

Secondly, we show that $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in (\rho_{1,\theta})_\sigma$ for $\sigma \in \mathcal{S}_h$. Let $\sigma \in \mathcal{S}_h$. Since $(a_{1i}, \dots, a_{hi}) \in (\rho_{1,\theta})_\sigma$ for $1 \leq i \leq n$, it follows that $(a_{\sigma^{-1}(1)i}, \dots, a_{\sigma^{-1}(h)i}) \in \rho_{1,\theta}$ for $1 \leq i \leq n$. Hence

$$(f(a_{\sigma^{-1}(1)1}, \dots, a_{\sigma^{-1}(1)n}), \dots, f(a_{\sigma^{-1}(h)1}, \dots, a_{\sigma^{-1}(h)n})) \in \rho_{1,\theta}.$$

Therefore $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in (\rho_{1,\theta})_\sigma$.

Thus $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \varsigma$.

And it follows that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\varsigma)$.

As $\rho \subsetneq \varsigma$ we have $\varsigma \setminus \rho \neq \emptyset$; then with $(u_1, u_2, \dots, u_h) \in \varsigma \setminus \rho$ and $(a, b) \notin \theta$ fixed; using the h -ary operation f defined by (5.1), and the same argument used in the proof of Lemma 5.22, it follows that $f \in \text{Pol}(\varsigma)$ and $f \notin \text{Pol}(\rho)$. Then $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\varsigma)$.

It is obvious to see that $\text{Pol}(\theta) \cap \text{Pol}(\varsigma) \subsetneq \text{Pol}(\theta)$ (because $\varsigma \neq E_k^h$).

Let us take a fixed element $(u_1, u_2, \dots, u_h) \in E_k^h \setminus \varsigma$ and $(a, b) \notin \theta$. Using the operation f defined above, we easily obtain $f \in \text{Pol}(\theta)$.

On the other hand

$$\begin{pmatrix} a & a & a & \cdots & a & a \\ a & b & a & \cdots & a & a \\ & & \cdots & \cdots & & \\ a & a & \cdots & a & b & a \\ a & a & \cdots & a & a & b \end{pmatrix} \subseteq \varsigma,$$

but by the definition of f we have

$$(u_1, u_2, \dots, u_h) = (f(a, a, \dots, a), f(a, b, a, \dots, a), \dots, f(a, a, \dots, a, a, b)) \notin \varsigma.$$

It follows that $f \notin \text{Pol}(\varsigma)$; thus $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\varsigma) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$.

□

Lemma 5.26. *If $\rho \subsetneq \varsigma = E_k^h$ and there exists $h < l \leq t$ with $\varsigma^l \neq E_k^l$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$, where $\varsigma^l = \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{1, \theta}^l)_\sigma$*

with

$$\rho_{1, \theta}^l = \{(a_1, \dots, a_l) \in E_k^l / \exists u_i \in [a_i]_\theta, 2 \leq i \leq l : \{a_1; u_2, u_3, \dots, u_l\}^h \subseteq \rho\}.$$

Proof. We set $m := \min\{l \in \{0; 1; \dots; t\} \setminus \{0; 1; 2; \dots; h\} / \varsigma^l \neq E_k^l\}$.

We want to prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\varsigma^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$.

We use the similar argument as in the proof of Lemma 5.25 to show that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\varsigma^m) \cap \text{Pol}(\theta)$.

Let $(u_1, u_2, \dots, u_h) \in \varsigma \setminus \rho$ and $(a, b) \notin \theta$ fixed.

Using the h -ary operation f defined by (5.1), we have again $f \notin \text{Pol}(\rho)$ and $f \in \text{Pol}(\varsigma^m)$; since $(u_1, u_2, \dots, u_h) \in E_k^h = \varsigma$, there exist $a_{ij} \in [u_i]_\theta$, $i, j \in \{1, \dots, s\}$ with $i \neq j$ such that for all $j \in \{1, \dots, h\}$,

$$(u_j, a_{j_1j}, a_{j_2j}, \dots, a_{j_{h-1}j}) \in \rho,$$

with $\{j_1, \dots, j_{h-1}\} = \{1, \dots, h\} \setminus \{j\}$. Then it follows that $\{u_1, u_2, \dots, u_h\}^m \subseteq \varsigma^m$. Moreover, as we have $\varsigma^m \neq E_k^m$, there exists $(u_1, \dots, u_m) \in E_k^m \setminus \varsigma^m$. Given $(u_1, \dots, u_m) \in E_k^m \setminus \varsigma^m$ and $(a, b) \notin \theta$.

Using the m -ary operation h defined by (5.2), and the same argument used in the proof of Lemma 5.23, we have $h \in \text{Pol}(\theta)$ but $h \notin \text{Pol}(\varsigma^m)$. Thus $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\varsigma^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

The previous lemma suggests to suppose that for each $l \in \{h; \dots; t\}$, $\varsigma^l = E_k^l$.

Let $m = \max\{|C_i|; 0 \leq i \leq t-1\}$ and we denote by γ' the relation

$$\gamma' = \{(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+t-1}) \in E_k^{m+t-1} / \forall \{i; j\} \subseteq \{1; \dots; m\}, a_i \theta a_j;$$

$$\exists u_i \in [a_i]_\theta, m+1 \leq i \leq m+t-1, \forall 1 \leq i \leq m,$$

$$\{a_i; u_{m+1}; \dots; u_{m+t-1}\}^h \subseteq \rho\}.$$

We have $\text{Pol}(\gamma') \subseteq \text{Pol}(\theta)$. We define two relations $\epsilon_{=}$ and ϵ'_{θ} on $\{1; \dots; m+t-1\}$ by:

$$(i, j) \in \epsilon_{=} \Leftrightarrow i = j \text{ and } (i, j) \in \epsilon'_{\theta} \Leftrightarrow (i = j \text{ or } \{i, j\} \subseteq \{1; \dots; m\}),$$

$\epsilon_{=}$ and ϵ'_{θ} are equivalence relations and $\gamma' \subseteq D_{\epsilon_{=}\epsilon'_{\theta}}$.

Lemma 5.27. *If $\forall l \in \{h, h+1, \dots, t\}, \varsigma^l = E_k^h$ and $\gamma' \neq D_{\epsilon_{=}\epsilon'_{\theta}}$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$.*

Proof. We just have to prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\gamma') \subsetneq \text{Pol}(\theta)$.

Let $f \in \mathcal{O}^n(E_k)$ be an n -ary operation on E_k such that $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$. Let

$$(a_{11}, \dots, a_{m+t-11}), \dots, (a_{1n}, \dots, a_{m+t-1n}) \in \gamma'.$$

According to the definition of γ' , for all $1 \leq j \leq n$, for all $\alpha, \beta \in \{1; \dots; m\}$, we have $[a_{\alpha j}]_{\theta} = [a_{\beta j}]_{\theta}$ and for all $j \in \{1; \dots; n\}$, there exists $u_{rj} \in [a_{rj}]_{\theta}$ with $r \in \{m+1; \dots; m+t-1\}$ such that for all $i \in \{1; \dots; m\}$,

$$\{a_{ij}; u_{m+1j}; u_{m+2j}; \dots; u_{m+t-1j}\}^h \subseteq \rho.$$

Since $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$, for all $\alpha, \beta \in \{1; \dots; m\}$, $[f(a_{\alpha 1}, \dots, a_{\alpha n})]_{\theta} = [f(a_{\beta 1}, \dots, a_{\beta n})]_{\theta}$ and for all $r \in \{m+1; \dots; m+t-1\}$, $f(u_{r1}, \dots, u_{rn}) \in [f(a_{r1}, \dots, a_{rn})]_{\theta}$. It follows that for all $i \in \{1; \dots; m\}$

$$\{f(a_{i1}, \dots, a_{in}); f(u_{m+11}, \dots, u_{m+1n}); \dots; f(u_{m+t-11}, \dots, u_{m+t-1n})\}^h \subseteq \rho.$$

Consequently,

$$(f(a_{11}, \dots, a_{1n}), \dots, f(a_{m+t-11}, \dots, a_{m+t-1n})) \in \gamma',$$

and then $f \in \text{Pol}(\gamma')$.

Let $(u_1, u_2, \dots, u_h) \in E_k^h \setminus \rho$ and $(a, b) \notin \theta$ fixed. Using the h -ary operation f defined by (5.1), we have $f \notin \text{Pol}(\rho)$ and $f \in \text{Pol}(\gamma')$; since $(u_1, u_2, \dots, u_h) \in E_k^h = \varsigma$, it follows that there exist $a_{ij} \in [u_i]_\theta$; $i, j \in \{1; \dots; h\}$ with $i \neq j$ such that for all $j \in \{1; \dots; h\}$,

$$(u_j, a_{j_1j}, a_{j_2j}, \dots, a_{j_{h-1}j}) \in \rho,$$

with $\{j_1, \dots, j_{h-1}\} = \{1; \dots; h\} \setminus \{j\}$. It follows that

$$\bigcup_{i \in \{1; \dots; h\}} \{u_i\}^m \times \{u_l; l \in \{1; \dots; h\} \setminus \{i\}\}^{t-1} \subseteq \gamma'.$$

From the fact that $\gamma' \neq D_{\epsilon=\epsilon'_\theta}$, we deduce that $D_{\epsilon=\epsilon'_\theta} \setminus \gamma' \neq \emptyset$.

Given $(v_1, \dots, v_{m+t-1}) \in D_{\epsilon=\epsilon'_\theta} \setminus \gamma'$, $(a, b) \notin \theta$ and consider the following $(m+t-1)$ -ary operation defined on E_k by:

$$h(x_1, \dots, x_{m+t-1}) = \begin{cases} v_1 & \text{if } (x_1, \dots, x_{m+t-1})\theta(a, \dots, a, a) = a_1, \\ v_{m+1} & \text{if } (x_1, \dots, x_{m+t-1})\theta(a, b, a, \dots, a) = a_2, \\ v_{m+2} & \text{if } (x_1, \dots, x_{m+t-1})\theta(a, a, b, a, \dots, a) = a_3, \\ \vdots & \\ v_{m+t-1} & \text{if } (x_1, \dots, x_{m+t-1})\theta(a, a, a, \dots, a, b) = a_t, \\ v_{m+t-1} & \text{otherwise.} \end{cases}$$

Since $a_i\theta a_j \Rightarrow i = j$, $h \in \text{Pol}(\theta)$. From the fact that $\{a, b\}^h \subseteq \rho$ we have $h \notin \text{Pol}(\gamma')$. Indeed,

$$\begin{pmatrix} a \\ \vdots \\ a \\ \vdots \\ a \\ a \\ a \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ \vdots \\ a \\ b \\ a \\ \vdots \\ a \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ \vdots \\ a \\ a \\ b \\ a \\ \vdots \\ a \\ a \end{pmatrix}, \dots, \begin{pmatrix} a \\ \vdots \\ a \\ a \\ a \\ \vdots \\ a \\ a \\ b \end{pmatrix} \in \gamma',$$

but

$$(v_1, \dots, v_1, v_{m+1}, \dots, v_{m+t-1}) = (h(a_1), \dots, h(a_1), h(a_2), \dots, h(a_t)) \notin \gamma'.$$

□

In the next step, we assume that

$$\gamma' = D_{\epsilon=\epsilon'_0}, \quad (5.3)$$

i.e., for all $i \in \{0; \dots; t-1\}$, there exists $u_{ji} \in C_j$, $j \in \{0; \dots; t-1\} \setminus \{i\}$ such that for all $a \in C_i$, $\{a; u_{1i}; \dots; u_{i-1i}, u_{i+1i}; \dots; u_{ti}\}^h \subseteq \rho$. Let ζ be the relation defined by:

$$\zeta = \{\mathbf{a} \in E_k^h \mid \exists u_i \in [a_i]_0, \text{ such that } (a_i, u_{i_1}, \dots, u_{i_{h-1}}) \in \rho, i \in \{1; \dots; h\},$$

$$\text{and } \{i_1; \dots; i_{h-1}\} = \{1; \dots; h\} \setminus \{i\}\}.$$

Our goal is to show that if $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$, then

$\zeta = E_k^h$. Before that, let us prove the following result.

Lemma 5.28. *If $\rho = \zeta$, then $\rho = E_k^h$.*

Proof. We suppose that $\rho = \zeta$. Let $a = (a_1, a_2, \dots, a_s) \in E_k^h$. If there exists $\{i, j\} \subseteq \{1; \dots; h\}$ with $i \neq j$ such that $(a_i, a_j) \in \theta$, then $a \in \rho$ since $\eta \subseteq \rho$. Else, according to (5.3), for all $i \in \{1, \dots, t\}$, there exist $u_{ji} \in C_j$, $j \in \{0; \dots; t-1\} \setminus \{i\}$ such that for all $b \in C_i$,

$$\{b; u_{0i}; \dots; u_{i-1i}, u_{i+1i}; \dots; u_{t-1i}\}^h \subseteq \rho,$$

where $C_j = [a_j]_\theta$, $j \in \{1; \dots; h\}$. Hence

$$(C_i \cup \{u_{0i}, \dots, u_{t-1i}\} \setminus \{u_{ii}\})^h \subseteq \rho, \quad 0 \leq i \leq t-1.$$

Therefore

$$(a_1, a_2, \dots, a_h) \in \zeta = \rho.$$

It follows that $\rho = E_k^h$. □

From Lemma 5.28 and the fact that $\rho \neq E_k^h$, we obtain $\rho \subsetneq \zeta \subseteq E_k^h$. Now we can prove the following lemma.

Lemma 5.29. *If $\zeta \neq E_k^h$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$.*

Proof. We have to prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\zeta) \subsetneq \text{Pol}(\theta)$. Let $f \in \mathcal{O}^n(E_k)$ be an n -ary operation on E_k such that $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$. Let

$$(a_{11}, a_{21}, \dots, a_{h1}), \dots, (a_{1n}, a_{2n}, \dots, a_{hn}) \in \zeta.$$

From the definition of ζ , for all $i \in \{1; \dots; h\}$, for all $j \in \{1, \dots, n\}$, there exist $u_{ij} \in [a_{ij}]_\theta$ such that

$$(a_{1j}, u_{2j}, \dots, u_{hj}), (u_{1j}, a_{2j}, \dots, u_{hj}), \dots, (u_{1j}, u_{2j}, \dots, u_{(h-1)j}, a_{hj}) \in \rho;$$

and as $f \in \text{Pol}(\rho)$ we have

$$(f(a_{11}, \dots, a_{1n}), f(u_{21}, \dots, u_{2n}), \dots, f(u_{h1}, \dots, u_{hn})) \in \rho,$$

$$(f(u_{11}, \dots, u_{1n}), f(a_{21}, \dots, a_{2n}), f(u_{31}, \dots, u_{3n}), \dots, f(u_{h1}, \dots, u_{hn})) \in \rho, \dots, \\ (f(u_{11}, \dots, u_{1n}), \dots, f(u_{(s-1)1}, \dots, u_{(h-1)n}), f(a_{h1}, \dots, a_{hn})) \in \rho.$$

Moreover, as $f \in \text{Pol}(\theta)$, we have $f(u_{i1}, \dots, u_{in})\theta f(a_{i1}, \dots, a_{in})$; therefore, using the definition of ζ ,

$$(f(a_{11}, \dots, a_{1n}), f(a_{21}, \dots, a_{2n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \zeta.$$

Then $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\theta) \cap \text{Pol}(\zeta)$.

As $\rho \subsetneq \zeta$ we have $\zeta \setminus \rho$ is not empty. Given $(u_1, u_2, \dots, u_h) \in \zeta \setminus \rho$ and $(a, b) \notin \theta$, using the h -ary operation f defined by (5.1), and the same argument used in the proof of Lemma 5.22, we obtain $f \in \text{Pol}(\zeta)$ and $f \notin \text{Pol}(\rho)$. Therefore, $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}(\zeta)$. It is obvious that $\text{Pol}(\theta) \cap \text{Pol}(\zeta) \subsetneq \text{Pol}(\theta)$ (because $\zeta \neq E_k^h$). Given $(u_1, u_2, \dots, u_h) \in E_k^h \setminus \zeta$ and $(a, b) \notin \theta$, $f \in \text{Pol}(\theta)$ and $f \notin \text{Pol}(\zeta)$; thus $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\zeta) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

Lemma 5.30. *If $\zeta = E_k^h$ and $\zeta^k \neq E_k^k$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$, where ζ^l is the l -ary relation on E_k specified by*

$$\zeta^l = \{(a_1, \dots, a_l) \in E_k^l / \exists u_i \in [a_i]_\theta / \forall i \in \{1; \dots; l\},$$

$$\{u_1; \dots; u_{i-1}; a_i; u_{i+1}; \dots; u_l\}^h \subseteq \rho\}.$$

Proof. We prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\zeta^k) \subsetneq \text{Pol}(\theta)$.

It is easy to see that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\zeta^k)$.

Given $(u_1, u_2, \dots, u_h) \in E_k^h \setminus \rho$ and $(a, b) \notin \theta$, using the h -ary operation f defined by (5.1), we have again $f \notin \text{Pol}(\rho)$ and $f \in \text{Pol}(\zeta^k)$; indeed, since $(u_1, u_2, \dots, u_h) \in E_k^h = \zeta$, there exist $v_i \in [u_i]_\theta$,

$i \in \{1; \dots; h\}$ such that for all $i \in \{1; \dots; h\}$, $\{u_i, v_{i_1}, \dots, v_{i_{h-1}}\}^h \subseteq \rho$, with $\{i_1, i_2, \dots, i_{h-1}\} = \{1; \dots; h\} \setminus \{i\}$, and it follows that $\{u_1, u_2, \dots, u_h\}^k \subseteq \zeta^k$.

Moreover, as we have $\zeta^k \neq E_k^k$, then $E_k^k \setminus \zeta^k \neq \emptyset$. Given $(w_1, \dots, w_k) \in E_k^k \setminus \zeta^k$, $(a, b) \notin \theta$, using the k -ary operation h defined on E_k , by:

$$h(x_1, \dots, x_k) = \begin{cases} w_1 & \text{if } (x_1, \dots, x_k)\theta(a, \dots, a, a) = a_1, \\ w_2 & \text{if } (x_1, \dots, x_k)\theta(a, b, a, \dots, a) = a_2, \\ w_3 & \text{if } (x_1, \dots, x_k)\theta(a, a, b, a, \dots, a) = a_3, \\ \vdots & \\ w_k & \text{if } (x_1, \dots, x_k)\theta(a, a, a, \dots, a, b) = a_k, \\ w_k & \text{otherwise;} \end{cases}$$

and the same argument used for the similar operation h in the proof of Lemma 5.23 yields $h \in \text{Pol}(\theta)$ and $h \notin \text{Pol}(\zeta^k)$.

Thus $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\zeta^k) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

Now, we continue our induction process with the assumption $\zeta^l = E_k^l$ for all $h \leq l \leq t$ and $\zeta^k = E_k^k$. Since $(0, \dots, k-1) \in E_k^k$, there exist $u_i \in [i]_\theta$ such that

$$\{u_0; u_1; \dots; u_{i-1}; i; u_{i+1}; \dots; u_{k-1}\}^h \subseteq \rho,$$

for all $i \in E_k$. We set

$$T_i := \min(C_i) \cap \{u_0; u_1; \dots; u_{k-1}\},$$

for all $i \in E_k$. Therefore $\{T_0; \dots; T_{t-1}\}$ is a transversal of order 1.

Before the main induction, we define the sequence $({}^h\beta_n)$ by: ${}^h\beta_0 = \eta$, ${}^h\beta_1 = \rho_{0, \theta}$, ${}^h\beta_2 = \varsigma$, and for $l \geq 3$,

$${}^h\beta_l = \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{l-1, \theta})_\sigma.$$

Let $n \in \{1; \dots; h-1\}$ and assume that there exists a transversal T of order $n-1$ for the θ -classes. Set $T = \{u_0; u_1; \dots; u_{t-1}\}$; then for every $a_1, a_2, \dots, a_{n-1} \in E_k$, $\{a_1; a_2; \dots, a_{n-1}; u_0; u_1; \dots; u_{t-1}\}^h \subseteq \rho$ and $\rho \subseteq {}^h\beta_{n+1}$.

Lemma 5.31. *If $\rho = {}^h\beta_{n+1} \subsetneq E_k^h$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$.*

Proof. If $\rho = {}^h\beta_{n+1} \subsetneq E_k^h$, then ρ is weakly θ -closed of order n . Hence ρ is a relation of type III. Using Proposition 5.13 $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$. $\square$

Lemma 5.32. *If $\rho \subsetneq {}^h\beta_{n+1} \subsetneq E_k^h$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$.*

Proof. We have to prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta) \cap \text{Pol}({}^h\beta_{n+1}) \subsetneq \text{Pol}(\theta)$. It is easy to show that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}(\theta) \cap \text{Pol}({}^h\beta_{n+1})$.

Using the h -ary operation f defined by (5.1), we show that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}({}^h\beta_{n+1}) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

Lemma 5.33. *If $\rho \subsetneq {}^h\beta_{n+1} = E_k^h$ and ${}^h\beta_{n+1}^l \neq E_k^l$ for some $l > h$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$; where ${}^h\beta_{n+1}^l = \bigcap_{\sigma \in \mathcal{S}_l} (\rho_{n, \theta}^l)_\sigma$ with*

$$\rho_{n, \theta}^l := \{(a_1, \dots, a_l) \in E_k^l / \exists u_i \in [a_i]_0, n+1 \leq i \leq l :$$

$$\{a_1; a_2; \dots; a_n; u_{n+1}, \dots, u_l\}^h \subseteq \rho\}.$$

Proof. Denote $m := \min\{l \in \mathbb{N} \setminus \{0; 1; 2; \dots; h\} / {}^h\beta_{n+1}^l \neq E_k^l\}$. Before proving that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}({}^h\beta_{n+1}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$, we first prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subseteq \text{Pol}({}^h\beta_{n+1}^m) \cap \text{Pol}(\theta)$.

Let $f \in \text{Pol}(\theta) \cap \text{Pol}(\rho)$ be an p -ary operation on E_k . Let

$$(a_{11}, \dots, a_{m1}), \dots, (a_{1p}, \dots, a_{mp}) \in {}^h\beta_{n+1}^m.$$

By definition of ${}^h\beta_{n+1}^m$, for all $c \in \{1; \dots; p\}$, for all $\sigma \in \mathcal{S}_m$, there exist $u_{\sigma(r)c} \in [a_{\sigma(r)c}]_\theta$, $n+1 \leq r \leq m$ such that

$$\{a_{\sigma(1)c}; \dots; a_{\sigma(n)c}; u_{\sigma(n+1)c}; \dots; u_{\sigma(m)c}\}^h \subseteq \rho,$$

and from the fact that $f \in \text{Pol}(\rho)$ it follows that for all $\sigma \in \mathcal{S}_m$,

$$\{f(a_{\sigma(1)1}, \dots, a_{\sigma(1)p}); \dots; f(a_{\sigma(n)1}, \dots, a_{\sigma(n)p});$$

$$f(u_{\sigma(n+1)1}, \dots, u_{\sigma(n+1)p}); \dots; f(u_{\sigma(m)1}, \dots, u_{\sigma(m)p})\}^h \subseteq \rho.$$

Since $f \in \text{Pol}(\theta)$, it follows that for all $i \in \{n+1; \dots; m\}$, and for all $d \in \{n+1; \dots; m\}$,

$$\sigma(d) = i \Rightarrow f(u_{\sigma(d)1}, u_{\sigma(d)2}, \dots, u_{\sigma(d)p}) \theta f(a_{i1}, \dots, a_{ip}).$$

From the definition of ${}^h\beta_{n+1}^m$, it follows that

$$(f(a_{11}, \dots, a_{1p}), \dots, f(a_{m1}, \dots, a_{mp})) \in {}^h\beta_{n+1}^m.$$

Then $f \in \text{Pol}({}^h\beta_{n+1}^m) \cap \text{Pol}(\theta)$.

Fix $(v_1, \dots, v_h) \in E_k^h \setminus \rho$ and let $(a, b) \notin \theta$. Using the h -ary operation f defined by (5.1) (where we replace u_i by v_i , $1 \leq i \leq h$), we have $f \notin \text{Pol}(\rho)$ and $f \in \text{Pol}({}^h\beta_{n+1}^m)$. Indeed, due to the fact that

$$(v_1, \dots, v_h) \in E_k^h = {}^h\beta_{n+1},$$

for all $\sigma \in \mathcal{S}_h$ there exist $d_r \in [v_r]_0$, $n+1 \leq r \leq h$ such that,

$$\{v_{\sigma(1)}; \dots; a_{\sigma(n)}; d_{\sigma(n+1)}; \dots; d_{\sigma(h)}\}^h \subseteq \rho.$$

Therefore, it follows that $\{v_1; v_2; \dots; v_h\}^m \subseteq {}^h\beta_{n+1}^m$.

Moreover, as we have ${}^h\beta_{n+1}^m \neq E_k^m$, then $E_k^m \setminus {}^h\beta_{n+1}^m$ is not an empty relation. Given $(v_1, \dots, v_m) \in E_k^m \setminus {}^h\beta_{n+1}^m$ and $(a, b) \notin \emptyset$; using the m -ary operation h defined by (5.2) (where we replace u_i by v_i , $1 \leq i \leq m$), and the same argument used in the proof of Lemma 5.23, it follows that $h \in \text{Pol}(\theta)$ and $h \notin \text{Pol}({}^h\beta_{n+1}^m)$. Thus $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}({}^h\beta_{n+1}^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

It is natural now to suppose that for each $l \in \{h; \dots; t\}$, ${}^h\beta_{n+1}^l = E_k^l$.

Let $m_1, \dots, m_n$ be integers such that $m_1 > m_2 > \dots > m_n$ and for each $i \in \{0; \dots; t-1\}$, $|C_i| \leq \min\{m_1, \dots, m_n\}$ or $|C_i| \in \{m_1, \dots, m_n\}$. We set $m = m_1 + \dots + m_n$. In this part we will use these notations

$$\mathbf{a} := (a_1, \dots, a_{m_1}, a_{m_1+1}, \dots, a_{m_1+m_2}, \dots, a_{m_1+\dots+m_n}, a_{m+1}, \dots, a_{m+t-n}),$$

$$\prod = \{1; \dots; m_1\} \times \{m_1 + 1; \dots; m_1 + m_2\} \times \dots \times \{m_1 + \dots + m_{n-1} + 1; \dots; m\},$$

and

$$\prod' = \{1; \dots; m_1\}^2 \cup \{m_1 + 1; \dots; m_1 + m_2\}^2 \cup \dots \cup \{m_1 + \dots + m_{n-1} + 1; \dots; m\}^2.$$

Let ${}^h\beta'_{n+1}$ be the relation

$${}^h\beta'_{n+1} = \{\mathbf{a} \in E_k^{m+t-n} / \forall (i, j) \in \prod', a_i \theta a_j;$$

$$\exists u_i \in [a_i]_0, m+1 \leq i \leq m+t-n, \text{ such that}$$

$$\forall (i_1, \dots, i_n) \in \prod, \{a_{i_1}, \dots, a_{i_n}, u_{m+1}, \dots, u_{m+t-n}\}^h \subseteq \rho\}.$$

We have $\text{Pol}({}^h\beta'_{n+1}) \subseteq \text{Pol}(\theta)$.

We define two relations ϱ_1 and ϱ_2 on $\{1; \dots; m+t-n\}$ by:

$$(i, j) \in \varrho_1 \Leftrightarrow i = j \text{ and } (i, j) \in \varrho_2 \Leftrightarrow (i = j \text{ or } (i, j) \in \prod'),$$

ϱ_1 and ϱ_2 are equivalence relations and ${}^h\beta'_{n+1} \subseteq D_{\varrho_1\varrho_2}$.

Lemma 5.34. *If $\forall l \in \{h, h+1, \dots, t\}$, ${}^h\beta_{n+1}^l = E_k^h$ and ${}^h\beta'_{n+1} \neq D_{\varrho_1\varrho_2}$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is not maximal in $\text{Pol}(\theta)$.*

Proof. It is similar to the proof of Lemma 5.27. $\square$

In what follows, we assume that

$${}^h\beta'_{n+1} \neq D_{\varrho_1\varrho_2}, \quad (5.4)$$

i.e., for all $i_1, \dots, i_n \in \{0; \dots; t-1\}$, there exist $u_{jI} \in C_j$, $j \in \{0; \dots; t-1\} \setminus \{i_1, \dots, i_n\}$, with $I = \{i_1, \dots, i_n\}$ such that for all $a_{i_r} \in C_{i_r}$, $r \in \{1; \dots; n\}$ we have

$$\{a_{i_1}; \dots; a_{i_n}, u_{1I}; \dots; u_{t-nI}\}^h \subseteq \rho.$$

We obtain a transversal of order n for the θ -classes. Hence we can continue the previous induction until

$${}^h\beta_h = \bigcap_{\sigma \in \mathcal{S}_h} (\rho_{h-1, \theta})_\sigma = E_k^h.$$

From now on, we suppose that ${}^h\beta_h = E_k^h$.

Lemma 5.35. *If ${}^h\beta_h = E_k^h$ and there exists $l > h$ such that ${}^h\beta_h^l \neq E_k^h$, then $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is not maximal in $\text{Pol}(\theta)$.*

Proof. Set $m = \min\{l \in \{1; \dots; k\} \setminus \{1; \dots; h\} / {}^h\beta_h^l \neq E_k^h\}$. It is easy to prove that $\text{Pol}(\rho) \cap \text{Pol}(\theta) \subsetneq \text{Pol}({}^h\beta_h^m) \cap \text{Pol}(\theta) \subsetneq \text{Pol}(\theta)$. $\square$

Lemma 5.36. *If ${}^h\beta_h^k = E_k^k$, then $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$.*

Proof. If ${}^h\beta_h^k = E_k^k$, then each equivalence class of θ has a central element of ρ . Since $\eta \subseteq \rho$, ρ is of type I and $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$ by Proposition 5.13. $\square$

We are ready now to give the proof of Proposition 5.18 and Theorem 5.9.

Proof of Proposition 5.18. Combining Lemmas 5.19, 5.22, 5.23, 5.25, 5.26, 5.27, 5.29, 5.30, 5.32, 5.33, 5.35, 5.36 and Remark 5.20, we obtain the result. $\square$

Proof of Theorem 5.9. It follows from Propositions 5.13 and 5.18. $\square$

We investigate the case of h -regular relations.

5.2. h -regular relations. As an h -regular relation is totally reflexive and totally symmetric, some results stated in the previous subsection can apply. Besides an h -regular generated relation does not contain a central element. We prove that there is submaximality if and only if ρ is θ -closed (or of type II). We begin this subsection with some examples of h -regular relations.

Example 5.37. Let $k \geq 3$ be an integer and $0 \leq i < j < r < n \leq k-1$, we denote by $A_{i,j,r}$ and $A_{i,j,r,n}$ the sets

$$A_{i,j,r} := \{(\sigma(i), \sigma(j), \sigma(r)); \sigma \in \mathcal{S}_{\{i;j;r\}}\},$$

and

$$A_{i,j,r,n} := \{(\sigma(i), \sigma(j), \sigma(r), \sigma(n)); \sigma \in \mathcal{S}_{\{i;j;r;n\}}\}.$$

We consider the following equivalence relations $\theta_6\theta_7\theta_8$ on E_{12} defined respectively by their equivalence classes denoted by C_m^i , $6 \leq i \leq 8$ as follows:

$$C_0^6 = \{0; 1; 2; 3; 4\}, C_1^6 = \{5; 6; 7\}, C_2^6 = \{8; 9; 10; 11\};$$

$$C_0^7 = \{0; 1; 5; 8\}, C_1^7 = \{2; 6; 9; 11\}, C_2^7 = \{3; 4; 7; 10\};$$

$$C_0^8 = \{0; 1\}, C_1^8 = \{2\}, C_2^8 = \{3; 4\}, C_3^8 = \{5\}, C_4^8 = \{6\}, C_5^8 = \{7\},$$

$$C_6^8 = \{8\}, C_7^8 = \{9\}, C_8^8 = \{10\}, C_9^8 = \{11\};$$

and the relation

$$\Upsilon_6 = \{(x_1, x_2, x_3) \in E_{12}^3 : (x_1, x_2) \in \theta_6, \text{ or } (x_2, x_3) \in \theta_6, \text{ or } (x_1, x_3) \in \theta_6\},$$

$$\Upsilon_7 = \bigcup_{\sigma \in \mathcal{S}_3} (\Upsilon)_\sigma,$$

where

$$\Upsilon = \{(x_1, x_2, x_3) \in E_{12}^3 : (x_1, x_2) \in \theta_6, (x_2, x_3) \in \theta_7\}.$$

It is easy to see that Υ_6 is a θ_6 -closed 3-regular relation associated to $T = \{\theta_6\}$, and Υ_7 is a θ_8 -closed 3-regular relation associated to $T = \{\theta_6; \theta_7\}$.

We continue with the characterization of θ -closed h -regular relation.

Lemma 5.38. *For $h \geq 3$, let ρ be an h -regular relation on E_k determined by the h -regular family $T = \{\theta_1; \dots; \theta_m\}$ and θ be a nontrivial equivalence relation on E_k . ρ is θ -closed iff $\theta \subseteq \theta_i$, for all $1 \leq i \leq m$.*

Proof. $\Rightarrow$ Firstly, we show that $\eta = \{(a_1, \dots, a_h) \in E_k^h : (a_1, a_2) \in \theta\} \subseteq \rho$. Let $(a_1, \dots, a_h) \in \eta$; since ρ is totally reflexive we have $(a_1, a_1, a_3, \dots, a_h) \in \rho$; we also have $(a_1, a_2, a_3, \dots, a_h) \theta (a_1, a_1, a_3, \dots, a_h)$. Hence $(a_1, \dots, a_h) \in \rho_{0, \theta} = \rho$. Our next step is to show that $\theta \subseteq \theta_i$ for all $1 \leq i \leq m$. Let $i \in \{1, \dots, m\}$ and $(a, b) \in \theta$; set $A_i = E_k / \theta_i \setminus \{[a]_{\theta_i}, [b]_{\theta_i}\}$. It is easy to see that $|A_i| \geq h - 2$; choose $(a_1, \dots, a_{h-2}) \in E_k^{h-2}$ such that $[a_p]_{\theta_i} \in A_i$ for all $1 \leq p \leq h - 2$ and $(a_p, a_q) \notin \theta_i$ for all $1 \leq p < q \leq h - 2$. Due to $\eta \subseteq \rho$, we have $(a, b, a_1, \dots, a_{h-2}) \in \rho$; therefore $(a, b) \in \theta$ and $\theta \subseteq \theta_i$.

$\Leftarrow$ It follows from the fact that $\theta \circ \theta_i \circ \theta \subseteq \theta_i$ for all $1 \leq i \leq m$. $\square$

Lemma 5.39. *For $h \geq 3$, let ρ be an h -ary relation and θ be a nontrivial equivalence relation on E_k . ρ is a θ -closed h -regular relation iff there exists an h -regular relation ψ on E_t such that $\rho = \varphi^{-1}(\psi)$.*

Proof. Firstly, let us assume that there exists an h -regular relation ψ on E_t and put $\perp = \{\nu_1; \dots; \nu_n\}$ the h -regular family associated to ψ . Clearly $\varphi^{-1}(\perp) = \{\varphi^{-1}(\nu_1); \dots; \varphi^{-1}(\nu_n)\}$ is an h -regular family and $\rho = \varphi^{-1}(\psi)$ is exactly the h -regular relation associated to $\varphi^{-1}(\perp)$. Moreover, for all $1 \leq i \leq n$ we have $\theta \subseteq \varphi^{-1}(\nu_i)$. Therefore, by Lemma 5.38, ρ is θ -closed.

Conversely, assume that ρ is an h -regular relation determined by the h -regular family $T = \{\theta_1; \dots; \theta_r\}$ and ρ is θ -closed. Clearly, for all $1 \leq i \leq r$, $\varphi(\theta_i)$ is an equivalence relation with exactly h equivalence classes whose are the images of equivalence classes of θ_i by φ . It follows that $\bigcap \{\varphi(B_i) \mid 1 \leq i \leq r\}$ is non-empty for arbitrary equivalence classes $\varphi(B_i)$ of $\varphi(\theta_i)$, $1 \leq i \leq r$. Therefore $\varphi(T) = \{\varphi(\theta_1); \dots; \varphi(\theta_r)\}$ is an

h -regular family and $\varphi(\rho)$ is an h -regular relation on E_t , associated to $\varphi(T)$. Moreover $\varphi^{-1}(\varphi(\rho)) = \rho$ because ρ is θ -closed. Therefore we have the result. $\square$

We end this subsection with the characterization of h -regular relations ρ such that $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$. We need the next lemma to prove our main result.

Lemma 5.40 ([7]). *Let $\chi \subseteq E_t^h$ such that $\text{Pol}(\chi)$ is maximal in $\mathcal{L}_t$. Then $\text{Pol}(\theta) \cap \text{Pol}(\varphi^{-1}(\chi))$ is submaximal in $\text{Pol}(\theta)$.*

Proof. See the proof of Lemma 18.2.5, page 565 in [7]. $\square$

Lemma 5.41. *If $\text{Pol}(\rho) \cap \text{Pol}(\theta)$ is maximal in $\text{Pol}(\theta)$, then $\eta \subseteq \rho$.*

Proof. See the proof of Lemma 5.19. $\square$

Now we give the result of this subsection.

Proposition 5.42. *Let θ be a nontrivial equivalence relation and ρ be an h -regular relation on E_k determined by the h -regular family $T = \{\theta_1; \dots; \theta_m\}$. $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$ if and only if ρ is θ -closed.*

Proof. $\Rightarrow$ It follows from Lemma 5.41 that $\eta \subseteq \rho$; and in the light of the proof of Lemma 5.38, we conclude that $\theta \subseteq \theta_i$ for all $1 \leq i \leq m$. It follows from Lemma 5.38 that ρ is θ -closed.

$\Leftarrow$ Combining Lemmas 5.38, 5.39, 5.40, we have the result. $\square$

6. Conclusion

In this paper, we characterized the relations ρ from the Rosenberg's list for which $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$, where θ is a nontrivial equivalence relation. The classification of all central relations

ρ on E_k such that the clone $\text{Pol}(\theta) \cap \text{Pol}(\rho)$ is maximal in $\text{Pol}(\theta)$ improves Temgoua and Rosenberg's results [18]. We plan in a future project to characterize the meet-irreducible submaximal clones of $\text{Pol}(\theta)$ for a nontrivial equivalence relation θ .

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