ERROR ANALYSIS FOR SPARSE TIME-FREQUENCY DECOMPOSITION OF SIGNALS BASED ON PROBABILITY

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Abstract

In this paper, after reviewing the data driven time-frequency analysis (DDTFA) and the matching pursuit method in decomposition of intrinsic mode functions (IMFs), we estimate the orthogonality properties between IMFs based on probability. By theoretical proofs and numerical experiments, we show that the estimation in [9] is the best.

Keywords: sparse time-frequency decomposition, scale separation, nonlinear matching pursuit.

1. Introduction

Time-frequency analysis for the signal can help us to further process the signal more easily, and we can have a better understanding of various complex phenomena. In recent years, the Joint Time-Frequency

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Analysis (JTFA) method has become a powerful tool for analyzing nonlinear non-stationary data. Common time-frequency analysis methods include wavelet transform (WT) [1], Wigner-Ville distribution [2], and Empirical Mode Decomposition (EMD) [3].

Inspired by the EMD method and compressive sensing [4, 5, 6], Hou and Shi proposed a data-driven time-frequency analysis method [7, 8]. The data-driven time-frequency analysis method based on the sparsest representation of multiscale data to decompose the signal into a finite number of IMF with a small residual:

$$f(t) = \sum_{k=1}^M a_k(t) \cos \theta_k(t) + r(t), \quad t \in [0, T], \quad (1)$$

where each IMF is a signal with amplitude modulation-frequency modulation (AM-FM). The amplitude $a_k(t) > 0$, the instantaneous frequency $\omega_k(t) = \theta_k'(t) > 0$, and $r(t)$ is a small residual.

The idea of this method is looking for the sparsest representation over the dictionary consisting of all IMFs by solving the following nonlinear optimization problem:

$$\begin{aligned} & \min_{(a_k)_{1 \leq k \leq M}, (\theta_k)_{1 \leq k \leq M}} M \\ & \text{subject to} \quad \left| f(t) - \sum_{k=1}^M a_k(t) \cos \theta_k(t) \right| \leq \varepsilon, \quad 0 \leq t \leq T, \quad a_k \cos \theta_k \in D. \quad (P_0) \end{aligned}$$

The dictionary is defined as follow:

$$D = \{a(t) \cos \theta(t): \theta'(t) \geq 0, a(t) \in V(\theta)\}, \quad (2)$$

where $V(\theta)$ is a collection of all the function that are less oscillatory than $\cos \theta(t)$. Under the assumption of the scale separation, the uniqueness of the optimization problem (P_0) problem is analyzed in the [9]. The definitions of scale separation and well-separated signal as follows.

Definition 1.1 (Scale separation [9]). A function $f(t) = a(t) \cos \theta(t)$ ($t \in [0, T]$) is said to satisfy a scale separation property with a separation factor $\varepsilon > 0$, if $a(t)$ and $\theta(t)$ satisfy the following conditions:

$$a(t) \in C^1(\mathbb{R}), \theta \in C^2(\mathbb{R}), \inf_{t \in [0, T]} \theta'(t) > 0,$$

$$\frac{\sup_{t \in [0, T]} \theta'(t)}{\inf_{t \in [0, T]} \theta'(t)} \leq M' < +\infty, \left| \frac{a'(t)}{a(t)\theta'(t)} \right| \leq \varepsilon, \left| \frac{\theta''(t)}{(\theta'(t))^2} \right| \leq \varepsilon, \quad \forall t \in [0, T]. \quad (3)$$

Definition 1.2 (Well-separated signal [9]). A signal $f : [0, T] \rightarrow \mathbb{R}$ is said to be well separated with separation factor ε and frequency ratio β if it can be written as

$$f(t) = \sum_{k=1}^M a_k(t) \cos \theta_k(t) + r(t), \quad (4)$$

where all $f_k(t) = a_k(t) \cos \theta_k(t)$ satisfy the scale separation property with separation factor ε , $r(t) = O(\varepsilon)$, and their phase function $\theta_k(t)$ satisfy

$$\theta'_k(t) \geq \beta \theta'_{k-1}(t), \quad \forall t \in [0, T], \quad (5)$$

and $\beta > 1$, $\beta - 1 = O(1)$.

Because the problem (P_0) is nonlinear and nonconvex, it is too difficult to solve. Based on matching pursuit [10], a nonlinear matching pursuit (NMP) is introduced. The NMP algorithm is as follows:

Task. To find an approximate optimizer of the problem (P_0) .

Initialization: Set the initial residual $r_0 = f$ and $k = 1$.

Main iteration:

Step 1. Solve the nonlinear least-squares problem as follows:

$$\begin{aligned} & \min_{a_k, \theta_k} \|r_{k-1} - a_k \cos \theta_k\|_{L^2}^2 \\ & \text{subject to } a_k \in V(\theta_k), \theta_k > 0. \end{aligned} \quad (6)$$

Step 2. Update the residual

$$r_k = f - \sum_{j=1}^k a_j \cos \theta_j. \quad (7)$$

Step 3. If $\|r_k\|_{L^2} < \varepsilon_0$, stop. Otherwise, set $k = k + 1$ and go back to Step 1.

In this paper, we decompose f over the dictionary

$$D_\varepsilon := \{a(\cdot) \cos \theta(\cdot) : (a, \theta) \in U_\varepsilon\}, \quad (8)$$

where

$$U_\varepsilon := \left\{ (a, \theta); a > 0, \theta' > 0; \left| \frac{a'}{a\theta'} \right| \leq \varepsilon, \left| \frac{\theta''}{[\theta']^2} \right| \leq \varepsilon, \frac{\sup_{t \in [0, T]} \theta'(t)}{\inf_{t \in [0, T]} \theta'(t)} \leq M' \right\}. \quad (9)$$

The main idea of the NMP algorithm is solving the following nonlinear least-squares problem:

$$\begin{aligned} & \text{minimize } p(a, \theta) := \|f(t) - a(t) \cos \theta(t)\|_{L^2}^2 \\ & \text{subject to } (a, \theta) \in U_\varepsilon. \end{aligned} \quad (P_2)$$

We consider the error of the sparse time-frequency decomposition of signals based on probability. By the proof of ([3], Theorem 3.1), we have the following result:

Lemma 1.3. *Let $f(t)$ be a function satisfying the scale separation property with separation factor ε and frequency ratio β as defined in Definition 1.2:*

$$f(t) = \sum_{k=1}^M a_k(t) \cos \theta_k(t) + r(t), \quad a_k \cos \theta_k \in U_\varepsilon, \quad a_k = O(1), \quad r = O(\varepsilon). \quad (10)$$

Suppose that $l \in \{1, \dots, M\}$. If

$$p(a, \theta) \leq p(a_l, \theta_l), \quad (11)$$

where $p(a, \theta)$ is given in (P₂), then we have

$$\frac{\|a \cos \theta - a_l \cos \theta_l\|_{L^2}}{\|a_l \cos \theta_l\|_{L^2}} \leq \delta_2 + \sqrt{\delta_2^2 + 2(\delta_1 + \delta_2)}, \quad (12)$$

where

$$\delta_1 = \sum_{k \neq l} \mu_{k,l} \frac{\|a_k \cos \theta_k\|_{L^2}}{\|a_l \cos \theta_l\|_{L^2}}, \quad \mu_{k,l} = \frac{|\langle a_l \cos \theta_l, a_k \cos \theta_k \rangle|}{\|a_l \cos \theta_l\|_{L^2} \|a_k \cos \theta_k\|_{L^2}}, \quad (13)$$

and

$$\delta_2 = \sum_{k \neq l} \gamma_{k,l} \frac{\|a_k \cos \theta_k\|_{L^2}}{\|a_l \cos \theta_l\|_{L^2}} + \frac{\|r(t)\|_{L^2}}{\|a_l \cos \theta_l\|_{L^2}}, \quad \gamma_{k,l} = \frac{|\langle a \cos \theta, a_k \cos \theta_k \rangle|}{\|a \cos \theta\|_{L^2} \|a_k \cos \theta_k\|_{L^2}}. \quad (14)$$

2. Error Analysis Based on Probability

In the actual information transmission process, the signal cannot arrive on time due to the long distance of transmission, such as the busy network or some faults in the transmission, which may result in phase delay of the signal. We introduce phase delay parameters and discuss f on the time domain $[0, T]$.

From Lemma 1.3, we know that the error of the sparse time-frequency decomposition of the signal depends on the coherences between $a_k \cos \theta_k$ and $a_l \cos \theta_l$. By ([9], Corollary 3.3), we have

$$\mu_{k,l} = \frac{|\langle a_l \cos \theta_l, a_k \cos \theta_k \rangle|}{\|a_l \cos \theta_l\|_{L^2} \|a_k \cos \theta_k\|_{L^2}} = O(\varepsilon), \quad (15)$$

and the error between the IMF computed by NMP algorithm and the true IMF is $O(\sqrt{\varepsilon})$. In this section, we introduce random variables to discuss

the error of signal sparse time-frequency decomposition from the perspective of probability. We assume that the phase delay parameter is a uniformly distributed random variable obeying $[0, 2\pi]$. Then we have

$$f(t) = \sum_{k=1}^M a_k(t) \cos(\theta_k(t) + X_k), \quad (16)$$

where $X_k \sim U[0, 2\pi]$, $k \in \{1, 2, \dots, M\}$. We define functions

$$\begin{aligned} g(X_k, X_l) &:= \langle a_k \cos(\theta_k + X_k), a_l \cos(\theta_l + X_l) \rangle \\ &= \int_0^T a_k(t) \cos(\theta_k(t) + x_k) a_l(t) \cos(\theta_l(t) + x_l) dt, \end{aligned} \quad (17)$$

for $k \neq l \in \{1, 2, \dots, M\}$. We have the following theorem.

Theorem 2.1. *Let $f(t)$ be a function satisfying the scale separation property with separation factor ε and frequency ratio β as defined in Definition 1.2:*

$$f(t) = \sum_{k=1}^M a_k(t) \cos(\theta_k(t) + X_k), \quad a_k = O(1), \quad \forall t \in [0, T], \quad (18)$$

where $(a_k, \theta_k)_{1 \leq k \leq M} \in U_\varepsilon$, Phase delay variable $X_k, X_l \sim U[0, 2\pi]$ ($k \neq l$). Then we have

$$E[g(X_k, X_l)] = 0, \quad (19)$$

$$\begin{aligned} D[g(X_k, X_l)] &= \frac{1}{4} \int_0^T \int_0^T a_k(t_1) a_l(t_1) a_k(t_2) a_l(t_2) \\ &\quad \cdot \cos(\theta_k(t_1) - \theta_k(t_2)) \cos(\theta_l(t_1) - \theta_l(t_2)) dt_1 dt_2. \end{aligned} \quad (20)$$

Proof. First, we have

$$\begin{aligned}
 E[g(X_k, X_l)] &= \frac{1}{2\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} dx_k dx_l \int_0^T a_k(t) \cos(\theta_k(t) + x_k) a_l(t) \cos(\theta_l(t) + x_l) dt \\
 &= \frac{1}{4\pi^2} \int_0^T a_k(t) a_l(t) dt \int_0^{2\pi} \cos(\theta_l(t) + x_k) dx_k \int_0^{2\pi} \cos(\theta_l(t) + x_l) dx_l \\
 &= 0.
 \end{aligned} \tag{21}$$

Now, we calculate $D[g(X_k, X_l)]$.

$$\begin{aligned}
 D[g(X_k, X_l)] &= \frac{1}{2\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} dx_k dx_l \left[\int_0^T a_k(t) \cos(\theta_k(t) + x_k) a_l(t) \cos(\theta_l(t) + x_l) dt \right]^2 \\
 &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} dx_k dx_l \left(\int_0^T a_k(t_1) \cos(\theta_k(t_1) + x_k) a_l(t_1) \cos(\theta_l(t_1) + x_l) dt_1 \right) \\
 &\quad \cdot \left(\int_0^T a_k(t_2) \cos(\theta_k(t_2) + x_k) a_l(t_2) \cos(\theta_l(t_2) + x_l) dt_2 \right).
 \end{aligned} \tag{22}$$

Since,

$$\begin{aligned}
 &\int_0^{2\pi} \cos(\theta_k(t_1) + x_k) \cos(\theta_k(t_2) + x_k) dx_k \\
 &= \int_0^{2\pi} \frac{1}{2} [\cos(\theta_k(t_1) + \theta_k(t_2) + 2x_k) + \cos(\theta_k(t_1) - \theta_k(t_2))] dx_k \\
 &= \pi \cos(\theta_k(t_1) - \theta_k(t_2)),
 \end{aligned} \tag{23}$$

and similarly,

$$\int_0^{2\pi} \cos(\theta_k(t_1) + x_k) \cos(\theta_k(t_2) + x_k) dx_k = \pi \cos(\theta_k(t_1) - \theta_k(t_2)), \tag{24}$$

we obtain (20) by exchange the integration order in (22). This completes the proof.

Noticing that

$$\begin{aligned}
 D[g(X_k, X_l)] = & \frac{1}{4} \left[\left(\int_0^T a_k(t_1) a_l(t_1) \cos \theta_k(t_1) \cos \theta_l(t_1) dt_1 \right) \right. \\
 & \cdot \left(\int_0^T a_k(t_2) a_l(t_2) \cos \theta_k(t_2) \cos \theta_l(t_2) dt_2 \right) \\
 & + \left(\int_0^T a_k(t_1) a_l(t_1) \cos \theta_k(t_1) \sin \theta_l(t_1) dt_1 \right) \\
 & \cdot \left(\int_0^T a_k(t_2) a_l(t_2) \cos \theta_k(t_2) \sin \theta_l(t_2) dt_2 \right) \\
 & + \left(\int_0^T a_k(t_1) a_l(t_1) \cos \theta_l(t_1) \sin \theta_k(t_1) dt_1 \right) \\
 & \cdot \left(\int_0^T a_k(t_2) a_l(t_2) \cos \theta_l(t_2) \sin \theta_k(t_2) dt_2 \right) \\
 & + \left(\int_0^T a_k(t_1) a_l(t_1) \sin \theta_k(t_1) \sin \theta_l(t_1) dt_1 \right) \\
 & \cdot \left. \left(\int_0^T a_k(t_2) a_l(t_2) \sin \theta_k(t_2) \sin \theta_l(t_2) dt_2 \right) \right], \quad (25)
 \end{aligned}$$

by ([9], Lemma 3.2), it is easy to check that

$$\begin{aligned}
 & \left| \int_0^T a_k(t_1) a_l(t_1) \cos \theta_k(t_1) \cos \theta_l(t_1) dt_1 \right| \\
 & = \left| \frac{1}{2} \int_0^T a_k(t_1) a_l(t_1) \cdot [\cos(\theta_k + \theta_l) + \cos(\theta_k - \theta_l)] dt_1 \right| \\
 & = O(\varepsilon). \quad (26)
 \end{aligned}$$

Other items in (24) have similar estimates. Thus, we get

$$D[g(X_k, X_l)] = O(\varepsilon^2). \quad (27)$$

Is this estimate the best? Let us study this issue by an instance.

Example 2.2. Let us consider the two IMFs which is defined as following:

$$imf_k = a_k \cos \theta_k(t), imf_l = a_l \cos \theta_l(t), \quad t \in [0, 1], \quad (28)$$

where

$$\begin{aligned} a_k &= c_k + \sum_{i=1}^{\lambda c_k} \left[a_{ik} \cos\left(\frac{\pi i}{T} t\right) + b_{ik} \sin\left(\frac{\pi i}{T} t\right) \right], \\ a_l &= c_l + \sum_{j=1}^{\lambda c_l} \left[a_{jl} \cos\left(\frac{\pi j}{T} t\right) + b_{jl} \sin\left(\frac{\pi j}{T} t\right) \right], \\ \theta_k &= 8\pi c_k t + \sum_{i=1}^{\lambda c_k} \left[a_{ik} \cos\left(\frac{4\pi i}{T} t\right) + b_{ik} \sin\left(\frac{4\pi i}{T} t\right) \right], \\ \theta_l &= 7.5\pi c_l t + \sum_{j=1}^{\lambda c_l} \left[a_{jl} \cos\left(\frac{3.5\pi j}{T} t\right) + b_{jl} \sin\left(\frac{3.5\pi j}{T} t\right) \right], \end{aligned}$$

and $c_k = 8, c_l = 6, \lambda = 0.5, T = 1; a_{ik}, b_{ik}, a_{jl}, b_{jl} (i = 1, \dots, \lambda c_k, j = 1, \dots, \lambda c_l)$ are random number from -1 to 1 such that $\varepsilon_0 \leq 0.1$, where

$$\varepsilon_0 = \max_{t \in [0, 1]} \left\{ \left| \frac{a'_k(t)}{a'_k(t)\theta'_k(t)} \right|, \left| \frac{\theta''_k(t)}{[\theta'_k(t)]^2} \right|, \left| \frac{a'_l(t)}{a'_l(t)\theta'_l(t)} \right|, \left| \frac{\theta'_l(t)}{[\theta'_l(t)]^2} \right| \right\}. \quad (29)$$

We present numerical experiments to show the relationship between $D[g(X_k, X_l)]$ and ε_0^2 in Example 2.2. We take 65000 different values for random numbers $a_{ik}, b_{ik}, a_{jl}, b_{jl}$, respectively and obtain the relationship between $D[g(X_k, X_l)]/(k_0 \varepsilon_0^2)$ and ε_0^2 in Figure 1, where

$$k_0 = (\|a_l \cos \theta_l\|_{L^2} \cdot \|a_k \cos \theta_k\|_{L^2})^2. \quad (30)$$

Figure 1. The scatter plot of $D[g(X_k, X_l)]/(k_0 \epsilon_0^2)$ and ϵ_0^2 .

It can be seen from Figure 1 that, in Example 2.2, there exist two positive constants M_1 and M_2 such that

$$M_1 \leq \left| D[g(X_k, X_l)]/(k \epsilon_0^2) \right| \leq M_2. \quad (31)$$

In Example 2.2, we know that the estimation result in (15) already be the best estimate. But in Theorem 2.1, we have $E[g(X_k, X_l)] = 0$, from the perspective of probability, $\mu_{k,l}$ may be smaller than that estimated in Equation (15). And the error between the computed IMF and the true IMF may be smaller than $O(\sqrt{\epsilon})$.

3. Concluding Remarks

In this paper, based on probability, we consider the sparse time-frequency decomposition of signals by nonlinear matching pursuit method and analyze the error of finding a sparse representation of signals. By the theoretical proof, we show the mean and variance of the inner product between IMFs. And we present numerical experiments and study the relationship between the variance and the separation factor ϵ .

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