THE BOLTZMANN EQUATION WITH POTENTIAL FORCES FOR INVERSE POWER LAW

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Abstract

We are concerned with the Cauchy problem of the non-cutoff Boltzmann equation with potential forces for inverse power law. The global existence of solutions with symmetric of origin and the exponential convergence rate in time to the steady state are obtained for any initially smooth, periodic, origin symmetric small perturbation.

Keywords: velocity weight, symmetric perturbation, convergence rate, non-cutoff.

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1. Introduction and the Statement of our Main Results

In this paper, we consider the following Boltzmann equation with potential forces:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x \phi \cdot \nabla_v F = Q(F, F), & t > 0, \\ F(0, x, v) = F_0(x, v), \end{cases} \quad (1.1)$$

where $F(t, x, v) \geq 0$ is the spatially periodic density function for the particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = \mathbf{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3$. $-\nabla_x \phi$ is the given external force generated by the stationary potential function $\phi = \phi(x)$. $Q(G_1, G_2)$ is defined by

$$\begin{aligned} Q(G_1, G_2)(v) &= \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(v - u, w) G_1(u') G_2(v') du dw \\ &\quad - \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(v - u, w) G_1(u) G_2(v) du dw, \end{aligned}$$

where $v' = \frac{v + u}{2} + \frac{|v - u|}{2} w$, $v' = \frac{v + u}{2} - \frac{|v - u|}{2} w$ denote the velocities of the particles after the collision, with u and v being their velocities before the collision. The Boltzmann collision kernel, $B(v - u, w)$ depends upon the relative velocity $|v - u|$ and on the deviation angle θ through $\cos \theta = \langle w, k \rangle$, where $k = \frac{v - u}{|v - u|}$ and $\langle \cdot, \cdot \rangle$ is the usual product in $\mathbf{R}^3$. Without restriction we suppose that $B(v - u, w)$ is supported on $\cos \theta \geq 0$, as in [1, 8].

The collision kernel. Our assumptions are the following:

(i) As usual, we suppose $B(v - u, w)$ takes product form in its argument as

$$B(v - u, w) = \Phi(|v - u|)b(\cos \theta).$$

It generally holds that both b and Φ are non-negative functions.

(ii) The angular function $x \rightarrow \mathfrak{b}(z)$ is not locally integrable, i.e.,

$$\int_{S^2} \mathfrak{b}(\cos \theta) dw = 2\pi \int_{-1}^1 \mathfrak{b}(z) dz = \infty.$$

Moreover, for $c_b > 0$, it satisfies

$$\frac{c_b}{\theta^{1+2s}} \leq \sin \theta \mathfrak{b}(\cos \theta) \leq \frac{1}{c_b \theta^{1+2s}}, \quad s \in (0, 1), \quad \theta \in \left(0, \frac{\pi}{2}\right].$$

(iii) The kinetic factor $z \rightarrow \Phi(z)$ satisfies for some $C_\Phi > 0$

$$\Phi(v - u) = C_\Phi |v - u|^\gamma,$$

where γ is determined by the potential of inverse power intermolecular forces, which is called soft potential when $-3 < \gamma < -2s$, and hard potential when $\gamma \geq -2s$. The present study is restricted to the case $-3 < \gamma < -2s$ and $s \in [\frac{1}{2}, 1)$.

We denote a normalized global Maxwellian by

$$\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp(-|v|^2/2).$$

It is straightforward to check that (1.1) has a stationary solution $F_s = F_s(x, v)$ given by

$$e^{-\phi(x)} \mu(v).$$

Throughout this paper, we look for solutions which are perturbations near such a stationary solution. Define the perturbation $f(t, x, v)$ as $F = F_s + \sqrt{F_s} f$, then the Boltzmann equation (1.1) for $f(t, x, v)$ takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f + e^{-\phi} Lf = e^{-\frac{\phi}{2}} \Gamma(f, f), \\ f(0, x, v) = f_0(x, v). \end{cases} \quad (1.2)$$

The linearized collision operator L is defined as

$$Lf = -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\}, \quad (1.3)$$

and Γ is given by

$$\begin{aligned} \Gamma(g_1, g_2) &= \mu^{-1/2} Q(\mu^{1/2} g_1, \mu^{1/2} g_2) \\ &= \int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} dw B(v-u, w) \mu^{1/2}(u) [g_1(u') g_2(v') - g_1(u) g_2(v)]. \end{aligned}$$

The main term of the linearized Boltzmann collision operator (1.3) can be split as follows:

$$\Gamma(\mu^{1/2}, f) = \int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} dw B(v-u, w) (f(v') - f(v)) \mu^{1/2}(u) \mu^{1/2}(u') - \tilde{v} f(v),$$

where

$$\tilde{v} = \int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} dw B(v-u, w) (\mu^{1/2}(u) - \mu^{1/2}(u')) \mu^{1/2}(u).$$

The first piece above contains a crucial Hilbert space structure, for the second one, we will use Pao's splitting as

$$\tilde{v} = \nu_1(v) + \nu_2(v),$$

and the following asymptotics are known:

$$\nu_1(v) \approx (1 + |v|^2)^{\frac{\gamma+2s}{2}}, \quad \nu_2(v) \lesssim (1 + |v|^2)^{\frac{\gamma}{2}}.$$

• Lg can be split as $Lg = \nu g + Kg$, where νg is the “norm part” and given by

$$\begin{aligned} \nu g &= -\mathcal{T}(\mu^{1/2}, g) - \nu_2(v)g \\ &= -\int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} dw B(v-u, w) (g(v') - g(v)) \mu^{1/2}(u) \mu^{1/2}(u') + \nu_1(v)g(v). \end{aligned}$$

Thus by using the pre-post collisional change of variables, this norm piece satisfies the following identity:

$$\begin{aligned} \langle \nu g, g \rangle &= \int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv \int_{\mathbf{S}^2} dw B(v-u, w) (g(v') - g(v))^2 \mu^{1/2}(u) \mu^{1/2}(v') \\ &\quad + \int_{\mathbf{R}^3} dv \nu_1(v) |g(v)|^2. \end{aligned}$$

These two quantities will define our designer norm, which is sharp for the linearized operator. We will also record here the definition of the “compact piece”

$$\begin{aligned} Kg &= -\mathcal{S}(g, \mu^{1/2}) + \nu_2(v)g \\ &= -\int_{\mathbf{R}^3} \int_{\mathbf{S}^2} du dv dw B(v-u, w) \mu^{1/2}(u) [g(u') \mu^{1/2}(v') - g(u) \mu^{1/2}(v)] \\ &\quad + \nu_2(v)g \pm(v). \end{aligned}$$

- The null space of L is given by

$$\mathcal{N} = \text{span}\{\mu^{1/2}, v_i \mu^{1/2} (1 \leq i \leq 3), |v|^2 \mu^{1/2}\}.$$

- For any fixed (t, x) , L is nonnegative and further L is known to be locally coercive in the sense that there is a constant $\delta_0 > 0$ such that [1, 8]

$$\langle f, Lf \rangle = \langle (\mathbf{I} - \mathbf{P})f, L(\mathbf{I} - \mathbf{P})f \rangle \geq \delta_0 \left| (1 + |v|^2)^{\frac{\gamma}{4}} (\mathbf{I} - \mathbf{P})f \right|^2,$$

where $\mathbf{P}$ denotes the orthogonal projection from $L_v^2 \times L_v^2$ to $\mathcal{N}$. Moreover, we can also define

$$\mathbf{P}f = \{a_f(t, x) + v \cdot b_f(t, x) + |v|^2 c_f(t, x)\} \sqrt{\mu}.$$

For notational simplicity, we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product in $\mathbf{R}_v^3$, with its L^2 norm given by $|\cdot|_2$, moreover $(\cdot, \cdot)$ is L^2 inner product either in $\mathbf{T}_x^3 \times \mathbf{R}_v^3$ or in $\mathbf{T}_x^3$ with corresponding L^2 norm $\|\cdot\|$. We

use the standard notation H^s or $W^{k,p}$ to denote the Sobolev space. We also define $B_C \subset \mathbf{R}^3$ to be the Euclidean ball of radius C centered at the origin, then $L^2(B_C)$ is the space L^2 on this ball and similarly for other spaces.

Let $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, $\beta = [\beta_1, \beta_2, \beta_3]$ denote multi-indices with length $|\alpha|$ and $|\beta|$, respectively, and let

$$\partial_\beta^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

Furthermore, define $\beta_1 \leq \beta$ if no component of β_1 is greater than the component of β , and $\beta_1 < \beta$ if $\beta_1 \leq \beta$ and $|\beta_1| < |\beta|$. Besides we use $C_\beta^{\beta_1}$

to denote the usual binomial coefficient $\binom{\beta}{\beta_1}$. We also denote by e_i the

multi-index whose i -th component is 1, and the others are 0. From now on, we use C , c or λ to denote a generic positive constant may be different from line to line. In addition, $B_1 \sim B_2$ means $\lambda B_1 \leq B_2 \leq \frac{1}{\lambda} B_1$ for a generic constant $0 < \lambda < 1$.

Our solution to the problem (1.2) rests heavily on our introduction into the Boltzmann theory of the weighted geometric fractional Sobolev space

$$N_\gamma^s = \{f \in \mathcal{S}'(\mathbf{R}^3) : |f|_{N_\gamma^s} < \infty\},$$

where $\mathcal{S}'(\mathbf{R}^3)$ denotes the space of the tempered distribution functions, and we specify the anisotropic norm by

$$|f|_{N_\gamma^s}^2 = |f|_{\gamma+2s}^2 + \int_{\mathbf{R}^3} dv \int_{\mathbf{R}^3} dv' (\langle v \rangle \langle v' \rangle)^{\frac{\gamma+2s+1}{2}} \frac{(f(v) - f(v'))^2}{d(v, v')^{3+2s}} 1_{d(v, v') \leq 1}. \quad (1.4)$$

This space includes the weighted L_ℓ^2 space, for $\ell \in \mathbf{R}$, with norm given by

$$|f|_\ell^2 = \int_{\mathbf{R}^3} dv \langle v \rangle^\ell |f(v)|^2, \quad \langle v \rangle = \sqrt{1 + |v|^2}.$$

The fractional differentiation effects are measured using the following anisotropic metric $d(v, v')$ on the “lifted” paraboloid:

$$d(v, v') = \sqrt{|v - v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}.$$

The analogous norms on $\mathbf{R}^3 \times \mathbf{R}^3$ will denoted by

$$\|h\|_{N_\gamma^s}^2 = \| |f|_{N_\gamma^s} \|_{L^2(\mathbf{R}^3)}^2.$$

For $0 < s < 1$, $\ell \in \mathbf{R}$, a reasonable definition of weighted fractional Sobolev norm is given by

$$|w^\ell h|_{H_\gamma^s}^2 = |w^\ell h|_\gamma^2 + \int_{\mathbf{R}^3} dv \int_{\mathbf{R}^3} dv' \frac{[\langle v \rangle^{\frac{\gamma}{2}} w^\ell(v) f(v) - \langle v' \rangle^{\frac{\gamma}{2}} w^\ell(v') f(v')]^2}{|v - v'|^{3+2s}} 1_{|v-v'| \leq 1}. \quad (1.5)$$

We mention that (1.5) has another equivalent form:

$$|w^\ell h|_{H_\gamma^s}^2 = \int_{\mathbf{R}^3} dv \langle v \rangle^\gamma \left| (\mathbf{I} - \Delta_v)^{\frac{s}{2}} (w^\ell(v) h(v)) \right|^2.$$

We also consider the general weighted anisotropic derivative space as in (1.4) by

$$|w^\ell h|_{N_\gamma^s}^2 = |w^\ell(v) h|_{\gamma+2s}^2 + \int_{\mathbf{R}^3} dv \int_{\mathbf{R}^3} dv' (\langle v \rangle \langle v' \rangle)^{\frac{\gamma+2s+1}{2}} w^{2\ell}(v) \\ \times \frac{(h(v) - h(v'))^2}{d(v, v')^{3+2s}} 1_{d(v, v') \leq 1}.$$

To this velocity space we associate the total spaces $\|w^\ell h\|_{N_\gamma^s} = \left\| |w^\ell h|_{N_\gamma^s} \right\|_{L^2(\mathbf{T}^3)}$. We also use the anisotropic space $N_{\gamma,K}^s(\mathbf{T}^3 \times \mathbf{R}^3)$ given by

$$\|w^\ell h\|_{N_{\gamma,K}^s}^2 = \|w^\ell h\|_{N_{\gamma,K}^s(\mathbf{T}^3 \times \mathbf{R}^3)}^2 = \sum_{|\alpha|+|\beta| \leq K} \|w^\ell \partial_\beta^\alpha h\|_{N_\gamma^s(\mathbf{T}^3 \times \mathbf{R}^3)}^2.$$

Furthermore, we use a version of this norm only in the velocity variables

$$|w^\ell h|_{N_{\gamma,K}^s}^2 = |w^\ell h|_{N_{\gamma,K}^s(\mathbf{R}^3)}^2 = \sum_{|\beta| \leq K} |w^\ell \partial_\beta h|_{N_\gamma^s(\mathbf{R}^3)}^2.$$

We note that

$$|w^\ell h|_{H_\gamma^s}^2 \lesssim |w^\ell h|_{N_\gamma^s}^2 \lesssim |w^\ell h|_{H_{\gamma+2s}^s}^2, \quad \|h\|_{N_\gamma^s}^2 \gtrsim \|\langle v \rangle^{\frac{\gamma+2s}{2}} h\|^2 + \|\langle v \rangle^{\frac{\gamma}{2}} (1 - \Delta_v)^{\frac{s}{2}} h\|^2. \quad (1.6)$$

For integer $K \geq 0$, we define the Sobolev space

$$|h|_{H^K} = \sum_{|\beta| \leq K} |\partial_\beta h|_{L^2(\mathbf{R}^3)}, \quad \|h\|_{H^K} = \sum_{|\alpha|+|\beta| \leq K} \|\partial_\beta^\alpha h\|_{L^2(\mathbf{T}^3 \times \mathbf{R}^3)}.$$

A weight function $w(\theta, q)(v)$ is given as

$$w(\theta, q)(v) = e^{\frac{q(v)}{4}} (1 + |v|^2)^{\frac{\gamma\theta}{2}}, \quad 0 \leq q < 1, \quad -3 < \gamma < 0, \quad \theta \in \mathbf{R}. \quad (1.7)$$

Let $l \geq N \geq 8$, for any function g , we define the instant energy functional to be any functional $\mathcal{E}_{N,l,q}(g)$, which satisfies the equivalent relation

$$\mathcal{E}_{N,l,q}(g) \sim \sum_{|\alpha|+|\beta| \leq N} \|w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha g\|^2, \quad -3 < \gamma < 0. \quad (1.8)$$

And define the dissipation rate $\mathcal{D}_{N,l,q}(g)$ as

$$\mathcal{D}_{N,l,q}(g) = \sum_{|\alpha|+|\beta|\leq N} \|w(-l+|\alpha|+|\beta|, q)\partial_{\beta}^{\alpha} g\|_{N_{\gamma}^s}^2, \quad -3 < \gamma < 0. \quad (1.9)$$

In addition, we write $\mathcal{E}_{N,l,0} = \mathcal{E}_{N,l}$ and $\mathcal{D}_{N,l} = \mathcal{D}_{N,l,0}$.

Throughout this paper, we shall assume $N \geq 8$. Our main results are as follows.

Theorem 1.1. *Let $F(0, x, v) = F_s + \sqrt{F_s}f_0(x, v) \geq 0$, $-3 < \gamma < -2s$, $0 \leq q < 1$, $l \geq N$. Assume that*

$$(A1) \quad \phi(-x) = \phi(x), f_0(-x, -v) = f_0(x, v);$$

$$(A2) \quad \int_{\mathbf{T}^3 \times \mathbf{R}^3} \left[1, v, \frac{1}{2}|v|^2 + \phi(x)\right] f_0 \sqrt{F_s} dx dv = 0;$$

(A3) $\|\phi\|_{L^\infty} \leq C_\phi$, for $C_\phi > 0$, $\|\nabla_x \phi\|_{W^{N,\infty}} \leq \epsilon_\phi$, for ϵ_ϕ is positive and small enough.

Then, there are constants $\lambda > 0$, $C > 0$, such that if $\mathcal{E}_{N,l,q}(f_0)$ is small enough, there exists a unique global solution $f(t, x, v)$ to (1.2) such that $F(t, x, v) = F_s + \sqrt{F_s}f(t, x, v) \geq 0$, $f(t, -x, -v) = f(t, x, v)$ and

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} \left[1, v, \frac{1}{2}|v|^2 + \phi(x)\right] f \sqrt{F_s} dx dv = 0;$$

$$\frac{d}{dt} \mathcal{E}_{N,l,q}(f) + \lambda \mathcal{D}_{N,l,q}(f)(t) \leq 0. \quad (1.10)$$

Moreover, for $q > 0$, there exists $\tilde{\lambda} > 0$ such that

$$\mathcal{E}_{N,l}(f)(t) \leq C e^{-\tilde{\lambda} t^p} \mathcal{E}_{N,l,q}(f_0),$$

where $p = \frac{1}{1-\gamma-2s}$.

Due to the great breakthrough of the study for the non-cutoff Boltzmann equation achieved by Alexandre-Morimoto-Ukai-Xu-Yang and Gressman-Strain, independently, there have been a lot of literatures concerning the investigation of the non-cutoff Boltzmann equation and related models, see [5, 6, 7] and reference therein. We also mention that the global unique solutions to the classical Vlasov-Poisson-Boltzmann and Vlasov-Poisson-Landau system are constructed in [9] and [10], respectively, by a unified energy estimate with velocity weight.

Recently, there are also extensive studies for the cutoff Boltzmann equation or Landau equation with external force, see [4, 11, 13, 14, 15]. However, to our knowledge, there are few results about the non-cutoff Boltzmann equation with external force, our paper is devoted to this problem. For this purpose, the main difficulty is created by the free streaming term $v \cdot \nabla_x f$ and the external forcing term $\nabla_x \phi \cdot \nabla_v f$. It is easy to see that the inner product (using Einstein's convention for repeated indices)

$$(\partial_{v_i}(v_j \partial_{x_j} f), \partial_{v_i} f) = \int \partial_{x_i} f \partial_{v_i} f,$$

and

$$(\partial_{x_i}(\partial_{x_j} \phi \partial_{v_j} f), \partial_{x_i} f) = \int (\partial_{x_i x_j}^2 \phi) \partial_{x_i} f \partial_{v_j} f,$$

can not be bounded in terms of $\|\partial_{x_i} f\| \|f\|_{N_\gamma^s}$. To overcome this difficulty

and to obtain the time decay rates, we design a new w -weighted norm (1.7) which contains an exponential growth and depends on both of v -derivatives and x -derivatives. The main trick is to do the splitting

$\partial_\beta^\alpha f = \partial_{e_i} \partial_{\beta - e_i}^\alpha f$ in the inner product term above, write in a rough way

the first-order velocity differentiation as $\partial_{e_i} = \partial_{e_i}^{1/2} \partial_{e_i}^{1/2}$, and further use

$w_l(\alpha, \beta) = w_l(\alpha, \beta - e_i) \langle v \rangle^\gamma$ to gain the degenerate velocity weight, so

that the delicate term $\left(\partial_{\beta-e_i}^{\alpha+e_i} f, w_l^2(\alpha, \beta) \partial_{\beta}^{\alpha} f\right)$ can be bounded by the dissipation norm

$$\|\langle v \rangle^{\frac{\gamma}{2}} (1 - \Delta_v)^{\frac{s}{2}} [w_l(\alpha, \beta - e_i) \partial_{\beta-e_i}^{\alpha} f]\|^2 + \|\langle v \rangle^{\frac{\gamma}{2}} (1 - \Delta_v)^{\frac{s}{2}} [w_l(\alpha + e_i, \beta - e_i) \partial_{\beta-e_i}^{\alpha+e_i} f]\|^2$$

up to some other controllable terms, where $s \geq 1/2$ and Fourier's transform are used.

The rest of this paper is arranged as follows. Section 2 is devoted to the estimates for the hydrodynamic part $\mathbf{P}f$, and the proof of Theorem 1.1 is given in Section 3. Finally, some basic estimates are collected in Appendix A.

2. The Estimates for the Macroscopic Part

In this section, we intend to estimate $a_f(t, x)$, $b_f(t, x)$, $c_f(t, x)$ (we denote them by $a(t, x)$, $b(t, x)$, $c(t, x)$ for brevity as follows) in terms of $(\mathbf{I} - \mathbf{P})f$. For results in this direction, we have

Lemma 2.1. *Let $f(t, x, v)$ be a solution constructed in Theorem 1.1, then there exists an instant interaction functional $G_{\alpha}(t)$ which satisfies $|G_{\alpha}(t)| \leq \mathcal{E}_{N,l}(f)$ such that*

$$\sum_{|\alpha| \leq N} \|\partial^{\alpha} \mathbf{P}f\|^2 \leq \frac{dG_{\alpha}(t)}{dt} + \tilde{C}_1 \sum_{|\alpha| \leq N} \|\partial^{\alpha} (\mathbf{I} - \mathbf{P})f\|_{N_{\gamma}^s}^2 + \tilde{C}_2 \mathcal{E}_{N,l}(f) \mathcal{D}_{N,l}(f). \quad (2.1)$$

Here $\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$ are constants.

Proof. The proof of Lemma 2.1 is similar to that of [11], we omit the details for brevity. $\square$

3. Global Existence and Exponential Decay Rate

In this section, we are going to prove our Theorem 1.1. The energy estimate is a crucial step in constructing the global solutions of (1.2). The global existence follows from the local existence together with the uniform a priori estimates as well as the standard continuum argument. Here, we skip the proof of the local existence for brevity. Now we state a energy estimate without the exponential weight ($q = 0$).

Lemma 3.1. *Let $f(t, x, v)$ be a unique solution of equation (1.2). For any given $N \geq 8$, $l \geq N$, $|\alpha| + |\beta| \leq N$, there exist $\lambda_i > 0$ ($1 \leq i \leq 5$) such that*

$$\begin{aligned} & \frac{d}{dt} \left\{ \lambda_1 \sum_{|\alpha|+|\beta| \leq N} \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|^2 + \lambda_2 \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2 + \lambda_3 G_\alpha(t) \right\} \\ & + \lambda_4 \sum_{|\alpha|+|\beta| \leq N} \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|_{N_\gamma^s}^2 \\ & \leq \lambda_5 \left\{ \mathcal{E}_{N,l}^{1/2}(f) + \mathcal{E}_{N,l}(f) \right\} \mathcal{D}_{N,l}(f). \end{aligned} \quad (3.1)$$

Proof. We divide our proof into following two steps. The first step is concerned with mixed derivatives with velocity weight. Letting $|\beta| + |\alpha| \leq N$, by taking ∂_β^α of (1.2), we obtain

$$\begin{aligned} & \partial_t \partial_\beta^\alpha f + v_i \partial_\beta^{\alpha+e_i} f - \partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f - \chi_\alpha \sum_{1 \leq |\alpha_1| \leq |\alpha|} C_\alpha^{\alpha_1} \partial^{\alpha_1+e_i} \phi \partial_{e_i+\beta}^{\alpha-\alpha_1} f + \chi_\beta \beta_i \partial_{\beta-e_i}^{\alpha+e_i} f \\ & + e^{-\phi} \partial_\beta^\alpha Lf + \chi_\alpha \sum_{1 \leq |\alpha_1| \leq |\alpha|} C_\alpha^{\alpha_1} \partial^{\alpha_1} e^{-\phi} \partial_\beta^{\alpha-\alpha_1} Lf = \sum_{\bar{\alpha}_1 \leq \alpha} C_{\alpha}^{\bar{\alpha}_1} \partial^{\bar{\alpha}_1} e^{-\frac{\phi}{2}} \partial_\beta^{\alpha-\bar{\alpha}_1} \Gamma(f, f), \end{aligned} \quad (3.2)$$

where

$$\chi_\alpha = \begin{cases} 0, & \alpha = 0, \\ 1, & \alpha > 0, \end{cases}$$

and χ_β is defined similarly. Taking inner product with $w^2(-l + |\alpha| + |\beta|, 0)\partial^\alpha f$ of Equation (3.2), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|^2 + (e^{-\phi}\partial_\beta^\alpha Lf, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f) \\ & \leq |(\partial^{e_i}\phi\partial_{e_i+\beta}^\alpha f, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f)| \\ & \quad + \chi_\beta \left| (\beta_i\partial_{\beta-e_i}^{\alpha+e_i} f, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f) \right| \\ & \quad + \sum_{1 \leq |\alpha_1| \leq |\alpha|} \chi_\alpha \left| C_\alpha^{\alpha_1} (\partial^{\alpha_1+e_i}\phi\partial_{e_i+\beta}^{\alpha-\alpha_1} f, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f) \right| \\ & \quad + \sum_{1 \leq |\alpha_1| \leq |\alpha|} \chi_\alpha \left| C_\alpha^{\alpha_1} (\partial^{\alpha_1} e^{-\phi}\partial_\beta^{\alpha-\alpha_1} Lf, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f) \right| \\ & \quad + \sum_{\bar{\alpha}_1 \leq \alpha} \left| C_{\bar{\alpha}_1}^{\bar{\alpha}_1} \left(\left(\partial^{\bar{\alpha}_1} e^{-\frac{\phi}{2}} \right) \partial_\beta^{\alpha-\bar{\alpha}_1} \Gamma(f, f), w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f \right) \right|. \end{aligned} \quad (3.3)$$

Next, applying Lemmas A.1 and A.2 and noticing $\|\phi\|_{L^\infty} \leq C_\phi$, we find that if $\beta > 0$,

$$\begin{aligned} & (e^{-\phi}\partial_\beta^\alpha Lf, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f) \\ & \geq \delta_1 \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|_{N_\gamma^s}^2 - C \sum_{\bar{\beta} < \beta} \|w(-l + |\alpha| + |\bar{\beta}|, 0)\partial_\beta^\alpha f\|_{N_\gamma^s}^2, \end{aligned}$$

if $\beta = 0$,

$$(e^{-\phi}L\partial^\alpha f, w^2(-l + |\alpha|, 0)\partial^\alpha f) \geq \tilde{\delta}_1 \|w(-l + |\alpha|, 0)\partial^\alpha f\|_{N_\gamma^s}^2 - C \|\partial^\alpha f\|_{N_\gamma^s}^2,$$

for some appropriate $\delta_1 > 0$, $\tilde{\delta}_1 > 0$ and $C > 0$.

For the first term in the second line of (3.3), by integration by parts, we discover

$$\begin{aligned} & \left(\partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right) \\ &= -\gamma(-l+|\alpha|+|\beta|) \left(\frac{v_i}{1+|v|^2} \partial^{e_i} \phi \partial_\beta^\alpha f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right), \end{aligned}$$

which implies

$$\begin{aligned} & \left| \left(\partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right) \right| \\ & \leq C \|\partial^{e_i} \phi\|_{L^\infty} \left| \left(\frac{v_i}{1+|v|^2} \partial^{e_i} \phi \partial_\beta^\alpha f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right) \right| \\ & \leq C \epsilon_\phi \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|_{N_\gamma^s}^2, \end{aligned}$$

where we have used the fact that $\left| \frac{v_i}{1+|v|^2} \right| \leq (1+|v|)^{\gamma+2}$, for $\gamma > -3$. In

light of the Cauchy-Schwartz inequality and Fourier's transform, the second term in the same line can be estimated as follows:

$$\begin{aligned} & \left| \beta_i \left(\partial_{\beta-e_i}^{\alpha+e_i} f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right) \right| \\ & \leq C \sum_{\substack{|\alpha'|+|\beta'| \leq N \\ \beta' < \beta}} \left\| w(-l+|\alpha'|+|\beta'|, 0) \partial_{\beta'}^{\alpha'} f \right\|_{N_\gamma^s}^2. \end{aligned}$$

Likewise, for the term in the third line, one can compute that

$$\begin{aligned} & \sum_{1 \leq |\alpha_1| \leq |\alpha|} \left| \left(\partial^{\alpha_1+e_i} \phi \partial_{e_i+\beta}^{\alpha-\alpha_1} f, w^2(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f \right) \right| \\ & \leq C \|\partial^{\alpha_1+e_i} \phi\|_{L^\infty} \left\{ \|w(-l+|\alpha|+|\beta|+1/2, 0) \partial_\beta^\alpha f\|^2 \right. \\ & \quad \left. + \|w(-l+|\alpha-\alpha_1|+|\beta|+1/2, 0) \partial_{e_i+\beta}^{\alpha-\alpha_1} f\|^2 \right\} \end{aligned}$$

$$\begin{aligned} &\leq C\epsilon_\phi \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|_{N_\gamma^s}^2 \\ &\quad + C\epsilon_\phi \|w(-l + |\alpha - \alpha_1| + |\beta|, 0)\partial_\beta^{\alpha - \alpha_1} f\|_{N_\gamma^s}^2. \end{aligned}$$

For the terms in the fourth line and fifth line, we get from Lemmas A.4 and A.3 that

$$\begin{aligned} &\sum_{0 < \alpha_1 \leq \alpha} C_\alpha^{\alpha_1} \left| \left(\partial^{\alpha_1} e^{-\phi} \partial_\beta^{\alpha - \alpha_1} Lf, w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f \right) \right| \\ &\leq C\epsilon_\phi \sum_{\bar{\beta} \leq \beta} \|w(-l + |\alpha - \alpha_1| + |\bar{\beta}|, 0)\partial_\beta^{\alpha - \alpha_1} f\|_{N_\gamma^s}^2 \\ &\quad + C\epsilon_\phi \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|_{N_\gamma^s}^2. \\ &\sum_{\bar{\alpha}_1 \leq \alpha} \left| \left(\partial^{\bar{\alpha}_1} e^{-\frac{\phi}{2}} \partial_\beta^{\alpha - \bar{\alpha}_1} \Gamma(f, f), w^2(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f \right) \right| \leq C\mathcal{E}_{N,l}^{1/2}(f) \mathcal{D}_{N,l}(f). \end{aligned}$$

Next, a suitable linear combination of the above estimates yields that

$$\begin{aligned} &\frac{d}{dt} \sum_{|\alpha|+|\beta| \leq N} \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|^2 + \delta_2 \sum_{|\alpha|+|\beta| \leq N} \|w(-l + |\alpha| + |\beta|, 0)\partial_\beta^\alpha f\|_{N_\gamma^s}^2 \\ &\leq C_1 \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{N_\gamma^s}^2 + C\mathcal{E}_{N,l}^{1/2}(f) \mathcal{D}_{N,l}(f), \end{aligned} \quad (3.4)$$

where $\delta_2 > 0$.

The second step is devoted to pure x -derivatives without velocity weight. Taking inner product with $\partial^\alpha f$ of Equation (3.2) and applying Lemma A.2, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial^\alpha f\|^2 + \delta_3 \sum_{|\alpha| \leq N} \|\partial^\alpha (\mathbf{I} - \mathbf{P})f\|_{N_\gamma^s}^2 \\
& \leq \chi_\alpha \sum_{1 \leq |\alpha_1| \leq |\alpha|} C_\alpha^{\alpha_1} |(\partial^{\alpha_1 + e_i} \phi \partial_{e_i}^{\alpha - \alpha_1} f, \partial^\alpha f)| \\
& \quad + \chi_\alpha \sum_{1 \leq |\alpha_1| \leq |\alpha|} C_\alpha^{\alpha_1} |(\partial^{\alpha_1} e^{-\phi} L \partial^{\alpha - \alpha_1} f, \partial^\alpha f)| \\
& \quad + \sum_{\bar{\alpha}_1 \leq \alpha} C_\alpha^{\bar{\alpha}_1} \left| \left(\partial^{\bar{\alpha}_1} e^{-\frac{\phi}{2}} \partial^{\alpha - \bar{\alpha}_1} \Gamma(f, f), \partial^\alpha f \right) \right|, \tag{3.5}
\end{aligned}$$

where $\delta_3 > 0$. For the first term on the right hand side of (3.5). We get from Cauchy-Schwartz's inequality that

$$|(\partial^{\alpha_1 + e_i} \phi \partial_{e_i}^{\alpha - \alpha_1} f, \partial^\alpha f)| \leq C_{\epsilon_\phi} \|\partial^\alpha f\|_{N_\gamma^s}^2 + C_{\epsilon_\phi} \|w^{-1} \partial^{\alpha - \alpha_1} f\|_{N_\gamma^s}^2. \tag{3.6}$$

Next, in view of Lemmas A.4 and A.3, we compute

$$\sum_{1 \leq |\alpha_1| \leq |\alpha|} |(\partial^{\alpha_1} e^{-\phi} L \partial^{\alpha - \alpha_1} f, \partial^\alpha f)| \leq C_{\epsilon_\phi} \|\partial^{\alpha - \alpha_1} f\|_{N_\gamma^s}^2 + C_{\epsilon_\phi} \|\partial^\alpha f\|_{N_\gamma^s}^2, \tag{3.7}$$

$$\sum_{\bar{\alpha}_1 \leq \alpha} \left| \left(\partial^{\bar{\alpha}_1} e^{-\frac{\phi}{2}} \partial^{\alpha - \bar{\alpha}_1} \Gamma(f, f), \partial^\alpha f \right) \right| \leq C \mathcal{E}_{N,l}^{1/2}(f) \mathcal{D}_{N,l}(f). \tag{3.8}$$

Now for some $\delta_4 > 0$, we conclude from (3.5), (3.6), (3.7), and (3.8) that

$$\begin{aligned}
& \frac{d}{dt} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2 + \delta_4 \sum_{|\alpha| \leq N} \|\partial^\alpha (I - P)f\|_{N_\gamma^s}^2 \\
& \leq C_{2\epsilon_\phi} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{N_\gamma^s}^2 + C_{3\epsilon_\phi} \sum_{|\alpha| \leq N-1} \|w^{-1} \partial^\alpha f\|_{N_\gamma^s}^2 + C \mathcal{E}_{N,l}^{\frac{1}{2}}(f) \mathcal{D}_{N,l}(f).
\end{aligned} \tag{3.9}$$

At this point, we note that if $l \geq N$,

$$\sum_{|\alpha'| \leq N-1} \|w^{-1} \partial^{\alpha'} f\|_{N_\gamma^s}^2 \leq \sum_{|\alpha'| \leq N-1} \|w^{-l+|\alpha'|} \partial^{\alpha'} f\|_{N_\gamma^s}^2.$$

This observation suggests that $l \geq N$ is required in our lemma.

Returning now to Lemma 3.2, we find

$$\frac{\delta_4}{2} \sum_{|\alpha| \leq N} \|\partial^\alpha (\mathbf{I} - \mathbf{P})f\|_{N_\gamma^s}^2 \geq \frac{\delta_4}{2\tilde{C}_1} \left\{ \sum_{|\alpha| \leq N} \|\partial^\alpha Pf\|^2 + G'_\alpha(t) \right\} - \frac{\tilde{C}_2 \delta_4}{2\tilde{C}_1} \mathcal{E}_{N,l}(f) \mathcal{D}_{N,l}(f).$$

From which and (3.9), we deduce that

$$\begin{aligned} & \sum_{|\alpha| \leq N} \frac{d}{dt} \left\{ \|\partial^\alpha f\|^2 + \frac{\delta_4}{2\tilde{C}_1} G_\alpha(t) \right\} + \left(\min \left\{ \frac{\delta_4}{2}, \frac{\delta_4}{2\tilde{C}_1} \right\} - C_{2\epsilon_\phi} \right) \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{N_\gamma^s}^2 \\ & \leq C_{3\epsilon_\phi} \sum_{|\alpha'| \leq N-1} \|w^{-1} \partial^{\alpha'} f\|_{N_\gamma^s}^2 + C \left\{ \mathcal{E}_{N,l}^{1/2}(f) + \mathcal{E}_{N,l}(f) \right\} \mathcal{D}_{N,l}(f). \end{aligned} \quad (3.10)$$

Now denote $\min \left\{ \frac{\delta_4}{2}, \frac{\delta_4}{2\tilde{C}_1} \right\} - C_{2\epsilon_\phi} = \delta_5$, and let $C_{3\epsilon_\phi}$ small enough.

Then $\frac{\delta_5}{2C_1} \times (3.4) + (3.10)$ gives

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{\delta_5}{2C_1} \sum_{|\alpha|+|\beta| \leq N} \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|^2 + \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2 + \frac{\delta_4}{2\tilde{C}_1} \sum_{|\alpha| \leq N} G_\alpha(t) \right\} \\ & + \left(\frac{\delta_5 \delta_2}{2C_1} - C_{3\epsilon_\phi} \right) \sum_{|\alpha|+|\beta| \leq N} \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|_{N_\gamma^s}^2 + \frac{\delta_5}{2} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{N_\gamma^s}^2 \\ & \leq C \left\{ \mathcal{E}_{N,l}^{1/2}(f) + \mathcal{E}_{N,l}(f) \right\} \mathcal{D}_{N,l}(f). \end{aligned} \quad (3.11)$$

Since $\|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|^2 \geq \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2$, (3.11) implies (3.1). This

completes the proof of Lemma 3.1. $\square$

Next we will bootstrap the energy estimate (3.1) to the Lemmas A.1 and A.2 to obtain the following general estimate:

Lemma 3.2. *Let $f(t, x, v)$ be a solution of Equation (1.2). For any given $N \geq 8$, $l \geq N$, $|\alpha| + |\beta| \leq N$, $0 \leq q < 1$, there exist $\lambda_i > 0$ ($6 \leq i \leq 11$) such that*

$$\begin{aligned} & \frac{d}{dt} \left\{ \lambda_6 \sum_{|\alpha|+|\beta| \leq N} \left\| e^{\frac{q\Phi}{2}} w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right\|^2 + \lambda_7 \sum_{|\alpha|+|\beta| \leq N} \|w(-l + |\alpha| + |\beta|, 0) \partial_\beta^\alpha f\|^2 \right. \\ & \quad \left. + \lambda_8 \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2 + \lambda_9 G_\alpha(t) \right\} + \lambda_{10} \sum_{|\alpha|+|\beta| \leq N} \|w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f\|_{N_\gamma^s}^2 \\ & \leq \lambda_{11} \left\{ \mathcal{E}_{N,l,q}^{1/2}(f) + \mathcal{E}_{N,l,q}(f) \right\} \mathcal{D}_{N,l,q}(f). \end{aligned} \quad (3.12)$$

Proof. Taking inner product with $e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f$ of Equation (3.2), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| e^{\frac{q\Phi}{2}} w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right\|^2 + (e^{-\Phi} \partial_\beta^\alpha Lf, e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f) \\ & \leq \left| (v_i \partial_\beta^{\alpha+e_i} f - \partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f, e^{q\Phi} w^2(-l + |\alpha|, q) \partial_\beta^\alpha f) \right| \\ & \quad + \chi_\beta \left| (\beta_i \partial_{\beta-e_i}^{\alpha+e_i} f, e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f) \right| \\ & \quad + \sum_{1 \leq |\alpha_1| \leq |\alpha|} \chi_\alpha \left| C_\alpha^{\alpha_1} \left(\partial^{\alpha_1+e_i} \phi \partial_{e_i-\beta}^{\alpha-\alpha_1} f, e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \right| \\ & \quad + \sum_{1 \leq |\alpha_1| \leq |\alpha|} \chi_\alpha \left| C_\alpha^{\alpha_1} \left(\partial^{\alpha_1} e^{-\Phi} \partial_\beta^{\alpha-\alpha_1} Lf, e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \right| \\ & \quad + \sum_{\bar{\alpha}_1 \leq \alpha} \left| C_\alpha^{\bar{\alpha}_1} \left(\left(\partial^{\bar{\alpha}_1} e^{-\frac{\Phi}{2}} \right) \partial_\beta^{\alpha-\bar{\alpha}_1} \Gamma(f, f), e^{q\Phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \right|, \end{aligned} \quad (3.13)$$

Next, employing Lemmas A.1 and A.2 and noticing $\|\phi\|_{L^\infty} \leq C_\phi$, we find that if $\beta > 0$,

$$\begin{aligned} & \left(e^{-\phi} \partial_\beta^\alpha Lf, e^{q\phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \\ & \geq \delta_6 \|w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f\|_{N_\gamma^s}^2 - C \sum_{\beta < \beta} \|w(-l + |\alpha| + |\bar{\beta}|, q) \partial_{\bar{\beta}}^\alpha f\|_{N_\gamma^s}^2, \end{aligned}$$

if $\beta = 0$,

$$(e^{-\phi} L \partial^\alpha f, e^{q\phi} w^2(-l + |\alpha|, q) \partial^\alpha f) \geq \tilde{\delta}_6 \|w(-l + |\alpha|, q) \partial^\alpha f\|_{N_\gamma^s}^2 - C \|\partial^\alpha f\|_{N_\gamma^s}^2,$$

for some appropriate $\delta_6 > 0$, $\tilde{\delta}_6 > 0$, and $C > 0$.

For the term in the second line of (3.13), by integration by parts, we obtain

$$\begin{aligned} & \left(v_i \partial_\beta^{\alpha+e_i} f - \partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f, e^{q\phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \\ & = \gamma(-l + |\alpha| + |\beta|) \left(\frac{v_i}{1 + |v|^2} \partial^{e_i} \phi \partial_\beta^\alpha f, e^{q\phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right), \quad (3.14) \end{aligned}$$

which implies

$$\begin{aligned} & \left| \left(v_i \partial_\beta^{\alpha+e_i} f - \partial^{e_i} \phi \partial_{e_i+\beta}^\alpha f, e^{q\phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right) \right| \\ & \leq C_{\epsilon_\phi} \|w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f\|_{N_\gamma^s}^2. \end{aligned}$$

We note that the weight function $e^{q\phi} w^2(-l + |\alpha| + |\beta|, q)$ is so designed such that there is an exact cancellation in (3.14). And we shall point out that as the discussions in Lemma 3.1, the L^∞ -norm of ϕ is frequently used in the following.

For the term in the third line of (3.13), by applying the Cauchy-Schwartz inequality, we compute

$$\begin{aligned} & \left| \beta_i \left(\partial_{\beta-e_i}^{\alpha+e_i} f, e^{q\phi} w^2(-l+|\alpha|+|\beta|, q) \partial_{\beta}^{\alpha} f \right) \right| \\ & \leq C \sum_{\substack{|\alpha'|+|\beta'| \leq N \\ \beta' < \beta}} \left\| w(-l+|\alpha'|+|\beta'|, q) \partial_{\beta'}^{\alpha'} (\mathbf{I}-\mathbf{P}) f \right\|_{N_{\gamma}^s}^2. \end{aligned}$$

For the term in the fourth line of (3.13), by Cauchy-Schwartz's inequality and Lemma 1.6, we obtain

$$\begin{aligned} & \sum_{1 \leq |\alpha_1| \leq |\alpha|} \left| \left(\partial^{\alpha_1+e_i} \phi \partial_{e_i+\beta}^{\alpha-\alpha_1} f, e^{q\phi} w^2(-l+|\alpha|+|\beta|, q) \partial_{\beta}^{\alpha} f \right) \right| \\ & \leq C \epsilon_{\phi} \left\| w(-l+|\alpha|+|\beta|, q) \partial_{\beta}^{\alpha} f \right\|_{N_{\gamma}^s}^2 \\ & \quad + C \epsilon_{\phi} \left\| w(-l+|\alpha-\alpha_1|+|\beta|, q) \partial_{\beta}^{\alpha-\alpha_1} f \right\|_{N_{\gamma}^s}^2. \end{aligned}$$

For the term in the fifth line of (3.13), we have from Lemma A.4 that

$$\begin{aligned} & \sum_{1 \leq |\alpha_1| \leq |\alpha|} \left| \left(\partial^{\alpha_1} e^{-\phi} \partial_{\beta}^{\alpha-\alpha_1} Lf, e^{q\phi} w^2(-l+|\alpha|+|\beta|, q) \partial_{\beta}^{\alpha} f \right) \right| \\ & \leq C \|e^{q\phi} \partial^{\alpha_1} e^{-\phi}\|_{L^{\infty}} \sum_{\beta \leq \beta} \left\| w(-l+|\alpha-\alpha_1| \right. \\ & \quad \left. + |\bar{\beta}|, q) \partial_{\beta}^{\alpha-\alpha_1} f \right\|_{N_{\gamma}^s} \left\| w(-l+|\alpha|+|\beta|) \partial_{\beta}^{\alpha} f \right\|_{N_{\gamma}^s} \\ & \leq C \epsilon_{\phi} \left\| w(-l+|\alpha|+|\beta|, q) \partial_{\beta}^{\alpha} f \right\|_{N_{\gamma}^s}^2 \\ & \quad + C \epsilon_{\phi} \sum_{\beta \leq \beta} \left\| w(-l+|\alpha-\alpha_1|+|\bar{\beta}|, q) \partial_{\beta}^{\alpha-\alpha_1} f \right\|_{N_{\gamma}^s}^2. \end{aligned}$$

For the remaining nonlinear term, Lemma A.3 yields that

$$\begin{aligned} \sum_{\alpha_1 \leq \alpha} \left| \left(\partial^{\alpha_1} e^{-\frac{\phi}{2}} \right) \partial^{\alpha - \alpha_1} \Gamma(f, f), e^{q\phi} w^2(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right| \\ \leq C \mathcal{E}_{N, l, q}^{1/2}(f) \mathcal{D}_{N, l, q}(f). \end{aligned} \quad (3.15)$$

Collect the estimates (3.13)-(3.15) and sum over $0 \leq |\beta| \leq N$, $|\alpha| + |\beta| \leq N$ to obtain

$$\begin{aligned} \sum_{|\alpha| + |\beta| \leq N} \frac{d}{dt} \left\| e^{\frac{q\phi}{2}} w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right\|^2 + \delta_7 \sum_{|\alpha| + |\beta| \leq N} \|w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f\|_{N_\gamma^s}^2 \\ \leq C \mathcal{E}_{N, l, q}^{1/2}(f) \mathcal{D}_{N, l, q}(f) + C \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{N_\gamma^s}^2, \end{aligned} \quad (3.16)$$

where $\delta_7 > 0$. Then a linear combination of (3.1) and (3.16) yields that (3.12) is true. Thus the proof of Lemma 3.2 is completed. $\square$

With Lemmas 3.1 and 3.2 in hand, now we turn to complete

The proof of Theorem 1.1. We first set

$$\begin{aligned} y(t) = \lambda_6 \sum_{|\alpha| + |\beta| \leq N} \left\| e^{\frac{q\phi}{2}} w(-l + |\alpha| + |\beta|, q) \partial_\beta^\alpha f \right\|^2 + \delta_7 \sum_{|\alpha| + |\beta| \leq N} \|w(-l + |\alpha| + |\beta|, 0) \partial_\beta^\alpha f\|^2 \\ + \lambda_8 \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2 + \lambda_6 G_\alpha(t). \end{aligned}$$

On the one hand, there exists a constant $C > 0$ such that

$$|G_\alpha(t)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha f\|^2.$$

On the other hand, since $\|\phi\|_{L^\infty} \leq C_\phi$ and $l \geq N$, one can see that

$$\begin{aligned} & \sum_{|\alpha|+|\beta|\leq N} \left\| e^{\frac{q\phi}{2}} w(-l+|\alpha|+|\beta|, q) \partial_\beta^\alpha f \right\|^2 \\ & \sim \sum_{|\alpha|+|\beta|\leq N} \|w(-l+|\alpha|+|\beta|, q) \partial_\beta^\alpha f\|^2 \\ & \geq \sum_{|\alpha|+|\beta|\leq N} \|w(-l+|\alpha|+|\beta|, 0) \partial_\beta^\alpha f\|^2 \geq \sum_{|\alpha|\leq N} \|\partial^\alpha f\|^2. \end{aligned}$$

Hence we can choose a constant $c_1 > 1$ such that for any $t \geq 0$

$$\frac{1}{c_1} y(t) \leq \mathcal{E}_{N,l,q}(t) \leq c_1 y(t).$$

Then by the standard continuous argument [9], one can verify that (1.10) is true. Now we turn to prove the exponential decay using the differential inequality (1.10). As in the work of Caflisch [2, 3] and Strain-Guo [12], the key point is to split $\mathcal{E}_{N,l}(f)(t)$ into a time dependent low velocity part

$$E = \{|v| \leq \rho t^{p'}\},$$

and its complementary high velocity part $E^c = \{|v| > \rho t^{p'}\}$, where $p' > 0$ and $\rho > 0$ will be chosen at the end of the proof. Let $\mathcal{E}_{N,l}^{\text{low}}(f)(t)$ be the instant energy restricted to E . Then from (1.9), (1.8), and (1.6), for $t > 0$, we obtain

$$\mathcal{D}_{N,l}(f)(t) \geq C\rho^{\gamma+2} t^{(\gamma+2s)p'} \mathcal{E}_{N,l}^{\text{low}}(f)(t).$$

Plugging this into the differential inequality (1.10), we get

$$\frac{d}{dt} \mathcal{E}_{N,l}(f)(t) + C\rho^{\gamma+2s} t^{(\gamma+2)p'} \mathcal{E}_{N,l}^{\text{low}}(f)(t) \leq 0.$$

Letting $\mathcal{E}_{N,l}^{\text{high}}(f)(t) = \mathcal{E}_{N,l}(f)(t) - \mathcal{E}_{N,l}^{\text{high}}(f)(t)$, we have

$$\frac{d}{dt} \mathcal{E}_{N,l}(f)(t) + C\rho^{\gamma+2s} t^{(\gamma+2s)p'} \mathcal{E}_{N,l}(f)(t) \leq C\rho^{\gamma+2s} t^{(\gamma+2s)p'} \mathcal{E}_{N,l}^{\text{high}}(f)(t).$$

Define $\tilde{\lambda} = \frac{C\rho^{\gamma+2s}}{p}$, where for now $p = (\gamma + 2s)p' + 1$ and $p' > 0$ is arbitrary. Then

$$\frac{d}{dt} \mathcal{E}_{N,l}(f)(t) + \tilde{\lambda} p t^{p-1} \mathcal{E}_{N,l}(f)(t) \leq \tilde{\lambda} p t^{p-1} \mathcal{E}_{N,l}^{\text{high}}(f)(t),$$

which implies

$$\mathcal{E}_{N,l}(f)(t) \leq e^{-\tilde{\lambda} t^p} \mathcal{E}_{N,l}(f)(0) + \tilde{\lambda} p e^{-\tilde{\lambda} t^p} \int_0^t s^{p-1} e^{\tilde{\lambda} s^p} \mathcal{E}_{N,l}^{\text{high}}(f)(s) ds. \quad (3.17)$$

Above $p > 0$ is assumed to guarantee the integral on the right-hand side is finite. Since $\mathcal{E}_{N,l}^{\text{high}}(f)(s)$ is on $E^c = \{|v| > \rho s^{p'}\}$, one can see that

$$\mathcal{E}_{N,l}^{\text{high}}(f)(s) = \mathcal{E}_{N,l,0}^{\text{high}}(f)(s) \leq e^{-\frac{q}{2}\rho^2 s^{p'}} \mathcal{E}_{N,l,q}^{\text{high}}(f)(s).$$

Substituting this into (3.17), we obtain

$$\mathcal{E}_{N,l}(f)(t) \leq e^{-\tilde{\lambda} t^p} \left(\mathcal{E}_{N,l}(f)(0) + \tilde{\lambda} p \mathcal{E}_{N,l,q}(f)(0) \int_0^t s^{p-1} e^{\tilde{\lambda} s^p - \frac{q}{2}\rho^2 s^{p'}} ds \right).$$

Then $p = p'$ can be allowed, on the other hand, $p = (\gamma + 2s)p' + 1$, so that

$$p = \frac{1}{1 - \gamma - 2s}.$$

Furthermore, choose $\rho > 0$ large enough so that $\tilde{\lambda} = \frac{C\rho^{\gamma+2s}}{p} < \frac{q}{2}\rho^2$ ($-3 < \gamma < -2s$) and hence

$$\int_0^t s^{p-1} e^{\tilde{\lambda} s^p - \frac{q}{2}\rho^2 s^{p'}} ds < +\infty.$$

Therefore, Theorem 1.1 follows. $\square$

Acknowledgements

Xuan Ma has received grant from the National Natural Science Foundation of China under contract 11601092.

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Appendix A

In this appendix, we collect some preliminary lemmas which will be used in the previous sections. The first one is on the linearized collision operators L .

Lemma A.1 ([12]). *Let $\beta > 0$, $\ell \in \mathbf{R}$, and any $\eta > 0$, there exists positive constant $C_\eta > 0$ such that*

$$(w^{2\ell} \partial_\beta^\alpha Lg, \partial_\beta^\alpha g) \geq \|w^\ell \partial_\beta^\alpha g\|_{N_\gamma^s}^2 - \eta \sum_{\beta_1 \leq \beta} \|w^\ell \partial_{\beta_1}^\alpha g\|_{N_\gamma^s}^2 - C_\eta \|\partial^\alpha g\|_{L^2(B_C)}^2. \quad (\text{A.1})$$

Remark A.1. If $|\beta| = 0$, (A.1) can be improved as

$$(w^{2\ell} \partial^\alpha Lg, \partial^\alpha g) \geq \|w^\ell \partial^\alpha g\|_{N_\gamma^s}^2 - C \|\partial^\alpha g\|_{L^2(B_C)}^2.$$

Moreover, one has

$$(\partial^\alpha Lg, \partial^\alpha g) = (\partial^\alpha L(\mathbf{I} - \mathbf{P})g, \partial^\alpha (\mathbf{I} - \mathbf{P})g) \geq \|\partial^\alpha (\mathbf{I} - \mathbf{P})g\|_{N_\gamma^s}^2.$$

The following result is devoted to the estimates on the x , v -derivatives of the linearized collision operators L and the nonlinear term $\Gamma(g, h)$.

Lemma A.1.1 ([6, 12]). *Let $\beta > 0$, $-3 < \gamma < 0$, $\theta \in \mathbf{R}$, for small $\eta > 0$, there exists a positive constant $C_\eta > 0$ such that*

$$|(\partial_\beta^\alpha Kg_1, w^2(\theta, q) \partial_\beta^\alpha g_2)| \leq \left\{ \eta \sum_{\beta_1 \leq \beta} \|w(\theta, q) \partial_{\beta_1}^\alpha g_1\|_{N_\gamma^s} + C_\eta \|\partial^\alpha g_1\|_{L^2(B_C)} \right\} \|w(\theta, q) \partial_\beta^\alpha g_2\|_{N_\gamma^s}.$$

Further if $\theta = |\beta| + \tau$, $\tau \in \mathbf{R}$, then

$$\begin{aligned} (\partial_{\beta}^{\alpha} Ag, w^2(\theta, q) \partial_{\beta}^{\alpha} g) &\geq \|w(\theta, q) \partial_{\beta}^{\alpha} g\|_{N_{\gamma}^s}^2 - \eta \sum_{|\bar{\beta}|=|\beta|} \|w(\theta, q) \partial_{\bar{\beta}}^{\alpha} g\|_{N_{\gamma}^s}^2 \\ &\quad - C_{\eta} \sum_{\beta_1 < \beta} \|w(|\beta_1| + \tau, q) \partial_{\beta_1}^{\alpha} g\|_{N_{\gamma}^s}^2, \end{aligned}$$

and

$$\begin{aligned} (\partial_{\beta}^{\alpha} Lg, w^2(\theta, q) \partial_{\beta}^{\alpha} g) &\geq \|w(\theta, q) \partial_{\beta}^{\alpha} g\|_{N_{\gamma}^s}^2 - \eta \sum_{|\bar{\beta}|=|\beta|} \|w(\theta, q) \partial_{\bar{\beta}}^{\alpha} g\|_{N_{\gamma}^s}^2 \\ &\quad - C_{\eta} \sum_{\beta_1 \leq \beta} \|w(|\beta_1| + \tau, q) \partial_{\beta_1}^{\alpha} g\|_{N_{\gamma}^s}^2. \end{aligned}$$

Lemma A.2 ([6]). *If $|\beta| = 0$, $-3 \leq \gamma < -2$, for $\eta > 0$ small enough, there exists $C(\eta) > 0$ such that*

$$(\partial^{\alpha} Lg, w^2(\theta, q) \partial^{\alpha} g) \geq \delta_q^2 \|w(\theta, q) \partial^{\alpha} g\|_{N_{\gamma}^s}^2 - C_{\eta} \|\partial^{\alpha} g|_{L^2(B_C)}\|^2,$$

where $\delta_q = 1 - q^2 - \eta > 0$.

Lemma A.3 ([6]). *If $\gamma > -3$, let $\theta \in \mathbf{R}$ and $0 \leq q < 1$, if $|\alpha| + |\beta| \leq N$, $\alpha_1 \leq \bar{\alpha} \leq \alpha$, there exists $C > 0$, such that*

$$\begin{aligned} &\langle \partial_{\beta}^{\bar{\alpha}} \Gamma(f, g), w^2(\theta, q) \partial_{\beta}^{\alpha} h \rangle \\ &\leq C \sum_{\bar{\beta}_1 \leq \beta_1 \leq \beta} \left(\left| w(\theta, 0) \partial_{\bar{\beta}_1}^{\alpha_1} f \right|_2 \left| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha} - \alpha_1} g \right|_{N_{\gamma}^s} + \left| w(\theta, 0) \partial_{\bar{\beta}_1}^{\alpha_1} f \right|_{N_{\gamma}^s} \left| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha} - \alpha_1} g \right|_2 \right) \\ &\quad |w(\theta, q) \partial_{\beta}^{\alpha} h|_{N_{\gamma}^s}, \end{aligned} \quad (\text{A.2})$$

where the summation is over $|\alpha_1| + |\beta_1| \leq N$, $\alpha_1 \leq \alpha$, $\bar{\beta}_1 \leq \beta_1 \leq \beta$.

Furthermore, if $|\alpha_1| + |\bar{\beta}_1| \leq \frac{N}{2}$, we have

$$\begin{aligned} & (\partial_{\bar{\beta}}^{\bar{\alpha}} \Gamma(f, g), w^2(\theta, q) \partial_{\beta}^{\alpha} h) \\ & \leq C \sum \left\{ \left\| w(\theta, 0) \partial_{\bar{\beta}_1}^{\bar{\alpha}_1} f \right\| \left\| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha} - \alpha_1} g \right\|_{N_{\gamma}^s} + \left\| w(\theta, 0) \partial_{\bar{\beta}_1}^{\bar{\alpha}} f \right\|_{N_{\gamma}^s} \left\| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha} - \alpha_1} g \right\| \right\} \\ & \quad \left\| w(\theta, q) \partial_{\beta}^{\alpha} h \right\|_{N_{\gamma}^s}, \end{aligned}$$

where the summation is over $|\bar{\alpha}_1| + |\bar{\beta}_1| \leq N$.

If $|\alpha_1| + |\bar{\beta}_1| \geq \frac{N}{2}$, we obtain

$$\begin{aligned} & (\partial_{\bar{\beta}}^{\bar{\alpha}} \Gamma(f, g), w^2(\theta, q) \partial_{\beta}^{\alpha} h) \\ & \leq C \sum \left\{ \left\| w(\theta, 0) \partial_{\bar{\beta}_1}^{\alpha_1} f \right\| \left\| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha}_2} g \right\|_{N_{\gamma}^s} + \left\| w(\theta, 0) \partial_{\bar{\beta}_1}^{\alpha_1} f \right\|_{N_{\gamma}^s} \left\| w(\theta, q) \partial_{\beta - \beta_1}^{\bar{\alpha}_2} g \right\| \right\} \\ & \quad \left\| w(\theta, q) \partial_{\beta}^{\alpha} h \right\|_{N_{\gamma}^s}, \end{aligned}$$

and here the summation is over $|\bar{\alpha}_2| + |\beta - \beta_1| \leq N$.

Recalling the definition (1.3) for L , and utilizing (A.2) we can get the following result.

Lemma A.4 ([12]). Assume $\theta \in \mathbf{R}$, then there exists $C > 0$ such that

$$\begin{aligned} & (\partial_{\beta} \nu g_1, w(\theta, q) \partial_{\beta} g_2) \leq C \left\| w(\theta, q) \partial_{\beta} g_2 \right\|_{N_{\gamma}^s}^2 \\ & \quad + C \sum_{\beta_1 \leq \beta} \left\{ \left\| w(\theta, q) \partial_{\beta_1} g_1 \right\|_2^2 + \left\| w(\theta, q) \partial_{\beta_1} g_1 \right\|_{N_{\gamma}^s}^2 \right\}. \end{aligned}$$

In addition, one can see that

$$\begin{aligned} & |\langle w^2(\theta, q) \partial_{\beta} L g_1, \partial_{\beta} g_2 \rangle| \\ & \leq C \sum_{|\beta_1| \leq |\beta|} \left\{ |w(\theta, q) \partial_{\beta_1} g_1|_2 + |w(\theta, q) \partial_{\beta_1} g_1|_{N_{\gamma}^s} \right\} |w(\theta, q) \partial_{\beta} g_2|_{N_{\gamma}^s}. \end{aligned}$$