Research Article

VBOC2($\alpha, 1-\alpha$) ACF Pure Signal Optimization

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This paper examines common optimization algorithms, such as sum and mean square sense applied to the optimization of the autocorrelation functions (ACF) of GPS M-code like signals such as the variable binary offset carrier (VBOC). Before the different versions are examined for efficiency, on GPS M-code like signals such as VBOC2 which vary with the parameter of signal design and optimization α , they are checked to make sure that their corresponding ACFs are valid via a set of conditions known as *continuity theorems*. Afterwards, by employing any of the optimization algorithms such as sum and mean square sense on GPS M-code like signals such as VBOC2 we are able to produce ACFs that are one *ninety four percent more efficient* than the corresponding ACFs of the GPS M-code signals.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

1 Introduction

The main objective of this paper is to introduce $VBOC2(\alpha, 1-\alpha)$ ACF pure signal optimization as a continuation of the discussion on $VBOC2(\alpha, 1-\alpha)$ generalized multidimensional geolocation modulation waveforms or $VBOC2(\alpha, 1-\alpha)$ pure signal design [1]-[7] methodology gaps created over the years as a results of incomplete signal design methodologies.

For the first time I introduced the idea of signal optimization in Chap. 7 of [5] and then for the second time in [2] it was not

until I started to design $VBOC2(\alpha, 1-\alpha)$ that it became apparent that $VBOC2(\alpha, 1-\alpha)$ pure signal design [3] and optimization offered unique challenges.

What are the unique and original challenges of $VBOC2(\alpha, 1-\alpha)$ pure signal design [3] and optimization that I believe so far have never been described in any other publication [7]-[27]?

If we take a look at $R_{VBOC2(4n,n,\alpha)}(\tau)$ in Appendix B of Chap. 7 of [5] we notice that $R_{VBOC2(4n,n,\alpha)}(\tau)$ contains only fifteen terms or fourteen nonzero terms and one zero term.

Therefore, initially this gave me the impression that the generalized ACF of $VBOC2(\alpha, 1 - \alpha)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ should also contain fifteen terms or fourteen nonzero terms and one zero term. Well coming up with the generalized ACF of $VBOC2(\alpha, 1 - \alpha)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ which contains fifteen terms or fourteen nonzero terms and one zero term was easy; however, it was *IMPOSSIBLE* to satisfy the continuity theorems discussed in this paper.

Therefore, I started to rethink the entire process. If you take a look at the generalized ACF of $VBOC2(\alpha, 1 - \alpha)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ in (38) or (39) of [3] for $p = 2$ we notice that it contains eight non-zero terms; therefore, the ACF of $VBOC2(\alpha, 1 - \alpha)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ in (38) or (39) of [3] for $p = 4$ needed to contain sixteen terms as shown in (38) or (39) of [3]. Hence, $VBOC2(\alpha, 1 - \alpha)$ pure signal design [3] and optimization offered unique and original signal design and optimization challenges not encountered anywhere else.

Finally, as a result of this work it is shown that $VBOC2(pn,n,\alpha)$ for $\alpha = 0.4$ offers 94 percent efficiency or for $\alpha = 0.4$ $VBOC2(pn,n,\alpha)$ is 94 percent more efficient than either $BOC(pn,n)/BOC(pn/2,n)(t)$ ($BPSK_0$ being a special case of $BOC(pn/2,n)(t)$) regardless of the optimization criteria in contrast, with the 100 percent efficiency for $VBOC1(pn,n,\alpha)$ we obtained in [2]. In the case of $VBOC1(\alpha)$ we were able to achieve a global optimum for $\alpha = 0.5$; hence, 100 percent efficiency; however, in the case of $VBOC2(\alpha, 1 - \alpha)$ we are very close to achieving a global optimum $\alpha = \sim 0.4$ **; hence, 94 percent efficiency.

This paper is organized as follows: $VBOC2(\alpha, 1 - \alpha)$ pure signal design is discussed next followed by numerical results. Conclusion is given afterwards and followed by a list of references.

2 $VBOC2(\alpha, 1 - \alpha)$ ACF Pure Signal Optimization

The detailed discussion on $VBOC2(\alpha, 1 - \alpha)$ ACF pure signal optimization includes: (1) $VBOC2(\alpha, 1 - \alpha)$ ACF continuity theorems; and (2) $VBOC2(\alpha, 1 - \alpha)$ ACF optimization theorems.

2.1 $VBOC2(\alpha, 1 - \alpha)$ ACF continuity theorems.

Theorem 1: Prove that the ACF $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the continuity theorem of the first

kind; i.e., it is a continuous function for all values of τ , $p = \{2, 4\}$, and $\alpha = [0, 1]$.

Proof of theorem 1: The proof of theorem 1 is straightforward. In order to prove that the ACF, $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the continuity theorem of the first kind; i.e., it is a continuous function for every values of τ and $p = \{2, 4\}$ it suffices to prove that $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (1) and (2) and $p = \{2, 4\}$ obeys the continuity theorem of the first kind; i.e., it is a continuous function for every values of τ .

We prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the continuity theorem of the first kind; i.e., it is a continuous function for every values of τ . Because the ACF $R_{VBOC2(pn,n,\alpha)}(\tau)$ is an even function of τ it is sufficient to check the continuity of $R_{VBOC2(pn,n,\alpha)}(\tau)$ for values of τ given by for $0 \leq \alpha \leq 1/3$

$$\tau_{\mp i}(\alpha) = \begin{array}{c} \begin{array}{cccc} \frac{T_c(1-\alpha)}{2p} \mp & \frac{T_c(1+\alpha)}{p} \mp & \frac{T_c(5-\alpha)}{2p} \mp & \frac{T_c(3+\alpha)}{p} \mp \\ \tau_{\mp 1}(\alpha) & \tau_{\mp 5}(\alpha) & \tau_{\mp 9}(\alpha) & \tau_{\mp 13}(\alpha) \\ \frac{T_c(1+\alpha)}{2p} \mp & \frac{T_c(3-\alpha)}{2p} \mp & \frac{T_c(5+\alpha)}{2p} \mp & \frac{T_c(7-\alpha)}{2p} \mp \\ \tau_{\mp 2}(\alpha) & \tau_{\mp 6}(\alpha) & \tau_{\mp 10}(\alpha) & \tau_{\mp 14}(\alpha) \\ \frac{T_c(1-\alpha)}{2p} \mp & \frac{T_c(3+\alpha)}{2p} \mp & \frac{T_c(5-\alpha)}{2p} \mp & \frac{T_c(7+\alpha)}{2p} \mp \\ \tau_{\mp 3}(\alpha) & \tau_{\mp 7}(\alpha) & \tau_{\mp 11}(\alpha) & \tau_{\mp 15}(\alpha) \\ \frac{T_c}{p} \mp & \frac{2T_c}{p} \mp & \frac{3T_c}{p} \mp & \frac{4T_c}{p} \mp \\ \tau_{\mp 4}(\alpha) & \tau_{\mp 8}(\alpha) & \tau_{\mp 12}(\alpha) & \tau_{\mp 16}(\alpha) \end{array} \\ p=2 \end{array} \quad (1)$$

and $1/3 \leq \alpha \leq 1$

$$\tau_{\mp i}(\alpha) = \begin{array}{c} \begin{array}{cccc} \frac{T_c(1-\alpha)}{2p} \mp & \frac{T_c(3-\alpha)}{2p} \mp & \frac{T_c(5-\alpha)}{2p} \mp & \frac{T_c(7-\alpha)}{2p} \mp \\ \tau_{\mp 1}(\alpha) & \tau_{\mp 5}(\alpha) & \tau_{\mp 9}(\alpha) & \tau_{\mp 13}(\alpha) \\ \frac{T_c(1-\alpha)}{2p} \mp & \frac{T_c(1+\alpha)}{2p} \mp & \frac{T_c(3-\alpha)}{2p} \mp & \frac{T_c(3+\alpha)}{2p} \mp \\ \tau_{\mp 2}(\alpha) & \tau_{\mp 6}(\alpha) & \tau_{\mp 10}(\alpha) & \tau_{\mp 14}(\alpha) \\ \frac{T_c(1+\alpha)}{2p} \mp & \frac{T_c(3+\alpha)}{2p} \mp & \frac{T_c(5+\alpha)}{2p} \mp & \frac{T_c(7+\alpha)}{2p} \mp \\ \tau_{\mp 3}(\alpha) & \tau_{\mp 7}(\alpha) & \tau_{\mp 11}(\alpha) & \tau_{\mp 15}(\alpha) \\ \frac{T_c}{p} \mp & \frac{2T_c}{p} \mp & \frac{3T_c}{p} \mp & \frac{4T_c}{p} \mp \\ \tau_{\mp 4}(\alpha) & \tau_{\mp 8}(\alpha) & \tau_{\mp 12}(\alpha) & \tau_{\mp 16}(\alpha) \end{array} \\ p=2 \end{array} \quad (2)$$

First, we substitute values of $\tau = \tau_{\mp 1}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] and we get

$$R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{\mp 1}(\alpha)] = 1 - \frac{(4p-1)(1-\alpha)}{2p} = \frac{1-2p+(4p-1)\alpha}{2p} \quad (3)$$

On the other hand

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+1}(\alpha)] = 2\alpha - 1 + \frac{1-\alpha}{2p} = \frac{1-2p+(4p-1)\alpha}{2p}}_{\left\{\frac{-3+7\alpha}{4}, \frac{-7+15\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC2(np,n,\alpha)}[\tau_{-1}(\alpha)] \quad (4)$$

Second, we substitute values of $\tau = \tau_{\mp 2}(\alpha)$ in $R_{VBOC2(pn=n,n,\alpha)}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-2}(\alpha)] = 2\alpha - 1 + \frac{1+\alpha}{2p} = \frac{1-2p+(4p+1)\alpha}{2p}}_{\left\{\frac{-3+9\alpha}{4}, \frac{-7+17\alpha}{8}\right\}_{p=2, p=4}} \quad (5)$$

On the other hand

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+2}(\alpha)] = \frac{-6p+4+4\alpha+(4p-3)(1+\alpha)}{2p} = \frac{1-2p+(4p+1)\alpha}{2p}}_{\left\{\frac{-3+9\alpha}{4}, \frac{-7+17\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC1(np,n,\alpha)}[\tau_{-2}(\alpha)] \quad (6)$$

Third, we substitute values of $\tau = \tau_{\mp 2}(\alpha)$ in $R_{VBOC2(pn=n,n,\alpha)}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-2}(\alpha)] = 2\alpha - 1 + \frac{1-\alpha}{p} = \frac{1-p+(2p-1)\alpha}{p}}_{\left\{\frac{-1+3\alpha}{2}, \frac{-3+7\alpha}{4}\right\}_{p=2, p=4}} \quad (7)$$

On the other hand

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+2}(\alpha)] = 1 - \frac{(2p-1)(1-\alpha)}{p} = \frac{1-p+(2p-1)\alpha}{p}}_{\left\{\frac{-1+3\alpha}{2}, \frac{-3+7\alpha}{4}\right\}_{p=2, p=4}} = R_{VBOC1(np,n,\alpha)}[\tau_{-2}(\alpha)] \quad (8)$$

Fourth, we substitute values of $\tau = \tau_{\mp 3}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-3}(\alpha)] = \frac{-3p+2+2\alpha+(4p-3)(1-\alpha)}{p} = \frac{1-3\alpha+3-11\alpha}{p}}_{\left\{\frac{1-3\alpha}{2}, \frac{3-11\alpha}{4}\right\}_{p=2, p=4}} \quad (9)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+3}(\alpha)] = \frac{2-p-2(p-1)\alpha-(2p-3)(1-\alpha)}{p} = \frac{1-3\alpha+3-11\alpha}{p}}_{\left\{\frac{1-3\alpha}{2}, \frac{3-11\alpha}{4}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-3}(\alpha)] \quad (10)$$

Fifth, we substitute values of $\tau = \tau_{\mp 3}(\alpha)$ in $R_{VBOC2(pn=n,n,\alpha)}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-3}(\alpha)] = 1 - \frac{(2p-1)(1+\alpha)}{2p} = \frac{1-(2p-1)\alpha}{2p}}_{\left\{\frac{1-3\alpha}{4}, \frac{1-7\alpha}{8}\right\}_{p=2, p=4}} \quad (11)$$

On the other hand

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+3}(\alpha)] = \frac{4-2p-4(p-1)\alpha+(2p-3)(1+\alpha)}{2p} = \frac{1-(2p-1)\alpha}{2p}}_{\left\{\frac{1-3\alpha}{4}, \frac{1-7\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-3}(\alpha)] \quad (12)$$

Sixth, we substitute values of $\tau = \tau_{\mp 4}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by either (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-4}(\alpha)] = \frac{2-p-2(p-1)\alpha+2p-3}{p} = \frac{1-2\alpha}{p}}_{\left\{\frac{1-2\alpha}{2}, \frac{3-6\alpha}{4}\right\}_{p=2, p=4}} \quad (13)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+4}(\alpha)] = \frac{(3p-2)-(2p-2)\alpha-2p+1}{p} = \frac{1-2\alpha}{p}}_{\left\{\frac{1-2\alpha}{2}, \frac{3-6\alpha}{4}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-4}(\alpha)] \quad (14)$$

Seventh, we substitute values of $\tau = \tau_{\mp 5}(\alpha)$ in $R_{VBOC2(pn=2n,n,\alpha)}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-5}(\alpha)] = \frac{3p-2-2(p-1)\alpha-(2p-1)(1+\alpha)}{p} = \frac{1-5\alpha}{p}}_{\left\{\frac{1-5\alpha}{2}, \frac{3-13\alpha}{4}\right\}_{p=2, p=4}} \quad (15)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+5}(\alpha)] = \frac{(5p-6)-2\alpha-(4p-5)(1+\alpha)}{p} = \frac{1-5\alpha}{p}}_{\left\{\frac{1-5\alpha}{2}, \frac{3-13\alpha}{4}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-5}(\alpha)] \quad (16)$$

Eight, we substitute values of $\tau = \tau_{\mp 5}(\alpha)$ in $R_{VBOC2(np=2n,n,\alpha)}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-5}(\alpha)] = \frac{6p-4-4(p-1)\alpha-(2p-1)(3-\alpha)}{2p} = \frac{-1-\alpha}{2p}}_{\left\{\frac{-1-\alpha}{4}, \frac{-1-5\alpha}{8}\right\}_{p=2, p=4}} \quad (17)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+5}(\alpha)] = \frac{-6p+8+(2p-3)(3-\alpha)}{2p} = \frac{-1-\alpha}{2p}}_{\left\{\frac{-1-\alpha}{4}, \frac{-1-5\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-5}(\alpha)] \quad (18)$$

Ninth, we substitute values of $\tau = \tau_{\mp 6}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-6}(\alpha)] = \frac{2(5p-6)-4\alpha-(4p-5)(3-\alpha)}{2p} = \frac{-1-\alpha}{2p}}_{\left\{\frac{-1-\alpha}{4}, \frac{-5+7\alpha}{8}\right\}_{p=2, p=4}} \quad (19)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+6}(\alpha)] = \frac{-2p+4(p-2)\alpha+3-\alpha}{2p} = \frac{-1-\alpha}{2p}}_{\left\{\frac{-1-\alpha}{4}, \frac{-5+7\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-6}(\alpha)] \quad (20)$$

Tenth, we substitute values of $\tau = \tau_{\mp 6}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-6}(\alpha)] = \frac{-3p+4+(2p-3)(1+\alpha)}{p}}_{\left\{\frac{-1+\alpha}{2}, \frac{-3+5\alpha}{4}\right\}_{p=2, p=4}} \quad (21)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+6}(\alpha)] = \frac{-p+2(p-2)\alpha+1+\alpha}{p}}_{\left\{\frac{-1+\alpha}{2}, \frac{-3+5\alpha}{4}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-6}(\alpha)] \quad (22)$$

Eleventh, we substitute values of $\tau = \tau_{\mp 7}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-7}(\alpha)] = \frac{-2p+4(p-2)\alpha+3+\alpha}{2p}}_{\left\{\frac{-1+\alpha}{4}, \frac{-5+9\alpha}{8}\right\}_{p=2, p=4}} \quad (23)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+7}(\alpha)] = \frac{-14p+24+(4p-7)(3+\alpha)}{2p}}_{\left\{\frac{-1+\alpha}{4}, \frac{-5+9\alpha}{8}\right\}_{p=2, p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-7}(\alpha)] \quad (24)$$

Twelfth, we substitute values of $\tau = \tau_{\mp 8}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-8}(\alpha)] = -\frac{7p-12}{p} + \frac{2(4p-7)}{p} = \frac{p-2}{p}}_{\left\{\frac{0}{p=2}, \frac{1}{2}, \frac{2}{p=4}\right\}} \quad (25)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+8}(\alpha)] = \frac{9p-20}{p} - \frac{2(4p-9)}{p} = \frac{p-2}{p}}_{\left\{\frac{0}{p=2}, \frac{1}{2}, \frac{2}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-8}(\alpha)] \quad (26)$$

Thirteenth, we substitute values of $\tau = \tau_{\mp 9}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-9}(\alpha)] = \frac{18p-40-(4p-9)(5-\alpha)}{2p}}_{\left\{\frac{-3+7\alpha}{8}, \frac{8}{p=4}\right\}} \quad (27)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+9}(\alpha)] = \frac{20-7p+(5p-12)\alpha+(5-\alpha)(p-3)}{2p}}_{\left\{\frac{-3+7\alpha}{8}, \frac{8}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-9}(\alpha)] \quad (28)$$

Fourteenth, we substitute values of $\tau = \tau_{\mp 10}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-10}(\alpha)] = \frac{20-7p+(5p-12)\alpha+(p-3)(5+\alpha)}{2p}}_{\left\{\frac{-3+9\alpha}{8}, \frac{8}{p=4}\right\}} \quad (29)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+10}(\alpha)] = \frac{60-22p+2(p-2)\alpha+(4p-11)(5+\alpha)}{2p}}_{\left\{\frac{-3+9\alpha}{8}, \frac{8}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-10}(\alpha)] \quad (30)$$

Fifteenth, we substitute values of $\tau = \tau_{\mp 10}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-10}(\alpha)] = \frac{20-7p+(5p-12)\alpha+2(p-3)(3-\alpha)}{2p}}_{\left\{\frac{-1+3\alpha}{4}, \frac{4}{p=4}\right\}} \quad (31)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+10}(\alpha)] = \frac{11p-28+(4-p)\alpha+2(5-2p)(3-\alpha)}{2p}}_{\left\{\frac{-1+3\alpha}{4}, \frac{4}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-10}(\alpha)] \quad (32)$$

Sixteenth, we substitute values of $\tau = \tau_{\mp 11}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-11}(\alpha)] = \frac{30-11p+(p-2)\alpha+(4p-11)(3-\alpha)}{p}}_{\left\{\frac{1-3\alpha}{4}, \frac{4}{p=4}\right\}} \quad (33)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+11}(\alpha)] = \frac{6-2p+(6-2p)\alpha+(p-3)(3-\alpha)}{p}}_{\left\{\frac{1-3\alpha}{4}, \frac{4}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-11}(\alpha)] \quad (34)$$

Seventeenth, we substitute values of $\tau = \tau_{\mp 11}(\alpha)$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-11}(\alpha)] = \frac{11p-28+(4-p)\alpha+(5-2p)(5+\alpha)}{2p}}_{\left\{\frac{1-3\alpha}{4}, \frac{8}{p=4}\right\}} \quad (35)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+11}(\alpha)] = \frac{-4(p-3)-4(p-3)\alpha+(p-3)(5+\alpha)}{2p}}_{\left\{\frac{1-3\alpha}{4}, \frac{8}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-11}(\alpha)] \quad (36)$$

Eighteenth, we substitute values of $\tau = \tau_{\mp 12}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-12}(\alpha)] = \frac{-2(p-3)-2(p-3)\alpha+3(p-3)}{=p-3+(6-2p)\alpha}}_{\left\{\frac{1-2\alpha}{4}\right\}_{p=4}} \quad (37)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+12}(\alpha)] = \frac{7p-18-2(p-3)\alpha+3(5-2p)}{=p-3+(6-2p)\alpha}}_{\left\{\frac{1-2\alpha}{4}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-12}(\alpha)] \quad (38)$$

Nineteenth, we substitute values of $\tau = \tau_{\mp 13}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-13}(\alpha)] = \frac{7p-18-2(p-3)\alpha+(5-2p)(3+\alpha)}{=p-3+(11-4p)\alpha}}_{\left\{\frac{1-5\alpha}{4}\right\}_{p=4}} \quad (39)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+13}(\alpha)] = \frac{13p-42-2\alpha+(13-4p)(3+\alpha)}{=p-3+(11-4p)\alpha}}_{\left\{\frac{1-5\alpha}{4}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-13}(\alpha)] \quad (40)$$

Twentieth, we substitute values of $\tau = \tau_{\mp 13}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-13}(\alpha)] = \frac{14p-36-4(p-3)\alpha+(5-2p)(7-\alpha)}{=-1+(7-2p)\alpha}}_{\left\{\frac{-1-\alpha}{8}\right\}_{p=4}} \quad (41)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+13}(\alpha)] = \frac{48-14p+(2p-7)(7-\alpha)}{=-1+(7-2p)\alpha}}_{\left\{\frac{-1-\alpha}{8}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-13}(\alpha)] \quad (42)$$

Twenty-first, we substitute values of $\tau = \tau_{\mp 14}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-14}(\alpha)] = \frac{26p-84-4\alpha+(13-4p)(7-\alpha)}{=7-2p+(4p-17)\alpha}}_{\left\{\frac{-1-\alpha}{8}\right\}_{p=4}} \quad (43)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+14}(\alpha)] = \frac{-2p+4(p-4)\alpha+7-\alpha}{=7-2p+(4p-17)\alpha}}_{\left\{\frac{-1-\alpha}{8}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-14}(\alpha)] \quad (44)$$

Twenty-second, we substitute values of $\tau = \tau_{\mp 14}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-14}(\alpha)] = \frac{24-7p+(2p-7)(3+\alpha)}{=3-p+(2p-7)\alpha}}_{\left\{\frac{-1+\alpha}{4}\right\}_{p=4}} \quad (45)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+14}(\alpha)] = \frac{-p+2(p-4)\alpha+3+\alpha}{=3-p+(2p-7)\alpha}}_{\left\{\frac{-1+\alpha}{4}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-14}(\alpha)] \quad (46)$$

Twenty-third, we substitute values of $\tau = \tau_{\mp 15}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-15}(\alpha)] = \frac{-2p+4(p-4)\alpha+7+\alpha}{=7-2p+(4p-15)\alpha}}_{\left\{\frac{-1+\alpha}{8}\right\}_{p=4}} \quad (47)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+15}(\alpha)] = \frac{102-30p+(4p-15)(7+\alpha)}{=7-2p+(4p-15)\alpha}}_{\left\{\frac{-1+\alpha}{8}\right\}_{p=4}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-15}(\alpha)] \quad (48)$$

Twenty-fourth and finally, we substitute values of $\tau = \tau_{\mp 16}$ in $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-16}(\alpha)] = \frac{56-15p}{p} + \frac{4(4p-15)}{p} = \frac{p-4}{p}}_{\left\{\frac{0}{p=4}\right\}} \quad (49)$$

On the other hand we have

$$\underbrace{R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{+16}(\alpha)] = 0}_{\left\{\frac{0}{p=4}\right\}} = R_{VBOC2(pn,n,\alpha)}[\tau_{-16}(\alpha)] \quad (50)$$

Equations (1) through (50) complete the proof of *Theorem 1*.

Theorem 2: Prove that the ACF $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the *continuity theorem of the second kind*; i.e., it is a continuous function for every values of τ , $p = \{2,4\}$, and $\alpha = \{0,1\}$.

Proof of theorem 2: The proof of theorem 2 is straightforward. In order to prove that the ACF, $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the *continuity theorem of the second kind*; i.e., it is a continuous function for all values of τ , $p = \{2,4\}$ and $\alpha = \{0,1\}$ it suffices to prove $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind*; i.e., it is a continuous function for all values of τ .

Since, we already proved in *theorem 1*; i.e., *continuity theorem of the first kind* that $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38)

and (39) [3] or for all values of τ , $p = \{2, 4\}$ and for $\alpha = [0, 1]$ we can employ the results of *theorem 1 to prove theorem 2*.

First, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (1) obeys the *continuity theorem of the second kind* for $\{\tau_1(\alpha), \tau_2(\alpha)\}$.

From (3) and (4), and $p = \{2, 4\}$ we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_1(0)] = \frac{1-2p+(4p-1)\alpha}{2p} = -1 + \frac{1}{2p} \quad (51)$$

$$= \left\{ \frac{-3}{4}, \frac{-7}{8} \right\}$$

One the other hand, from (5) and (6) and $p = \{2, 4\}$ we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_2(0)] = \frac{1-2p+(4p+1)\alpha}{2p} = -1 + \frac{1}{2p} \quad (52)$$

$$= \left\{ \frac{-3}{4}, \frac{-7}{8} \right\}$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_1(0)]$$

Again, from (3) and (4) and $p = \{2, 4\}$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_1(1)] = \frac{1-2p+4p-1}{2p} = 1 = R_{BOC(n,n)}[0] \quad (53)$$

Finally, from (5) and (6) and $p = \{2, 4\}$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_2(1)] = \frac{1-p+2p-1}{p} = 1 = R_{BOC(n,n)}[0] \quad (54)$$

Equations (53) and (54) result from *definition 2*, Appendix B, and Chap. 7 of [5].

Second, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (1) obeys the *continuity theorem of the second kind* for $\{\tau_3(\alpha), \tau_4(\alpha), \tau_5(\alpha)\}$.

From (9) and (10), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_3(0)] = \frac{p-1+(5-4p)\alpha}{p} = 1 - \frac{1}{p} = \left\{ \frac{1}{2}, \frac{3}{4} \right\} \quad (55)$$

One the other hand, from (13) and (14), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_4(0)] = \frac{p-1+2(1-p)\alpha}{p} = 1 - \frac{1}{p} = \left\{ \frac{1}{2}, \frac{3}{4} \right\} \quad (56)$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_3(0)]$$

Again from (11) and (12), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_3(1)] = \frac{1-(2p-1)\alpha}{2p} = -1 + \frac{1}{p} = \left\{ \frac{-1}{2}, \frac{-3}{4} \right\} \quad (57)$$

Again, from (13) and (14), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_4(1)] = \frac{p-1+2(1-p)\alpha}{p} = \frac{1}{p} - 1 = \left\{ \frac{-1}{2}, \frac{-3}{4} \right\} \quad (58)$$

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_3(1)]$$

Finally, from (15) and (16), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_5(0)] = \frac{p-1+(3-4p)\alpha}{p} = 1 - \frac{1}{p} = \left\{ \frac{1}{2}, \frac{3}{4} \right\} \quad (59)$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_3(0)]$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_4(0)]$$

Finally, again, from (17) and (18), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_5(1)] = \frac{-1+(3-2p)\alpha}{2p} = \frac{1}{p} - 1 = \left\{ \frac{-1}{2}, \frac{-3}{4} \right\} \quad (60)$$

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_3(1)]$$

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_4(1)]$$

Third, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_6(\alpha), \tau_7(\alpha)\}$.

From (19), and (20), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_6(0)] = \frac{3-2p+(4p-9)\alpha}{2p} = \frac{3}{2p} - 1 = \left\{ \frac{-1}{4}, \frac{-5}{8} \right\} \quad (61)$$

One the other hand, from (23) and (24), for $p = \{2, 4\}$ we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_7(0)] = \frac{3-2p+(4p-7)\alpha}{2p} = \frac{3}{2p} - 1 = \left\{ \frac{-1}{4}, \frac{-5}{8} \right\} \quad (62)$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_6(0)]$$

From (21), and (22), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_6(1)] = \frac{1-p+(2p-3)\alpha}{p} = 1 - \frac{2}{p} = \left\{ 0, \frac{1}{2} \right\} \quad (63)$$

One the other hand, from (23) and (24), for $p = \{2, 4\}$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_7(1)] = \frac{3-2p+(4p-7)\alpha}{2p} = 1 - \frac{2}{p} = \left\{ 0, \frac{1}{2} \right\} \quad (64)$$

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_6(1)]$$

Fourth, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_9(\alpha), \tau_{10}(\alpha)\}$.

From (27) and (28), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_9(0)] = \frac{5-2p+(4p-9)\alpha}{2p} = \frac{5}{2p} - 1 = \left\{ \frac{-3}{8} \right\} \quad (65)$$

One the other hand, from and (29) and (30), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_{10}(0)] = \frac{5-2p+(6p-15)\alpha}{2p} = \frac{5}{2p} - 1 = \left\{ \frac{-3}{8} \right\} \quad (66)$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_9(0)]$$

Again from (27) and (28) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_9(1)] = \frac{5-2p+(4p-9)\alpha}{2p} = 1 - \frac{2}{p} = \left\{ \frac{1}{4} \right\} \quad (67)$$

One the other hand, from (31) and (32), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{10}(1)] = \frac{2-p+(3p-6)\alpha}{2p} = 1 - \frac{2}{p} = \left\{ \frac{1}{2} \right\} \quad (68)$$

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_9(1)]$$

Fifth, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_{11}(\alpha), \tau_{12}(\alpha), \tau_{13}(\alpha)\}$.

From (33) and (34), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_{11}(0)] = \frac{p-3+(9-3p)\alpha}{p} = 1 - \frac{3}{p} = \left\{ \frac{1}{4} \right\} \quad (69)$$

One the other hand, from and (37) and (38), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_{12}(0)] = \frac{p-3+(6-2p)\alpha}{p} = 1 - \frac{3}{p} = \left\{ \frac{1}{4} \right\} \quad (70)$$

$$R_{VBOC2(pn,n,\alpha=0)}[\tau_{11}(0)]$$

Again from (35) and (36) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{11}(1)] = \frac{p-3+(9-3p)\alpha}{2p} = -1 + \frac{3}{p} = \left\{ \frac{-1}{4} \right\} \quad (71)$$

One the other hand, from (37) and (38), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{12}(1)] = \frac{(p-3)-2(p-3)\alpha}{p} = \frac{3}{p} - 1 = \left\{\frac{-1}{4}\right\} \quad (72)$$

Again from (39) and (40) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{13}(0)] = \frac{p-3-(4p-11)\alpha}{p} = 1 - \frac{3}{p} = \left\{\frac{1}{4}\right\} \quad (73)$$

Again from (41) and (42) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{13}(1)] = \frac{-1-(2p-7)\alpha}{2p} = -1 + \frac{3}{p} = \left\{\frac{-1}{4}\right\} \quad (74)$$

Equations (55) through (71) complete the proof of *theorem 4*.

From (43) and (44), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{14}(0)] = \frac{7-2p+(4p-17)\alpha}{2p} = \frac{7}{2p} - 1 = \left\{\frac{-1}{8}\right\} \quad (75)$$

From (45) and (46), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{14}(1)] = \frac{3-p+(2p-7)\alpha}{p} = 1 - \frac{4}{p} = \{0\} \quad (76)$$

From (47) and (48), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{15}(0)] = \frac{7-2p+(4p-15)\alpha}{2p} = \frac{7}{2p} - 1 = \left\{\frac{-1}{8}\right\} \quad (77)$$

From (47) and (48), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=1)}[\tau_{15}(1)] = \frac{7-2p+(4p-15)\alpha}{2p} = 1 - \frac{4}{p} = \{0\} \quad (78)$$

Again from (49) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-16}(0)] = \frac{p-4}{p} = 1 - \frac{4}{p} = \{0\} \quad (79)$$

Unfortunately, this is also only one data point.

Finally, from (50), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha)}[\tau = \tau_{-16}(1)] = \frac{p-4}{p} = 1 - \frac{4}{p} = \{0\} \quad (80)$$

Unfortunately, this is only one data point.

Equations (51) through (80) complete the proof of *theorem 2* or *continuity theorem of the second kind of the ACF of VBOC2(pn,n,α)*.

Theorem 3: Prove that the ACF $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the continuity theorem of the third kind; i.e., it is a continuous function for all values of τ and $p = \{2,4\}$ and $\alpha = \{1/3\}$.

The *proof of theorem 3* is straightforward. In order to prove that the ACF, $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the *continuity theorem of the third kind*; i.e., it is a continuous function for every values of τ , $p = \{2,4\}$ and $\alpha = \{1/3\}$ it suffices to prove $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind*; i.e., it is a continuous function for every values of τ .

Since, we already proved in *theorem 1*; i.e., *continuity theorem of the first kind* that $R_{VBOC2(pn,n,\alpha)}(\tau)$ given by (38) and (39) [3] for *or every values of τ , $p = \{2,4\}$ and for $\alpha = [0,1]$* we can employ the results of *theorem 1* to prove *theorem 3*.

First, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_2(\alpha = \frac{1}{3}), \tau_3(\alpha = \frac{1}{3})\}$.

From (5) and (6), and $p = \{2,4\}$ we have

$$R_{VBOC2(pn,n,\alpha=0)}\left[\tau_2\left(\frac{1}{3}\right)\right] = \frac{1-2p+(4p+1)\frac{1}{3}}{2p} = -\frac{1}{3} + \frac{2}{3p} = \left\{0, \frac{-1}{3}\right\} \quad (81)$$

On the other hand, from (11) and (12), and $p = \{2,4\}$ we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_3\left(\frac{1}{3}\right)\right] = \frac{1-(2p-1)\frac{1}{3}}{2p} = \frac{2}{3p} - \frac{1}{3} = \left\{0, \frac{-1}{3}\right\} \quad (82)$$

Again, from (10) and (11), and for $p = \{2,4\}$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_3\left(\frac{1}{3}\right)\right] = \frac{p-1+(5-4p)\frac{1}{3}}{p} = \frac{2}{3p} - \frac{1}{3} = \left\{0, \frac{-1}{3}\right\} \quad (83)$$

Finally, from (7) and (8), and $p = \{2,4\}$ we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_2\left(\frac{1}{3}\right)\right] = \frac{1-p+(2p-1)\frac{1}{3}}{p} = \frac{2}{3p} - \frac{1}{3} = \left\{0, \frac{-1}{3}\right\} \quad (84)$$

Second, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_5(\alpha = \frac{1}{3}), \tau_6(\alpha = \frac{1}{3})\}$.

From (15) and (16), and for $p = \{2,4\}$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_5\left(\frac{1}{3}\right)\right] = \frac{p-1+(3-4p)\frac{1}{3}}{p} = \frac{-p}{3p} = \left\{\frac{-1}{3}, \frac{-1}{3}\right\} \quad (85)$$

On the other hand, from (21) and (22), and for $p = \{2,4\}$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_6\left(\frac{1}{3}\right)\right] = \frac{3-2p+(4p-9)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3}, \frac{-1}{3}\right\} \quad (86)$$

Again, from (17) and (18), and for $p = \{2,4\}$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_5\left(\frac{1}{3}\right)\right] = \frac{-1+(3-2p)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3}, \frac{-1}{3}\right\} \quad (87)$$

On the other hand, from (21), and (22), and for $p = \{2,4\}$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})}\left[\tau_6\left(\frac{1}{3}\right)\right] = \frac{3-2p+(4p-9)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3}, \frac{-1}{3}\right\} \quad (88)$$

Third, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_{10}(\alpha = \frac{1}{3}), \tau_{11}(\alpha = \frac{1}{3})\}$.

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{10} \left(\frac{1}{3} \right) \right] = \frac{5-2p+(6p-15)\frac{1}{3}}{2p} = 0 = \{0\} \quad (89)$$

From and (29) and (30), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{11} \left(\frac{1}{3} \right) \right] = \frac{p-3+(9-3p)\frac{1}{3}}{p} = 0 = \{0\} \quad (90)$$

Again, from (33) and (34), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{11} \left(\frac{1}{3} \right) \right] = \frac{p-3+(9-3p)\frac{1}{3}}{p} = 0 = \{0\} \quad (91)$$

On the other hand, from (35) and (36) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{10} \left(\frac{1}{3} \right) \right] = \frac{2-p+(3p-6)\frac{1}{3}}{2p} = 0 = \{0\} \quad (92)$$

Fourth, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_{12}(\alpha = \frac{1}{3})\}$.

From and (37) and (38), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{12} \left(\frac{1}{3} \right) \right] = \frac{p-3+(6-2p)\frac{1}{3}}{p} = \frac{p-3}{3p} = \left\{ \frac{1}{12} \right\} \quad (93)$$

Fifth, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_{13}(\alpha = \frac{1}{3})\}$.

From (39) and (40), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{13} \left(\frac{1}{3} \right) \right] = \frac{(p-3)-(4p-11)\frac{1}{3}}{p} = \frac{2-p}{3p} = \left\{ \frac{-1}{6} \right\} \quad (94)$$

On the other hand, from (41) and (42), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{13} \left(\frac{1}{3} \right) \right] = \frac{-1-(2p-7)\frac{1}{3}}{2p} = \frac{2-p}{3p} = \left\{ \frac{-1}{6} \right\} \quad (95)$$

Sixth, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_{14}(\alpha = \frac{1}{3})\}$.

From (43) and (44) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{14} \left(\frac{1}{3} \right) \right] = \frac{7-2p+(4p-17)\frac{1}{3}}{2p} = \frac{2}{3p} - \frac{1}{3} = \left\{ \frac{-1}{6} \right\} \quad (96)$$

One the other hand, from (45) and (46), for $p = 4$, we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau_{14} \left(\frac{1}{3} \right) \right] = \frac{3-p+(2p-7)\frac{1}{3}}{p} = \frac{-1}{3} + \frac{2}{3p} = \left\{ \frac{-1}{6} \right\} \quad (97)$$

Seventh, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [[3]] obeys the *continuity theorem of the third kind* for $\{\tau_{15}(\alpha = \frac{1}{3})\}$.

From (47) and (48) for $p = 4$ we have

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau = \tau_{15} \left(\frac{1}{3} \right) \right] = \frac{7-2p+(4p-15)\frac{1}{3}}{2p} = \frac{1}{p} - \frac{1}{3} = \left\{ \frac{-1}{12} \right\} \quad (98)$$

Eight and finally, we prove that $R_{VBOC2(pn,n,\alpha)}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\{\tau_{16}(\alpha = \frac{1}{3})\}$.

Finally, from (49) for $p = 4$

$$R_{VBOC2(pn,n,\alpha=\frac{1}{3})} \left[\tau = \tau_{16} \left(\frac{1}{3} \right) \right] = \frac{p-4}{p} = 1 - \frac{4}{p} = \{0\} \quad (99)$$

Equations (81) through (99) complete the proof of *theorem 3* or *continuity theorem of the second kind of the ACF of VBOC2(pn,n,α)*.

Corollary 1: Based on *theorems 2* [1], 2 and 3 we conclude that $R_{VBOC1(pn,n,\alpha)}(0) = 1 = \max$ regardless of p and/or α ; i.e., $R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)] \leq 1$ regardless of p and/or α .

2.2 VBOC1(α) ACF Optimization Theorems

Theorem 4: Prove that the ACF $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, given by (38) and (39) [3], obeys the *optimization theorem of the first kind*; i.e., find the values of α that minimizes the out of phase ACF,

$$\left| \begin{array}{c} R_{VBOC2(pn,n,\alpha)}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC2(pn,n,\alpha)}[\tau^{4p-2}(\alpha)] \end{array} \right| \xrightarrow[\min_{\alpha}]{\sum_{i=1}^{2p-1} R_{VBOC2(pn,n,\alpha)}[\tau^i(\alpha)]} \quad (100)$$

$$\tau^i(\alpha)_{i=\{1,2,\dots,2p\}} = \begin{array}{c} \begin{array}{cccc} T_c(1-\alpha) & T_c(1+\alpha) & T_c(5-\alpha) & T_c(3+\alpha) \\ \hline \frac{2p}{\tau^1(\alpha)} & \frac{p}{\tau^8(\alpha)} & \frac{2p}{\tau^8(\alpha)} & \frac{p}{\tau^{12}(\alpha)} \\ \hline T_c(1+\alpha) & T_c(3-\alpha) & T_c(5+\alpha) & T_c(7-\alpha) \\ \hline \frac{2p}{\tau^2(\alpha)} & \frac{2p}{\tau^6(\alpha)} & \frac{2p}{\tau^9(\alpha)} & \frac{2p}{\tau^{13}(\alpha)} \\ \hline T_c(1-\alpha) & T_c(3+\alpha) & T_c(3-\alpha) & T_c(7+\alpha) \\ \hline \frac{p}{\tau^3(\alpha)} & \frac{2p}{\tau^7(\alpha)} & \frac{p}{\tau^{10}(\alpha)} & \frac{2p}{\tau^{14}(\alpha)} \\ \hline \frac{T_c}{\tau^4(\alpha)} & & \frac{3T_c}{\tau^{11}(\alpha)} & \\ \hline \end{array} \\ p=2 \end{array} \quad (101)$$

$$\tau^i(\alpha)_{i=\{1,2,\dots,2p\}} = \begin{array}{c} \begin{array}{cccc} T_c(1-\alpha) & T_c(3-\alpha) & T_c(5-\alpha) & T_c(7-\alpha) \\ \hline \frac{2p}{\tau^1(\alpha)} & \frac{2p}{\tau^5(\alpha)} & \frac{2p}{\tau^8(\alpha)} & \frac{2p}{\tau^{12}(\alpha)} \\ \hline T_c(1+\alpha) & T_c(3+\alpha) & T_c(5+\alpha) & T_c(7+\alpha) \\ \hline \frac{p}{\tau^2(\alpha)} & \frac{p}{\tau^6(\alpha)} & \frac{p}{\tau^9(\alpha)} & \frac{p}{\tau^{13}(\alpha)} \\ \hline T_c(1-\alpha) & T_c(3+\alpha) & T_c(5-\alpha) & T_c(7+\alpha) \\ \hline \frac{2p}{\tau^3(\alpha)} & \frac{2p}{\tau^7(\alpha)} & \frac{2p}{\tau^{10}(\alpha)} & \frac{2p}{\tau^{14}(\alpha)} \\ \hline \frac{T_c}{\tau^4(\alpha)} & & \frac{3T_c}{\tau^{11}(\alpha)} & \\ \hline \end{array} \\ p=2 \end{array} \quad (102)$$

Proof of theorem 4: The *proof of theorem 4* is also straightforward. Because the min-max of the ACF of $VBOC2(pn,n,\alpha)(t)$, or $R_{VBOC1(pn,n,\alpha)}(\tau)$, that depends α on for

values of $p = \{2, 4\}$ and $\tau^i(\alpha)$ given by (101) and (102);
 $i = \{1, 2, \dots, 2p\}$

hence, it suffices to show that:

For $p = \{2, 4\}$ find the value of α that minimizes $\tau^1(\alpha)$,
 $\dots, \tau^{13}(\alpha)$

$$\min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1(p,n,\alpha)}[\tau^{14}(\alpha)] \end{bmatrix} = \min_{\alpha} \begin{bmatrix} 1-2p+(4p-1)\alpha & 1-2p+(4p-1)\alpha \\ 1-2p+(4p+1)\alpha & 2-2p+(4p-2)\alpha \\ 2p-2+(10-8p)\alpha & 1+(1-2p)\alpha \\ 2p-2+(4-4p)\alpha & 2p-2+(4-4p)\alpha \\ 2p-2+(6-8p)\alpha & -1+(3-2p)\alpha \\ 3-2p+(4p-9)\alpha & 2-2p+(4p-6)\alpha \\ 3-2p+(4p-7)\alpha & 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha & 5-2p+(4p-9)\alpha \\ 5-2p+(6p-15)\alpha & 2-p+(3p-6)\alpha \\ 2p-6+(18-6p)\alpha & p-3+(9-3p)\alpha \\ 2p-6+(12-4p)\alpha & 2p-6+(12-4p)\alpha \\ 2p-6+(22-8p)\alpha & -1+(7-2p)\alpha \\ 7-2p+(4p-17)\alpha & 6-2p+(4p-14)\alpha \\ 7-2p+(4p-15)\alpha & 7-2p+(4p-15)\alpha \end{bmatrix} \quad (103)$$

Optimization theorem of the first kind says that proving (103) is equivalent with proving the following

$$\min_{\alpha} \left\{ \sum_{i=1}^{4p-2=14} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] \right\} \quad (104)$$

$$\min_{\alpha} \left\{ \sum_{i=1}^{4p-2=14} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] \right\} \quad (105)$$

The first solution is $0 \leq \alpha \leq 1/3$

$$2 - p - p\alpha = 0 \text{ or } \alpha = \frac{2-p}{p} = \left\{ \frac{-1}{2} = -0.5 \right\} \quad (106)$$

And the second solution is

$$15 - 8p - (21 - 8p)\alpha = 0 \text{ or } \alpha = \frac{8p-15}{8p-21} = \left\{ \frac{17}{9} > 1 \right\} \quad (107)$$

Both the first solution for $\alpha = -0.5$ and the second solution for $\alpha = \frac{17}{9} > 1$ are unacceptable. In order to fix the optimization problem we consider the following:

$$\min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}[\tau^1(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^2(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^3(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^4(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^5(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^6(\alpha)] \\ \vdots \\ R_{VBOC1(p,n,\alpha)}[\tau^9(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^{10}(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^{11}(\alpha)] \\ -R_{VBOC1(p,n,\alpha)}[\tau^{12}(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^{13}(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^{14}(\alpha)] \end{bmatrix} = \min_{\alpha} \begin{bmatrix} 1-2p+(4p-1)\alpha & 1-2p+(4p-1)\alpha \\ 1-2p+(4p+1)\alpha & 2-2p+(4p-2)\alpha \\ 2-2p+(8p-10)\alpha & -1+(2p-1)\alpha \\ 2-2p+(4p-4)\alpha & 2-2p+(4p-4)\alpha \\ 2-2p+(8p-6)\alpha & 1+(2p-3)\alpha \\ 3-2p+(4p-9)\alpha & 2-2p+(4p-6)\alpha \\ 3-2p+(4p-7)\alpha & 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha & 5-2p+(4p-9)\alpha \\ 5-2p+(6p-15)\alpha & 2-p+(3p-6)\alpha \\ 6-2p+(6p-18)\alpha & 3-p+(3p-9)\alpha \\ 6-2p+(4p-12)\alpha & 6-2p+(4p-12)\alpha \\ 6-2p+(8p-22)\alpha & 1+(2p-7)\alpha \\ 7-2p+(4p-17)\alpha & 6-2p+(4p-14)\alpha \\ 7-2p+(4p-15)\alpha & 7-2p+(4p-15)\alpha \end{bmatrix} \quad (108)$$

Optimization theorem of the first kind says that proving (108) is equivalent with proving the following

$$\min_{\alpha} \left\{ \sum_{i=1, \neq 3, 4, 5, 10, 11, 12}^{4p-2=14} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] - \sum_{i=3, \neq 6, 7, 7, 9}^{12} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] \right\} \quad (109)$$

$$\min_{\alpha} \left\{ \sum_{i=1, \neq 3, 4, 5, 10, 11, 12}^{4p-2=14} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] - \sum_{i=3, \neq 6, 7, 7, 9}^{12} R_{VBOC2(p,n,\alpha)}[\tau^i(\alpha)] \right\} \quad (110)$$

The first solution in $0 \leq \alpha \leq 1/3$ is

$$56 - 28p - (144 - 72p)\alpha = 0 \text{ or } \alpha = \frac{28}{72} \times \frac{2-p}{2-p} = \frac{7}{18} \quad (111)$$

And the second solution in $1/3 \leq \alpha \leq 1$ is

$$40 - 20p - (96 - 48p)\alpha = 0 \text{ or } \alpha = \frac{20p-40}{48p-96} = \frac{20(p-2)}{48(p-2)} = \frac{5}{12} = 0.4166666667 \quad (112)$$

The first solution in $0 \leq \alpha \leq 1/3$ is $\alpha = \frac{28}{72} \times \frac{2-p}{2-p} = \frac{7}{18} =$

0.3888889 and it is independent of p . However, since this is outside of the interval then the optimum solution for $\alpha = 1/3$ and it is independent of p .

The second solution for in $1/3 \leq \alpha \leq 1$ is independent of p and it is equal to $\alpha = \frac{5}{12} = 0.4166666667$; hence, the

optimum α is in fact independent of p and it is given by

$$\alpha_{\text{opt}} = \frac{5}{12} = 0.4166666667 \quad (113)$$

Hence, the optimization theorem of the first kind proves that $\alpha = 0.4166666667$, independent of p , minimizes the out-of-

phase autocorrelation peaks of the ACF of $VBOC2_{(pn,n,\alpha)}(t)$, or $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, for all values of p and this completes the proof of *theorem 4*.

Theorem 5: Prove that the ACF $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3], obeys the optimization theorem of the fourth kind¹ (or in the minimum mean square error (MMSE) sense [28]); i.e., find the values of α that minimizes the out-of-phase ACF,

$$\min_{\alpha} \begin{bmatrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{2p-2}(\alpha)] \end{bmatrix} \rightarrow 0 \quad (114)$$

which is equivalent with

$$\alpha_{min} = \frac{\sum_{i=1}^{2p-2} a_i(p)b_i(p)}{\sum_{i=1}^{2p-2} [a_i(p)]^2} = \frac{a(p)^T b(p)}{a(p)^T a(p)} = \frac{a(p)^T b(p)}{\|a(p)\|^2} \quad (115)$$

$\tau^i(\alpha)$ for $i = \{1, 2, \dots, 2p-2\}$ given by (101) and (102) in *Theorem 4* and $p = \{2, 4\}$ and $a_i(p)$ and $b_i(p)$ are given by

$$\min_{\alpha} \begin{bmatrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{bmatrix} = \frac{\begin{bmatrix} 1-2p \\ 1-2p \\ 2p-2 \\ 2p-2 \\ 2p-2 \\ 3-2p \\ 3-2p \\ p=\{2,4\} \\ 5-2p \\ 5-2p \\ 2p-6 \\ 2p-6 \\ 2p-6 \\ 2p-6 \\ 7-2p \\ 7-2p \\ p=\{4\} \end{bmatrix}}{\frac{2p}{b(p)}} - \frac{\begin{bmatrix} 1-4p \\ -1-4p \\ 8p-10 \\ 4p-4 \\ 8p-6 \\ 9-4p \\ 7-4p \\ p=\{2,4\} \\ 9-4p \\ 15-6p \\ 6p-18 \\ 4p-12 \\ 8p-22 \\ 17-4p \\ 15-4p \\ p=\{4\} \end{bmatrix}}{\frac{2p}{a(p)}} \alpha \quad (116)$$

$$\min_{\alpha} \begin{bmatrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{bmatrix} = \frac{\begin{bmatrix} 1-2p \\ 2-2p \\ 1 \\ 2p-2 \\ -1 \\ 2-2p \\ 3-2p \\ p=\{2,4\} \\ 5-2p \\ 2-p \\ p-3 \\ 2p-6 \\ -1 \\ 6-2p \\ 7-2p \\ p=\{4\} \end{bmatrix}}{\frac{2p}{b(p)}} - \frac{\begin{bmatrix} 1-4p \\ 2-4p \\ 2p-1 \\ 4p-4 \\ 2p-3 \\ 6-4p \\ 7-4p \\ p=\{2,4\} \\ 9-4p \\ 6-3p \\ 3p-9 \\ 4p-12 \\ 2p-7 \\ 14-4p \\ 15-4p \\ p=\{4\} \end{bmatrix}}{\frac{2p}{a(p)}} \alpha \quad (117)$$

Proof of theorem 5 : The proof of *theorem 5* is also straight forward.

For $p = 4$ from the *optimization theorem of the second kind* (or in the MMSE sense); i.e., find the values of α that minimizes the out-of-phase ACF, based on (116),

$$\alpha_{min} = \frac{a(p)^T b(p)}{\|a(p)\|^2} = \frac{744-574p+144p^2}{2056-1588p+408p^2} = \frac{372-287p+72p^2}{1028-794p+204p^2} \quad (118)$$

or

$$\alpha_{min} = \frac{a(p)^T b(p)}{\|a(p)\|^2} = \frac{376}{1116} \cong 0.3369 \quad \text{for } 0 \leq \alpha \leq 1/3 \quad (119)$$

and based on (117) we have

$$\alpha_{min} = \frac{a(p)^T b(p)}{\|a(p)\|^2} = \frac{400-308p+78p^2}{928-694p+174p^2} \quad (120)$$

or

$$\alpha_{min} = \frac{a(p)^T b(p)}{\|a(p)\|^2} = \frac{416}{936} = 0.4444 \quad \text{for } 1/3 \leq \alpha \leq 1 \quad (121)$$

In order to find that the optimum α we consider only the p^2 coefficients from (118) and (120) as given by

$$\alpha_{opt} = \frac{72}{204} = 0.352941176470588; \text{ for } 0 \leq \alpha \leq 1/3 \quad (122)$$

or

$$\alpha_{opt} = \frac{78}{174} = 0.448275862068966; \text{ for } 1/3 \leq \alpha \leq 1 \quad (123)$$

Since, $\alpha_{opt} = 0.352941176470588 > 1/3$ outside the range; hence, $\alpha_{opt} = 0.448275862068966$ is the optimum solution in terms of mean square sense. This completes the proof of *theorem 5*.

Corollary 2: $\alpha = 0.448275862068966$ is the asymptotic value for which the MMSE optimum α s from the *optimization theorem of the second kind* of the ACF of the generalized $VBOC2_{(pn,n,\alpha)}(t)$ approach the asymptotic optimum α as indicated in (123).

Corollary 3: or *performance enhancement or efficiency* both in terms of optimization theorems of the *first kind* and *fourth kind*.

Performance enhancement or efficiency based on the *optimization theorem of the first kind* or $Y_{2(p,\alpha)}$ is defined as the ratio or $Y_{2(p,\alpha)}$ the sum of all out-of-phase autocorrelation peaks that depend on α by their number as follows:

$$Y_{2(p,\alpha)} = \frac{\left| \frac{\sum_{i=1}^{14} R_{VBOC1_{(pn,n,\alpha)}}[\tau^i(\alpha)]}{0 \leq \alpha \leq 1/3} \right|}{14 \times 2p} = \frac{|56-28p-(144-72p)\alpha|}{14 \times 2p} \quad (124)$$

and

$$Y_{2(p,\alpha)} = \frac{\left| \frac{\sum_{i=1}^{14} R_{VBOC1_{(pn,n,\alpha)}}[\tau^i(\alpha)]}{0 \leq \alpha \leq 1/3} \right|}{14 \times 2p} = \frac{|56-28p-(144-72p)\alpha|}{14 \times 2p} \quad (125)$$

or

$$Y_{2(p,0 \leq \alpha \leq 1/3)} = \frac{|56-28p-(144-72p)\alpha|}{14 \times 2p} = \frac{|2-p-(\frac{36}{7}-\frac{18}{7}p)\alpha|}{p} \quad (126)$$

and

$$Y_{2(p,1/3 \leq \alpha \leq 1)} = \frac{|40-20p-(96-48p)\alpha|}{14 \times 2p} = \frac{|\frac{10}{7}-\frac{5}{7}p-(\frac{24}{7}-\frac{12}{7}p)\alpha|}{p} \quad (127)$$

¹ Optimization theorems of the second and third kind are not considered for

$VBOC2_{(pn,n,\alpha)}(t)$ only for $VBOC1_{(pn,n,\alpha)}(t)$ [2].

As we take the limit as $p \rightarrow \infty$ we find efficiency only as function of α as follows:

$$Y_{2(0 \leq \alpha \leq 1/3)} = \lim_{p \rightarrow \infty} Y_{2(p, 0 \leq \alpha \leq 1/3)} = \left| \frac{18}{7} \alpha - 1 \right| \quad (128)$$

and

$$Y_{2(1/3 \leq \alpha \leq 1)} = \lim_{p \rightarrow \infty} Y_{2(p, 1/3 \leq \alpha \leq 1)} = \left| \frac{12}{7} \alpha - \frac{5}{7} \right| \quad (129)$$

or $\frac{12}{7} \alpha - \frac{5}{7} = 0$ or $\alpha = \frac{5}{12} = 0.416666666666667$ gives efficiency equal to zero (or infinite, or 100 percent) improvements of the performance enhancement.

Performance enhancement or efficiency based on the optimization theorem of the second kind is defined as the ratio of the sum of all out-of-phase autocorrelation peaks squared by their number and denoted by $Y_{2MS(p, \alpha)}$ as follows:

$$Y_{2MS(p, 0 \leq \alpha \leq 1/3)} = \frac{\sum_{i=1}^{14} (R_{VBOC1(p, n, \alpha)}[\tau_i(\alpha)])^2}{14} \quad (130)$$

and

$$Y_{2MS(p, 1/3 \leq \alpha \leq 1)} = \frac{\sum_{i=1}^{14} (R_{VBOC1(p, n, \alpha)}[\tau_i(\alpha)])^2}{14} \quad (131)$$

or

$$Y_{2MS(p, 0 \leq \alpha \leq 1/3)} = \frac{[b(p) - a(p)\alpha]^T [b(p) - a(p)\alpha]}{14} \quad (132)$$

and

$$Y_{2MS(p, 1/3 \leq \alpha \leq 1)} = \frac{[b(p) - a(p)\alpha]^T [b(p) - a(p)\alpha]}{16} \quad (133)$$

or

$$Y_{2MS(p, \alpha)} = \frac{b(p)^T b(p) - [b(p)^T a(p) + a(p)^T b(p)]\alpha + a(p)^T a(p)\alpha^2}{14} \quad (134)$$

and

$$Y_{2MS(p, \alpha)} = \frac{b(p)^T b(p) - [b(p)^T a(p) + a(p)^T b(p)]\alpha + a(p)^T a(p)\alpha^2}{14} \quad (135)$$

or

$$Y_{2,1MS(p, 0 \leq \alpha \leq 1/3)} = \frac{(288 - 224p + 56p^2) - (1288 - 1016p + 288p^2)\alpha + (1968 - 1460p + 364p^2)\alpha^2}{56p^2} \quad (136)$$

and

$$Y_{2MS(p, 1/3 \leq \alpha \leq 1)} = \frac{(184 - 146p + 38p^2) - (703 - 558p + 156p^2)\alpha + (928 - 694p + 174p^2)\alpha^2}{56p^2} \quad (137)$$

As we take the limit as $p \rightarrow \infty$ we find efficiency only as function of α as follows:

$$Y_{2MS(0 \leq \alpha \leq 1/3)} = \lim_{p \rightarrow \infty} Y_{2MS(p, \alpha)} = 1 - \frac{36}{7} \alpha + \frac{93}{14} \alpha^2 \quad (138)$$

and

$$Y_{2MS(1/3 \leq \alpha \leq 1)} = \lim_{p \rightarrow \infty} Y_{2MS(p, \alpha)} = \frac{19}{28} - \frac{78}{28} \alpha + \frac{87}{28} \alpha^2 \quad (139)$$

This concluded the discussion on the $VBOC2(\alpha, 1 - \alpha)$ ACF optimization theorems.

(a) α optimum vs. order p of $VBOC2_{(pn, n, \alpha)}(t)$

(b) Efficiency $Y_{1(\alpha)}$ (+) and $Y_{1MS(\alpha)}$ (◇) vs α

FIGURE 1: Illustration of $VBOC2_{(pn, n, \alpha)}(t)$ ACF optimization theorems.

3 Numerical, Theoretical Results

Now that we have produced a closed form expression of the efficiency $Y_{2MS(p, \alpha)}$ and $Y_{2MS(\alpha)}$ as a function of the signal design and optimization parameter α and the generalized parameter p (or subcarrier frequency) let us consider two examples which illustrate that: (1) $Y_{2MS(p, \alpha)}$ is in fact independent of p ; and (2) $Y_{2MS(\alpha)}$ has a minimum for $\alpha = 0.4$.

Figure 1(a) presents the α optimum vs. order p of $VBOC2_{(pn, n, \alpha)}(t)$. There are four curves shown in Figure 1(a) correspond to $VBOC2_{(pn, n, \alpha)}(t)$ optimization theorems 4 and 5 however for different range of α : (1)-(2) first and second for $0 \leq \alpha \leq 1/3$; and (3)-(4) third and fourth $1/3 \leq \alpha \leq 1$. The fifth and sixth curves which corresponds to $VBOC2_{(pn, n, \alpha)}(t)$ shows that as p changes from two to one hundred α_{opt} is either equal to or converges to 0.416666666666667 or 0.448275862068966 to regardless of generalized

optimization parameter p (or of sub-carrier frequency) and regardless of the optimization theorem(s).

The argument can be made that the choice we make for α will equally affect the standardization of the waveforms regardless of sub-carrier frequency; i.e., even though the choice on $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC2_{(2,1,\alpha=0)}(t)$ or the GPS military M-code [11], [18] on both GPS L1 and L2 frequencies is arbitrary it only happened by accident that they have the same efficiency as $BOC_{(1,1)}(t) = VBOC2_{(1,1,\alpha=0)}(t)$ [1], [2]. Next, we consider how efficient are these choices?

Figure 1(b) depicts the efficiency $Y_{2MS(\alpha)}$ corresponding to $VBOC2_{(pn,n,\alpha)}(t)$ respectively vs. signal design and optimization parameter α . As shown in Figure 1(b) efficiency for $Y_{2MS(\alpha)}$ does have a minimum exactly at $\alpha = 13/29$ equal to $Y_{2MS(\alpha=0.4)} = 0.054187192118227$ in corollary 3.

For $\alpha = \{0,1\}$ we have $Y_{2MS(\alpha=\{0,1\})} = 1$ as opposed to for $\alpha = 13/29$ we have $Y_{2MS(\alpha=13/29)} = 0.054187192118227$. Hence, $\alpha = 13/29$ offers 94 percent efficiency or for $\alpha = 13/29$ $VBOC2_{(pn,n,\alpha)}$ is 94 percent more efficient than either $BOC_{(pn,n)}/BOC_{(pn/2,n)}(t)$ ($BPSK_0$ being a special case of $BOC_{(pn/2,n)}(t)$) in terms of mean square sense.

4 Conclusions

This journal paper is the first complete (i.e., in its entirety) discussion on ACF pure signal optimization for $VBOC2(\alpha, 1 - \alpha)$.

ACF pure signal optimization as a function of a signal design and optimization parameter α is very laborious and creative work; however, the results are fantastic: it is shown that $VBOC2_{(pn,n,\alpha)}$ for $\alpha = 13/29$ offers 94 percent efficiency or for $\alpha = 0.44$ $VBOC2_{(pn,n,\alpha)}$ is 94 percent more efficient than either $BOC_{(pn,n)}/BOC_{(pn/2,n)}(t)$ ($BPSK_0$ being a special case of $BOC_{(pn/2,n)}(t)$) in terms of MMSE sense and $p \geq 2$; i.e., regardless of the subcarrier frequency.

Future work: Is it possible that from optimization of the system (transmitter, channel, and receiver) one can obtain a different optimum α than $\alpha = 0.44$ as a result of the optimization of the received out-of-phase ACF? The results of this laborious investigation will be presented in the future.

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