Research Article

VBOC2($\alpha, 1-\alpha$) Generalized Multidimensional Geolocation Modulation Waveforms

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This paper presents the complete original definition of first generation Variable Binary Offset Carrier $VBOC2(\alpha, 1 - \alpha)$ generalized multidimensional geolocation modulation waveforms, to improve the standardization of the United States DoD GPS, European Galileo, Russian GLONASS, Chinese Compass, Indian IRNSS in the L-band (1-2 GHz), and the United Nations International Telecommunications Union (ITU) GNSS or geolocation waveforms in the S-band (2-4 GHz) and C-band (4-8 GHz).

In the paper it is argued that the selection of BOC(10,5) (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC2(\alpha, 1 - \alpha)$ for $\alpha = 0$ or $\alpha = 1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

$VBOC2(\alpha, 1 - \alpha)$ pure signal design or broad definition of generalized autocorrelation function (ACF) and power spectral density (PSD) offers a unique signal design methodology and provides the necessary framework for $VBOC2(\alpha, 1 - \alpha)$ ACF pure signal optimization to fill in substantial signal design gaps; hence, improving the GNSS signal design and standardization.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

1 Introduction

The main objective of this paper is to introduce the first

generation $VBOC2(\alpha, 1 - \alpha)$ generalized multidimensional geolocation modulation waveforms so as to fill in substantial signal design [1]-[27] methodology gaps created over the years

as a results of incomplete signal design methodologies.

The main motivation or the argument of the signal design waveforms has already been established that $VBOC1(\alpha)$ generalized multidimensional geolocation modulation waveforms which is a type of asymmetric non-return-to-zero (or As-NRZ) BOC for $0 \leq \alpha \leq 1$, generalizes the transition from the BPSK waveform $BPSK(t) = VBOC1_{(m,n,\alpha=1)}(t)$ to the current symmetric NRZ (or S-NRZ) BOC modulation, used extensively in GPS L1 and L2 frequencies [5]-[17], because $BOC_{(m,n)}(t) = VBOC1_{(m,n,\alpha=0)}(t)$; however, in the current GNSS standard the choice $\alpha = 0$ in the context of $VBOC1_{(m,n,\alpha)}(t)$ was made entirely arbitrary [1]-[7].

The first argument was made that $VBOC1_{(1,1,\alpha=0.5)}(t)$ is probably better choice on the GPS L1C data signal than $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ because the current choice on GPS L1C data signal instead of $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ is arbitrary and just as inefficient as the BPSK modulation [1], [2] in terms of out of phase ACF.

The second argument was made that $VBOC1_{(2,1,\alpha=0.5)}(t)$ is also a better choice than $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code on both GPS L1 and L2 frequencies [1], [2], because either $VBOC1_{(1,1,\alpha=0.5)}(t)$ or $VBOC1_{(2,1,\alpha=0.5)}(t)$ is 100 percent more efficient than either $BOC_{(1,1)}(t)$ or $BOC_{(2,1)}(t)$ respectively in terms of out of phase ACF [2].

The question now is as follows: can we find a better choice for: (1) $BOC_{(1,1)}(t)$ than $VBOC1_{(1,1,\alpha=0.5)}(t)$? (2) $BOC_{(2,1)}(t) = BOC_{(10,5)}(t)$ than $VBOC1_{(2,1,\alpha=0.5)}(t)$?

For the purposes of this work $VBOC2(\alpha, 1 - \alpha)$ really means $VBOC2_{(pn,n,\alpha)}(t)$ for $0 \leq \alpha \leq 1$ which is a special case of the second generation of generalized multidimensional geolocation modulation waveforms $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$.

We notice that $VBOC1(\alpha) = VBOC2(\alpha_1 = \alpha, \alpha_2 = \alpha)$ for $0 \leq \alpha \leq 1$; i.e., $VBOC1(\alpha)$ that is already being discussed in [1], [2] is a special case of $VBOC2(\alpha_1, \alpha_2)$. Since, we have already discussed $VBOC1(\alpha)$ [1], [2] instead of discussing $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$; why not consider $VBOC2(\alpha, 1 - \alpha)$ and show if in fact $VBOC2(\alpha, 1 - \alpha)$ is better than $VBOC1(\alpha)$ as a first step and if this is the case [3] and then later on consider $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$.

This publication contain substantially original and innovative research, in contrast to my earlier signal design work [7], [21]-[24] and end user work [18]-[20], [25]-[27], in the

context of $VBOC2(\alpha, 1 - \alpha)$ pure signal design methodology which includes: (1) original and innovative formalized definition and discussion of the $VBOC2(\alpha, 1 - \alpha)$ signal waveform; (2) include original and innovative formalized definition and discussion of $VBOC2(\alpha, 1 - \alpha)$ the generalized ACF and its complete set of properties; (3) include original and innovative formalized definition and discussion of the $VBOC2(\alpha, 1 - \alpha)$ generalized PSD; (4) all (1) through (4) are analyzed as a function of the signal design and optimization parameter, α , and generalized parameter, p which denotes the order of the $VBOC2(\alpha, 1 - \alpha)$.

Why is the discussion of all of these so important? First, signal design and optimization parameter, α , exploits a particular asymmetry of the As-NRZ signal coding modulation in order to improve $VBOC2(\alpha, 1 - \alpha)$ signal design and optimization. Second, the derivation of generalized ACFs and PSDs as a function of α and p provides for the first time the opportunity to understand the properties of all individual $VBOC2(\alpha, 1 - \alpha)$ such as for all positive integer values of p without having to analyze all individual $VBOC2(\alpha, 1 - \alpha)$ separately.

This paper is organized as follows: in Sect. 2 $VBOC2(\alpha, 1 - \alpha)$ pure signal design is discussed. Sect. 3 contains numerical results; Conclusion is provided in Sect. 4 along with a list of references.

2 $VBOC2(\alpha, 1 - \alpha)$ Pure Signal Design

The detailed discussion on $VBOC2(\alpha, 1 - \alpha)$ pure signal design starts with the: (1) definition of the $VBOC2_{(m,n,\alpha)}(t)$ waveform; (2) $VBOC2(\alpha, 1 - \alpha)$ ACF definition and discussion; and (3) $VBOC2(\alpha, 1 - \alpha)$ PSD definition and discussion.

2.1 $VBOC2(\alpha, 1 - \alpha)$ Signal Definition and Discussion

Definition 1: $VBOC2_{(m,n,\alpha)}(t)$ waveform is a generalized, periodic function of the $BOC_{(m,n)}(t)$ waveform with period $2T_s$, (T_s : the subcarrier period), for all values of $t -\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = VBOC2_{(m,n,\alpha)}(t \pm 2T_s) \quad (1)$$

and the relation of integers m, n is given by

$$2mT_s = nT_c \quad (2)$$

where T_c is the defined as the chipping period.

Definition 2: $VBOC2_{(m,n,\alpha)}(t)$ waveform is an odd function for all values of $-\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = -VBOC2_{(m,n,\alpha)}(-t) \quad (3)$$

Definition 3: $VBOC2_{(m,n,\alpha)}(t)$ waveform is a periodic function with period T_s for all values of m even and $-\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = VBOC2_{(m,n,\alpha)}(t \pm T_s) \quad (4)$$

Definition 4: The odd/even sub-carrier period of $VBOC2_{(m,n,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ or $+1/w_f$ and $-1/w_r$ or $-1/w_f$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ or $-1/w_r$ and $+1/w_r$ or $+1/w_f$ respectively that satisfy the following

$$w_r = 0.5T_s(1 \pm \alpha) = 0.5 \frac{nT_c(1 \pm \alpha)}{2m} = \frac{T_c(1 \pm \alpha)}{4p} \quad (5)$$

$$w_f = 0.5T_s(1 \mp \alpha) = 0.5 \frac{nT_c(1 \mp \alpha)}{2m} = \frac{T_c(1 \mp \alpha)}{4p} \quad (6)$$

Hence, the waveform $VBOC2_{(m,n,\alpha)}(t)$ can be obtained as

$$VBOC2_{(m,n,\alpha)}(t) = \sum_{q: \text{odd}}^{\infty} [p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)] + \sum_{q: \text{even}}^{\infty} [p_{w_f}(t_1, w_f) - p_{w_r}(t_2, w_r)] \quad (7)$$

where

$$t_1 = t - 0.5w_r - qT_s \quad (8)$$

$$t_2 = t - 0.5w_f - w_r - qT_s \quad (9)$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

It is straightforward to show that $VBOC2_{(m,n,\alpha)}(t)$ obeys *Theorem 1* of [1]; i.e., for $VBOC2_{(m,n,\alpha)}(t)$ $0 \leq \alpha \leq 1$.

Corollary 1: $BOC_{(m,n)}(t)$ is simply $VBOC2_{(m,n,\alpha)}(t)$ for $\alpha = 0$.

Corollary 2: $BOC_{(m/2,n)}(t)$ is simply $VBOC2_{(m,n,\alpha)}(t)$ for $\alpha = 1$.

Even though we have defined the $VBOC2_{(m,n,\alpha)}(t)$ signal waveform we want to describe the properties of the signal with itself for values of $0 \leq \tau \leq T$ in the observation interval. One means of achieving this by correlating the $VBOC2_{(m,n,\alpha)}(t)$ signal waveform with itself; hence, the definition of the ACF, $R_{VBOC2_{(m,n,\alpha)}}(\tau)$.

This concludes $VBOC2(\alpha, 1 - \alpha)$ signal definition and discussion; next we continue with $VBOC2(\alpha, 1 - \alpha)$ generalized ACF definition and discussion.

2.2 $VBOC1(\alpha)$ Generalized ACF Definition and Discussion

Definition 5: The ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m,n,\alpha)}}(\tau, 2T)$ for m odd with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ ; $t_0 - 2T \leq \tau \leq t_0 + 2T$ is given by

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0) = \frac{v(t) * v(t)}{2T} \quad (10)$$

or in integral form

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0) = \frac{\int_{t_0}^{t_0+2T} v(t)v(t \pm \tau) dt}{2T} \quad (11)$$

$$v(t) = VBOC2_{(m:o,n,\alpha)}(t) \quad (12)$$

Definition 6: The ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m,n,\alpha)}}(\tau, T)$ for m even with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ ; $t_0 - T \leq \tau \leq t_0 + T$ is given by

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0) = \frac{v(t) * v(t)}{T} \quad (13)$$

or in integral form

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0) = \frac{\int_{t_0}^{t_0+T} v(t)v(t \pm \tau) dt}{T} \quad (14)$$

$$v(t) = VBOC2_{(m:e,n,\alpha)}(t) \quad (15)$$

There are a couple of things that we should describe about the ACF, $R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0)$ and $R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0)$ with respect to $\tau = 0$ and $\tau = \infty$.

It is straight forward to show that the ACF of $VBOC2_{(m,n,\alpha)}(t)$ ($R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0)$ given by (10) and (11) and $R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0)$ given by (13) and (14)) obey *theorems 2 and 3*, and *corollaries 3 and 4* of [1].

Theorem 1 (“*Fundamental theorem of the ACF of the generalized $VBOC2_{(m,n,\alpha)}(t)$* ”). Show that if $T = T_s$ starting with:

(1) **Definition 5** the ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ the following is obtained:

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+2T_s} v(t)v(t \pm \tau) dt}{2T_s} \quad (16)$$

(2) **Definition 6** the ACF of $VBOC2_{(m:e,n,\alpha)}(t)$ the following is obtained:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T_s, t_0) = \frac{\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt}{T_s} \quad (17)$$

(3) prove that when $VBOC2_{(m:e,n,\alpha)}(t)$ the following holds

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T_s, t_0) \quad (18)$$

¹ For ease of analysis we will assume that the $+$ for w_r and $-$ for w_f .

² o: odd and e: even. This short hand notation will be used throughout the

(4) prove that when $VBOC2_{(m:o,n,\alpha)}(t)$ the following holds

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T_s, t_0) \neq R_{VBOC2_{(m:o,n,\alpha)}}(\tau, T_s, t_0) \quad (19)$$

Proof of theorem 1 : The *proof of theorem 1* is straightforward. First, by substituting $T = T_s$ from *definition 5* (16) is obtained. Second, from *definition 6* (17) is obtained. Third, now, let us assume that $v(t) = VBOC2_{(m:e,n,\alpha)}(t)$ starting with *definition 5* or (16) we have:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+2T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt + \int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau) dt}{2T_s} \quad (20)$$

By changing the variables $t' = t - T_s$ we notice that

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt' \quad (21)$$

where $t = t' + T_s$.

Based on *definition 3* (19) becomes:

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau) dt' \quad (22)$$

or

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt \quad (23)$$

Finally, substituting (23) into (21) the following is obtained:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+2T_s} v(t)v(t \pm \tau) dt = 2 \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt}{2T_s} \quad (24)$$

or

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt}{T_s} \quad (25)$$

which completed the proof of (18).

Fourth, it is also very straightforward to show that (25) holds also. *Definition 5* is known as the necessary condition of the ACF of $VBOC2_{(m,n,\alpha)}(t)$ and *definition 6* as the sufficient condition the ACF of $VBOC2_{(m,n,\alpha)}(t)$.

Next, we will exploit the ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ a bit further; i.e., (21) and (23). Because (21) and (23) are functions of τ

$$\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau) dt \quad (26)$$

$$\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt \quad (27)$$

for values of $-2T_s \leq \tau \leq 2T_s$ we recognize the following special cases: (a) $0 \leq |\tau| \leq T_s$ and (b) $T_s \leq |\tau| \leq 2T_s$.

First, let us consider (a) $0 \leq |\tau| \leq T_s$ for (26) and (b) $T_s \leq |\tau| \leq 2T_s$ for (26); hence,

$$v(t' \pm \tau)|_{T_s \leq |\tau| \leq 2T_s} = v(t \pm \tau)|_{0 \leq |\tau| \leq T_s} \quad (28)$$

which leads to the following:

$$\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t')v(t \pm \tau) dt \quad (29)$$

$$\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t)v(t' \pm \tau) dt \quad (30)$$

Second, let us consider (b) $T_s \leq |\tau| \leq 2T_s$ for (26) and (a) $0 \leq |\tau| \leq T_s$ for (27); hence,

$$v(t \pm \tau)|_{T_s \leq |\tau| \leq 2T_s} = v(t' \pm \tau)|_{0 \leq |\tau| \leq T_s} \quad (31)$$

which leads to the following:

$$\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t')v(t \pm \tau) dt \quad (32)$$

$$\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau) dt = \int_{t_0}^{t_0+T_s} v(t)v(t' \pm \tau) dt \quad (33)$$

Finally, let us compare and contrast (29) with (32) and (30) and (33). Typically, either (29) or (30) is different from (32) or (33) respectively; which leads to the fundamental following *corollary* for the ACF of $VBOC2_{(m,n,\alpha)}(t)$.

Corollary 3: The ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ can either be computed directly as in *definition 5* in which case requires integration along $2T_s$, as given by (16), or indirectly via four coherent integrations along T_s , as given by (29), (30), (32) and (33); i.e., four independent half ACF (or HACF) periods.

Corollary 4: The ACF of $VBOC2_{(m:e,n,\alpha)}(t)$ is four times more efficient than ACF of $VBOC2_{(m:o,n,\alpha)}(t)$.

Corollary 5: $R_{VBOC2_{(m,n,\alpha)}}(\tau, T, t_0)$ is periodic with period $T = nT_c = 2mT_s$ corresponding to the observation interval $t_0 \leq \tau \leq t_0 + T_c$ and in fact $R_{VBOC2_{(m,n,\alpha)}}(\tau)$ is only one period of ACF $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ when

$$R_{VBOC2_{(m:o,2n,\alpha)}}(\tau) = \frac{\int_0^{2nT_c} v(t)v(t \pm \tau) dt}{2nT_c} \quad (34)$$

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau) = \frac{\int_0^{nT_c} v(t)v(t \pm \tau) dt}{nT_c} \quad (35)$$

which is the definition of the ACF whose shape we are most familiar with and it has the largest autocorrelation peak.³

Corollary 6: The ACF of $VBOC2_{(m,in,\alpha)}(t)$ or $R_{VBOC2_{(m,in,\alpha)}}(\tau)$ for $i = \{1,2\}$ is in general an even function with respect to τ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ ; $-\infty < \tau < \infty$; i.e.,

$$R_{VBOC2_{(m,in,\alpha)}}(-\tau) = R_{VBOC2_{(m,in,\alpha)}}(\tau) \text{ for } i = \{1,2\} \quad (36)$$

³ In general there are an infinite number of shapes of ACFs; however, only one

of them has the largest ACF peak.

The closed form expression of the generalized ACF, $R_{VBOC2(m=pn,n,\alpha)}(\tau)$, with respect to p , is given subject to the following theorem.

Theorem 2: Show that, for values of $p = \{2, 4\}$, $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ is the generalized ACF with respect to p ; i.e., (1) $R_{VBOC2(2 \times n,n,\alpha)}(\tau) = R_{VBOC2(pn=2n,n,\alpha)}(\tau)$; $p = 2$; (2) $R_{VBOC2(4 \times n,n,\alpha)}(\tau) = R_{VBOC2(pn=4n,n,\alpha)}(\tau)$; $p = 4$; For $0 \leq \alpha \leq 1/3$

$$\tau_1^\alpha = \left[\frac{T_c}{2p} = T_s \right] (1 - \alpha); \tau_2^\alpha = \frac{T_c(1+\alpha)}{2p}; \tau_3^\alpha = \frac{T_c(1-\alpha)}{p}; \quad (37a)$$

$$\tau_4 = \frac{T_c}{p}; \tau_5^\alpha = \frac{T_c(1+\alpha)}{p}; \tau_6^\alpha = \frac{T_c(3-\alpha)}{2p}; \quad (37b)$$

$$\tau_7^\alpha = \frac{T_c(3+\alpha)}{2p}; \tau_8 = \frac{2T_c}{p}; \tau_9^\alpha = \frac{T_c(5-\alpha)}{2p}; \quad (37c)$$

$$\tau_{10}^\alpha = \frac{T_c(5+\alpha)}{2p}; \tau_{11}^\alpha = \frac{T_c(3-\alpha)}{p}; \tau_{12} = \frac{3T_c}{p}; \quad (37d)$$

$$\tau_{13}^\alpha = \frac{T_c(3+\alpha)}{p}; \tau_{14}^\alpha = \frac{T_c(7-\alpha)}{2p}; \tau_{15}^\alpha = \frac{T_c(7+\alpha)}{2p}; \tau_{16} = \frac{4T_c}{p} \quad (37e)$$

and $1/3 \leq \alpha \leq 1$

$$\tau_1^\alpha = \left[\frac{T_c}{2p} = T_s \right] (1 - \alpha); \tau_2^\alpha = \frac{T_c(1-\alpha)}{p}; \tau_3^\alpha = \frac{T_c(1+\alpha)}{2p}; \quad (37f)$$

$$\tau_4 = \frac{T_c}{p}; \tau_5^\alpha = \frac{T_c(3-\alpha)}{2p}; \tau_6^\alpha = \frac{T_c(1+\alpha)}{p}; \quad (37g)$$

$$\tau_7^\alpha = \frac{T_c(3+\alpha)}{2p}; \tau_8 = \frac{2T_c}{p}; \tau_9^\alpha = \frac{T_c(5-\alpha)}{2p}; \quad (37h)$$

$$\tau_{10}^\alpha = \frac{T_c(3-\alpha)}{p}; \tau_{11}^\alpha = \frac{T_c(5+\alpha)}{2p}; \tau_{12} = \frac{3T_c}{p}; \quad (37i)$$

$$\tau_{13}^\alpha = \frac{T_c(7-\alpha)}{2p}; \tau_{14}^\alpha = \frac{T_c(3+\alpha)}{p}; \tau_{15}^\alpha = \frac{T_c(7+\alpha)}{2p}; \tau_{16} = \frac{4T_c}{p} \quad (37j)$$

if $R_2(\tau) = R_{VBOC2(m=pn,n,\alpha)}(\tau)$ is given by the expression below:

$$R_2(\tau) = \begin{aligned} & \underbrace{\frac{1}{a_{1(p,n,\alpha)}} + \frac{-(-4p-1)}{T_c} |\tau|}_{r_{1(pn,n,\alpha)}; p=\{2,4\}} \leq \tau_1^\alpha \\ & \underbrace{\frac{-1+2\alpha}{a_{2(p,n,\alpha)}} + \frac{1}{T_c}}_{b_{2(p,n,\alpha)}} |\tau|, \tau_1^\alpha \leq |\tau| \leq \tau_2^\alpha \\ & \underbrace{\frac{-3p+2+2\alpha}{p}}_{a_{3(p,n,\alpha)}} + \underbrace{\frac{4p-3}{T_c}}_{b_{3(p,n,\alpha)}} |\tau|, \tau_2^\alpha \leq |\tau| \leq \tau_3^\alpha \\ & \underbrace{\frac{2-p-2(p-1)\alpha}{p}}_{a_{4(p,n,\alpha)}} + \underbrace{\frac{2p-3}{T_c}}_{b_{4(p,n,\alpha)}} |\tau|, \tau_3^\alpha \leq |\tau| \leq \tau_4 \\ & \underbrace{\frac{3p-2-2(p-1)\alpha}{p}}_{a_{5(p,n,\alpha)}} + \underbrace{\frac{-(2p-1)}{T_c}}_{b_{5(p,n,\alpha)}} |\tau|, \tau_4 \leq |\tau| \leq \tau_5^\alpha \\ & \underbrace{\frac{(5p-6)-2\alpha}{p}}_{a_{6(p,n,\alpha)}} + \underbrace{\frac{-(4p-5)}{T_c}}_{b_{6(p,n,\alpha)}} |\tau|, \tau_5^\alpha \leq |\tau| \leq \tau_6^\alpha \\ & \underbrace{-1 + \frac{2(p-2)\alpha}{p}}_{a_{7(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{7(p,n,\alpha)}} |\tau|, \tau_6^\alpha \leq |\tau| \leq \tau_7^\alpha \\ & \underbrace{-\frac{7p-12}{p}}_{a_{8(p,n,\alpha)}} + \underbrace{\frac{4p-7}{T_c}}_{b_{8(p,n,\alpha)}} |\tau|, \tau_7^\alpha \leq |\tau| \leq \tau_8 \\ & \underbrace{\frac{9p-20}{p}}_{a_{9(p,n,\alpha)}} + \underbrace{\frac{-(4p-9)}{T_c}}_{b_{9(p,n,\alpha)}} |\tau|, \tau_8 \leq |\tau| \leq \tau_9^\alpha \quad (38) \\ & \underbrace{\frac{20-7p+(5p-12)\alpha}{2p}}_{a_{10(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{10(p,n,\alpha)}} |\tau|, \tau_9^\alpha \leq |\tau| \leq \tau_{10}^\alpha \\ & \underbrace{\frac{30-11p+(p-2)\alpha}{p}}_{a_{11(p,n,\alpha)}} + \underbrace{\frac{4p-11}{T_c}}_{b_{11(p,n,\alpha)}} |\tau|, \tau_{10}^\alpha \leq |\tau| \leq \tau_{11}^\alpha \\ & \underbrace{\frac{-4(p-3)-4(p-3)\alpha}{2p}}_{a_{12(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{12(p,n,\alpha)}} |\tau|, \tau_{11}^\alpha \leq |\tau| \leq \tau_{12} \\ & \underbrace{\frac{7p-18-2(p-3)\alpha}{p}}_{a_{13(p,n,\alpha)}} + \underbrace{\frac{-(2p-5)}{T_c}}_{b_{13(p,n,\alpha)}} |\tau|, \tau_{12} \leq |\tau| \leq \tau_{13}^\alpha \\ & \underbrace{\frac{13p-42-2\alpha}{p}}_{a_{14(p,n,\alpha)}} + \underbrace{\frac{13-4p}{T_c}}_{b_{14(p,n,\alpha)}} |\tau|, \tau_{13}^\alpha \leq |\tau| \leq \tau_{14}^\alpha \\ & \underbrace{-1 + \frac{2(p-4)\alpha}{p}}_{a_{15(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{15(p,n,\alpha)}} |\tau|, \tau_{14}^\alpha \leq |\tau| \leq \tau_{15}^\alpha \\ & \underbrace{\frac{56-15p}{p}}_{a_{16(p,n,\alpha)}} + \underbrace{\frac{4p-15}{T_c}}_{b_{16(p,n,\alpha)}} |\tau|, \tau_{15}^\alpha \leq |\tau| \leq \tau_{16} \\ & 0, |\tau| \geq T_c \end{aligned}$$

$$\begin{aligned}
R_2(\tau) = & \left\{ \begin{array}{l} \underbrace{\frac{1}{a_{1(p,n,\alpha)}} + \frac{-(4p-1)|\tau|}{b_{1(p,n,\alpha)} T_c}}_{r_{1(p,n,\alpha)}; p=\{2,4\}} \quad |\tau| \leq \tau_1^\alpha \\ \underbrace{\frac{-1+2\alpha}{a_{2(p,n,\alpha)}} + \frac{1}{b_{2(p,n,\alpha)} T_c}}_{r_{2(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_1^\alpha \leq |\tau| \leq \tau_2^\alpha \\ \underbrace{\frac{1}{a_{3(p,n,\alpha)}} + \frac{-(2p-1)|\tau|}{b_{3(p,n,\alpha)} T_c}}_{r_{3(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_2^\alpha \leq |\tau| \leq \tau_3^\alpha \\ \underbrace{\frac{2-p-2(p-1)\alpha}{a_{4(p,n,\alpha)}} + \frac{2p-3}{b_{4(p,n,\alpha)} T_c}}_{r_{4(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_3^\alpha \leq |\tau| \leq \tau_4^\alpha \\ \underbrace{\frac{3p-2-2(p-1)\alpha}{a_{5(p,n,\alpha)}} + \frac{-(2p-1)|\tau|}{b_{5(p,n,\alpha)} T_c}}_{r_{5(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_4^\alpha \leq |\tau| \leq \tau_5^\alpha \\ \underbrace{\frac{-3p-4}{a_{6(p,n,\alpha)}} + \frac{2p-3}{b_{6(p,n,\alpha)} T_c}}_{r_{6(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_5^\alpha \leq |\tau| \leq \tau_6^\alpha \\ \underbrace{\frac{-1+\frac{2(p-2)\alpha}{p}}{a_{7(p,n,\alpha)}} + \frac{1}{b_{7(p,n,\alpha)} T_c}}_{r_{7(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_6^\alpha \leq |\tau| \leq \tau_7^\alpha \\ \underbrace{\frac{-\frac{7p-12}{p}}{a_{8(p,n,\alpha)}} + \frac{4p-7}{b_{8(p,n,\alpha)} T_c}}_{r_{8(p,n,\alpha)}; p=\{2,4\}} \quad |\tau|, \tau_7^\alpha \leq |\tau| \leq \tau_8^\alpha \\ \underbrace{\frac{9p-20}{a_{9(p,n,\alpha)}} + \frac{-(4p-9)|\tau|}{b_{9(p,n,\alpha)} T_c}}_{r_{9(p,n,\alpha)}; p=\{4\}} \quad |\tau|, \tau_8^\alpha \leq |\tau| \leq \tau_9^\alpha \\ \underbrace{\frac{20-7p+(5p-12)\alpha}{2p}}_{a_{10(p,n,\alpha)}} + \frac{p-3}{b_{10(p,n,\alpha)} T_c} \quad |\tau|, \tau_9^\alpha \leq |\tau| \leq \tau_{10}^\alpha \\ \underbrace{\frac{11p-28+(4-p)\alpha}{2p}}_{a_{11(p,n,\alpha)}} + \frac{-(2p-5)|\tau|}{b_{11(p,n,\alpha)} T_c} \quad |\tau|, \tau_{10}^\alpha \leq |\tau| \leq \tau_{11}^\alpha \\ \underbrace{\frac{-4(p-3)-4(p-3)\alpha}{2p}}_{a_{12(p,n,\alpha)}} + \frac{p-3}{b_{12(p,n,\alpha)} T_c} \quad |\tau|, \tau_{11}^\alpha \leq |\tau| \leq \tau_{12}^\alpha \\ \underbrace{\frac{7p-18-2(p-3)\alpha}{p}}_{a_{13(p,n,\alpha)}} + \frac{-(2p-5)|\tau|}{b_{13(p,n,\alpha)} T_c} \quad |\tau|, \tau_{12}^\alpha \leq |\tau| \leq \tau_{13}^\alpha \\ \underbrace{\frac{24-7p}{p}}_{a_{14(p,n,\alpha)}} + \frac{2p-7}{b_{14(p,n,\alpha)} T_c} \quad |\tau|, \tau_{13}^\alpha \leq |\tau| \leq \tau_{14}^\alpha \\ \underbrace{\frac{-1+\frac{2(p-4)\alpha}{p}}{a_{15(p,n,\alpha)}} + \frac{1}{b_{15(p,n,\alpha)} T_c}}_{r_{15(p,n,\alpha)}; p=\{4\}} \quad |\tau|, \tau_{14}^\alpha \leq |\tau| \leq \tau_{15}^\alpha \\ \underbrace{\frac{56-15p}{p}}_{a_{16(p,n,\alpha)}} + \frac{4p-15}{b_{16(p,n,\alpha)} T_c} \quad |\tau|, \tau_{15}^\alpha \leq |\tau| \leq \tau_{16}^\alpha \\ \underbrace{\frac{a_{16(p,n,\alpha)}}{r_{16(p,n,\alpha)}}}_{r_{16(p,n,\alpha)}; p=\{4\}} \quad 0, |\tau| \geq T_c \end{array} \right. \quad (39)
\end{aligned}$$

$$T = nT_c = 2mT_s, \text{ and } m, n \text{ integers} \quad (39a)$$

For $0 \leq \alpha \leq 1/3$ and $1/3 \leq \alpha \leq 1$ the ACF $R_{VBOC2(pn,n,\alpha)}(\tau)$ is given by (38) and (39) respectively.

The *proof of theorem 2* is straightforward.

First, we recognize that substituting $p = 2$ in (38) we get

$$\begin{aligned}
R_1(\tau) = & \left\{ \begin{array}{l} 1 - \frac{(4p-1)|\tau|}{T_c}, \quad |\tau| \leq \tau_1^\alpha \\ -1 + 2\alpha - \frac{|\tau|}{T_c}, \quad \tau_1^\alpha \leq |\tau| \leq \tau_2^\alpha \\ \frac{-3p+2+2\alpha}{p} + \frac{(4p-3)|\tau|}{T_c}, \quad \tau_2^\alpha \leq |\tau| \leq \tau_3^\alpha \\ \frac{2-p-2(p-1)\alpha}{p} - \frac{(2p-3)|\tau|}{T_c}, \quad \tau_3^\alpha \leq |\tau| \leq \tau_4^\alpha \\ \frac{(5p-6)-(2p-2)\alpha}{p} - \frac{(2p-1)|\tau|}{T_c}, \quad \tau_4^\alpha \leq |\tau| \leq \tau_5^\alpha \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, \quad \tau_5^\alpha \leq |\tau| \leq \tau_6^\alpha \\ 0, \quad |\tau| \geq T_c \end{array} \right. \quad (40)
\end{aligned}$$

and $p = 2$ and $1/3 \leq \alpha \leq 1$ in (39) we get

$$\begin{aligned}
R_1(\tau) = & \left\{ \begin{array}{l} 1 - \frac{(4p-1)|\tau|}{T_c}, \quad |\tau| \leq \tau_1^\alpha \\ -1 + 2\alpha - \frac{|\tau|}{T_c}, \quad \tau_1^\alpha \leq |\tau| \leq \tau_2^\alpha \\ 1 - \frac{(2p-1)|\tau|}{T_c}, \quad \tau_2^\alpha \leq |\tau| \leq \tau_3^\alpha \\ \frac{2-p-2(p-1)\alpha}{p} + \frac{(2p-3)|\tau|}{T_c}, \quad \tau_3^\alpha \leq |\tau| \leq \tau_4^\alpha \\ \frac{3p-2-2(p-1)\alpha}{p} - \frac{(2p-1)|\tau|}{T_c}, \quad \tau_4^\alpha \leq |\tau| \leq \tau_5^\alpha \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, \quad \tau_5^\alpha \leq |\tau| \leq \tau_6^\alpha \\ 0, \quad |\tau| \geq T_c \end{array} \right. \quad (41)
\end{aligned}$$

Equations (40) and (41), for $p = 2$, are the same as in Chap. 7 of [5]. Also, (38) and (39), for $p = 4$, are the same as Chap. 7 of [5]. This completed the *proof of theorem 2*.

Although we did not show the enormous convolution computational process for computing the generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ with respect to the signal design and optimization parameter α and generalized parameter p there is no way of knowing that by simple inspection this expression is correct or not especially considering the fact that $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ takes an entire page.

However, the following material which discusses the broad properties of generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ and the material discussed in [3] which discusses in great detail the complete set of the continuity and optimization theorems guarantees that generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2(m=pn,n,\alpha)}(\tau)$ is the correct expression.

Corollary 7: From *Corollary 1* $VBOC2_{(m,n,\alpha=0)}(t) = BOC_{(m,n)}(t)$ and from *Theorem 2*, the ACF $R_{VBOC2(m,n,\alpha=0)}(\tau) = R_{BOC(m,n)}(\tau)$ hence, prove that the generalized ACF with respect to p ; i.e., is

- (1) $R_{BOC(2n,n)}(\tau) = R_{VBOC2(pn=2n,n,\alpha=0)}(\tau)$; $p = 2$;
- (2) $R_{BOC(4n,n)}(\tau) = R_{VBOC2(pn=4n,n,\alpha=0)}(\tau)$; $p = 4$;

$$R_{BOC(pn,n)}(\tau) = \begin{cases} \underbrace{\frac{1}{a_{1(p,n)}} + \frac{-(4p-1)}{b_{1(p,n)}} |\tau|}_{r_{1(pn,n)}; p=\{1,2,3,4\}}, & |\tau| \leq \frac{T_c}{2p} = \frac{T_s}{\tau_1} \\ \underbrace{-\frac{3p-2}{p} + \frac{4p-3}{T_c} |\tau|}_{a_{3(p,n)} b_{3(p,n)}}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{\frac{5p-6}{p} + \frac{-(4p-5)}{T_c} |\tau|}_{a_{4(p,n)} b_{4(p,n)}}, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ \underbrace{-\frac{7p-12}{p} + \frac{(4p-7)}{T_c} |\tau|}_{a_{6(p,n)} b_{6(p,n)}}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{\frac{9p-20}{p} + \frac{-(4p-9)}{T_c} |\tau|}_{a_{7(p,n)} b_{7(p,n)}}, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \\ \underbrace{-\frac{(11p-30)}{p} + \frac{(4p-11)}{T_c} |\tau|}_{a_{9(p,n)} b_{9(p,n)}}, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \underbrace{\frac{13p-42}{p} + \frac{-(4p-13)}{T_c} |\tau|}_{a_{10(p,n)} b_{10(p,n)}}, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ \underbrace{-\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c} |\tau|}_{a_{12(p,n)} b_{12(p,n)}}, & \frac{7T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (42)$$

$$T = nT_c = 2mT_s, \text{ and } m, n \text{ integers} \quad (42a)$$

Proof of corollary 7: The *proof of corollary 7* is straightforward.

First, we recognize that substituting $p = 2$ in (38) we get

$$R_{BOC(pn,n)}(\tau) = \begin{cases} \frac{1 - \frac{(4p-1)|\tau|}{T_c}}{p}, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (43)$$

$$T = nT_c = 2mT_s, \text{ and } m = np, n \text{ integers} \quad (43a)$$

Equation (43), for $p = 2$, is the same as in Chap. 7 [5] and (82) in [6].

Equation (42), for $p = 4$, is the same as in Chap. 7 [5]. This completed the *proof of corollary 7*.

Corollary 8: From *corollary 2* $VBOC2_{(m,n,\alpha=1)}(t) = BOC_{(m/2,n)}(t)$ and from *theorem 2*, the ACF $R_{VBOC2_{(m,n,\alpha=1)}}(\tau) = R_{BOC_{(m/2,n)}}(\tau)$ hence, prove that the generalized ACF with respect to p ; i.e., is

- (1) $R_{BOC_{(n,n)}}(\tau) = R_{VBOC2_{(np=2n,n,\alpha=1)}}(\tau)$; $p = 2$;
- (2) $R_{BOC_{(2n,n)}}(\tau) = R_{VBOC2_{(np=4n,n,\alpha=1)}}(\tau)$; $p = 4$;

$$R_{BOC(pn/2,n)}(\tau) = \begin{cases} \underbrace{\frac{1}{a_{3(p,n)}} - \frac{2p-1}{b_{3(p,n)}} \frac{|\tau|}{T_c}}_{r_{3(pn,n)}; p=\{2,4\}}, & 0 \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{\frac{4-3p}{p} + \frac{2p-3}{T_c} \frac{|\tau|}{1}}_{a_{6(p,n)} b_{6(p,n)}}, & \frac{T_c}{p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{\frac{5p-12}{p} + \frac{5-2p}{T_c} \frac{|\tau|}{1}}_{a_{11(p,n)} b_{11(p,n)}}, & \frac{2T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ \underbrace{\frac{24-7p}{p} + \frac{2p-7}{T_c} \frac{|\tau|}{1}}_{a_{14(p,n)} b_{14(p,n)}}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (44)$$

$$T = nT_c = 2mT_s, \text{ and } m, n \text{ integers} \quad (44a)$$

Proof of corollary 8: The *proof of corollary 8* is straightforward. First, we recognize that that for up to values of $p = 2$ in (44) or for $\alpha = 1$ at (41) we get

$$R_{BOC(pn/2,n)}(\tau) = \begin{cases} \underbrace{\frac{1}{a_{3(p,n)}} - \frac{2p-1}{b_{3(p,n)}} \frac{|\tau|}{T_c}}_{r_{3(pn,n)}; p=\{2\}}, & 0 \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{\frac{4-3p}{p} + \frac{2p-3}{T_c} \frac{|\tau|}{1}}_{a_{6(p,n)} b_{6(p,n)}}, & \frac{T_c}{p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{\frac{5p-12}{p} + \frac{5-2p}{T_c} \frac{|\tau|}{1}}_{r_{6(pn,n)}; p=\{2\}}, & \frac{2T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (45)$$

Equation (45), for $p = 2$, is the same as in Chap. 7 [5] and (100) in [6].

Equation (45), for $p = 4$, is the same as in Chap. 7 [5]. This completed the *proof of corollary 8*.

This completes the first discussion on $VBOC2_{(m,n,\alpha)}(t)$ and $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$; further material with regards to $VBOC2_{(m,n,\alpha)}(t)$ pure signal optimization $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ continues in [3].

Next, we continue with $VBOC2(\alpha, 1 - \alpha)$ PSD definition and discussion.

2.3 $VBOC2(\alpha, 1 - \alpha)$ Generalized PSD Definition and Discussion

Definition 7: The PSD of $VBOC2_{(m,n,\alpha)}(t)$ or $G_{VBOC2_{(m,n,\alpha)}}(f)$ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of f ; $-\infty < f < \infty$ is given by

$$G_{VBOC2_{(m,n,\alpha)}}(f, T) = \mathcal{F} \left\{ \frac{R_{VBOC2_{(m,n,\alpha)}}(\tau, T, t_0)}{\frac{V(f)V(f)}{T} = \frac{V^2(f)}{T}} \right\} \quad (46)$$

where $\mathcal{F}\{v(t)\}$ denotes the Fourier Transform (FT) of $v(t)$; i.e.,

$$V(f) = \mathcal{F}\{VBOC2_{(m,n,\alpha)}(t)\} = \mathcal{F}\{v(t)\} \quad (47)$$

Now that we have defined the PSD of $VBOC2_{(m,n,\alpha)}(t)$, it is straightforward to show that $G_{VBOC2_{(m,n,\alpha)}}(f, T)$ obeys *theorem 11* in [1] so we leave it as an exercise to the reader.

Corollary 9: The PSD of $VBOC2_{(m,n,\alpha)}(t)$, $G_{VBOC2_{(m,n,\alpha)}}(f, T)$, is in general NOT periodic; simply, because $V(f)$ is in general not periodic, even if $v(t)$ is periodic.

Corollary 10: The PSD of $VBOC2_{(m,n,\alpha)}(t)$; i.e., $G_{VBOC2_{(m,n,\alpha)}}(f, T)$ is always an even function of $-\infty < f < \infty$; i.e., from (46) the following holds

$$G_{VBOC2_{(m,n,\alpha)}}(-f) = G_{VBOC2_{(m,n,\alpha)}}(f) \quad (48)$$

Theorem 3: Show that, for values of $p = \{2, 4\}$, $G_{VBOC2_{(m=np,n,\alpha)}}(f)$ is the generalized ACF with respect to p ; i.e.,

$$(1) G_{VBOC2_{(2n,n,\alpha)}}(f) = G_{VBOC2_{(pn=2n,n,\alpha)}}(f); p = 2;$$

$$(2) G_{VBOC2_{(4n,n,\alpha)}}(f) = G_{VBOC2_{(pn=4n,n,\alpha)}}(f); p = 4.$$

Find the generalized PSD of $G_{VBOC1_{(m=np,n,\alpha)}}(f)$ given by

$$G_{VBOC2_{(pn,n,\alpha)}}(f) = \left| \mathcal{F} \left\{ R_{VBOC2_{(pn,n,\alpha)}}(\tau) \right\} \right| = \frac{V(f)V(f)}{T} = \frac{V^2(f)}{T} \quad (49)$$

or

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2 \int_0^\infty R_{VBOC2_{(pn,n,\alpha)}}(\tau) \cos(\omega\tau) d\tau \quad (50)$$

where $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ is given by (38) and (39), $T = nT_c = 2mT_s$, and $\omega = 2\pi f$.

The *proof of theorem 3* is straightforward and is very similar to the proof of *Theorems 12* and *4* in [1].

Hence, we compute the generalized PSD of $VBOC2_{(pn,n,\alpha)}$, or $G_{VBOC2_{(pn,n,\alpha)}}(\omega)$, as a Fourier transform of the ACF such as

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2 \int_0^\infty \underbrace{R_{VBOC2_{(pn,n,\alpha)}}(\tau) \cos(\omega\tau) d\tau}_{\substack{\sum_{i=1}^{4p} \int_{\tau_{i-1}}^{\tau_i} r_{i(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau \\ g_{i(np,n,\alpha)}(\omega)}} \quad (51)$$

where t_i and $r_{i(pn,n,\alpha)}(\tau)$ are given below in the recall of $R_{VBOC2_{(np,n,\alpha)}}(\tau)$ given by (38) and (39).

As we can see from (51) and (38) and (39) the computation of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$ is in general a laborious process because it involves the computation of twelve integrals to obtain the generalized expression of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$.

Let us provide the details for obtaining the twelve integrals that lead to the generalized expression of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$ given by (51) by considering the integral of the general expression as follows:

$$g_{i(pn,n,\alpha)}(\omega) = \int_{\tau_{i-1}}^{\tau_i} r_{i(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau \quad (52)$$

where $r_{i(pn,n,\alpha)}(\tau)$ is given by

$$r_{i(pn,n,\alpha)}(\tau) = a_{i(p,n,\alpha)} + \tau b_{i(p,n,\alpha)} \quad (53)$$

where $a_{i(p,n,\alpha)}$ and $b_{i(p,n,\alpha)}$ are independent of τ given in (38) and (39).

Substituting (53) into (52) the following is obtained:

$$g_{i(pn,n,\alpha)}(\omega) = \left| \begin{array}{l} a_{i(p,n,\alpha)} \int_{\tau_{i-1}}^{\tau_i} \cos(\omega\tau) d\tau + \\ b_{i(p,n,\alpha)} \int_{\tau_{i-1}}^{\tau_i} \tau \cos(\omega\tau) d\tau \end{array} \right| \quad (54)$$

In order to compute (54) we compute two integrals $q_a(\tau_i, \tau_{i-1}, \omega)$ as a direct integral as follows:

$$q_a(\tau_i, \tau_{i-1}, \omega) = \int_{\tau_{i-1}}^{\tau_i} \cos(\omega\tau) d\tau = \frac{\sin(\omega\tau_i) - \sin(\omega\tau_{i-1})}{\omega} \quad (55)$$

and $q_b(\tau_i, \tau_{i-1}, \omega)$ using integration by parts

$$q_b(\tau_i, \tau_{i-1}, \omega) = \left| \begin{array}{l} \int_{\tau_{i-1}}^{\tau_i} \tau \cos(\omega\tau) d\tau = \\ \frac{\tau_i \sin(\omega\tau_i) - \tau_{i-1} \sin(\omega\tau_{i-1})}{\omega} + \\ \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right| \quad (56)$$

Substituting (56) and (55) into (54) we obtain the detailed expression of $g_{i(pn,n,\alpha)}(\omega)$ as follows:

$$g_{i(pn,n,\alpha)}(\omega) = \left| \begin{array}{l} a_{i(p,n,\alpha)} \frac{\sin(\omega\tau_i) - \sin(\omega\tau_{i-1})}{\omega} + \\ b_{i(p,n,\alpha)} \left[\frac{\tau_i \sin(\omega\tau_i) - \tau_{i-1} \sin(\omega\tau_{i-1})}{\omega} + \right. \\ \left. \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2} \right] \end{array} \right| \quad (57)$$

or

$$g_{i(pn,n,\alpha)}(\omega) = \left| \begin{array}{l} \frac{[a_{i(p,n,\alpha)} + b_{i(p,n,\alpha)} \tau_i] \sin(\omega\tau_i) - \\ [a_{i(p,n,\alpha)} + b_{i(p,n,\alpha)} \tau_{i-1}] \sin(\omega\tau_{i-1})}{\omega} + \\ b_{i(p,n,\alpha)} \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right| \quad (58)$$

Finally, substituting (58) into (51) we obtain the final expression for PSD of the generalized $VBOC2_{(pn,n,\alpha)}$ or $G_{VBOC2_{(pn,n,\alpha)}}(\omega)$ as follows

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2 \left| \begin{array}{l} \frac{\sum_{i=1}^{2p} g_{i(np,n,\alpha)}(\omega)}{G_1; p=\{2,4\}} \\ + \frac{\sum_{i=2p+1}^{4p} g_{i(np,n,\alpha)}(\omega)}{G_2; p=\{4\}} \end{array} \right| \quad (59)$$

Leaving the computation of G_1 and G_2 to the reader we obtain

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2(G_s + G_c) \quad (60)$$

where

$$G_s = \sum_{i=1}^2 G_{is} \text{ and } G_c = \sum_{i=1}^2 G_{ic} \quad (61)$$

or

$$G_s = \frac{\left| \begin{array}{l} \frac{p-2}{p} \sin\left(\frac{\omega 2T_c}{p}\right) \\ + \frac{p-2}{p} \sin\left(\frac{\omega 2T_c}{p}\right) + \frac{p-4}{p} \sin\left(\frac{\omega 4T_c}{p}\right) \end{array} \right|}{p=\{4\}} = 0 \quad (62)$$

and

$$G_C = \frac{\begin{matrix} -4p\cos\left[\frac{\omega T_c(1-\alpha)}{2p}\right] + 4(1-p)\cos\left[\frac{\omega T_c(1+\alpha)}{2p}\right] \\ + 2p\cos\left[\frac{\omega T_c(1-\alpha)}{p}\right] + 4(p-1)\cos\left(\frac{\omega T_c}{p}\right) \\ + 2(p-2)\cos\left[\frac{\omega T_c(1+\alpha)}{p}\right] + 4(1-p)\cos\left[\frac{\omega T_c(3-\alpha)}{2p}\right] \\ + 4(2-p)\cos\left[\frac{\omega T_c(3+\alpha)}{2p}\right] + 4(p-7)\cos\left(\frac{\omega 2T_c}{p}\right) + 4p-1 \end{matrix}}{\begin{matrix} p=\{2,4\} \\ (12-5p)\cos\left[\frac{\omega T_c(5-\alpha)}{2p}\right] + (8-3p)\cos\left[\frac{\omega T_c(5+\alpha)}{2p}\right] \\ + (3p-8)\cos\left[\frac{\omega T_c(3-\alpha)}{p}\right] + (3p-8)\cos\left(\frac{\omega 3T_c}{p}\right) \\ + (2p-8)\cos\left[\frac{\omega T_c(3+\alpha)}{p}\right] + (12-4p)\cos\left[\frac{\omega T_c(7-\alpha)}{2p}\right] \\ + (16-4p)\cos\left[\frac{\omega T_c(7+\alpha)}{2p}\right] \\ + (4p-15)\cos\left(\frac{\omega 4T_c}{p}\right) + (4p-9)\cos\left(\frac{\omega 2T_c}{p}\right) \end{matrix}} \quad (63)$$

Since, $G_S = 0$ for any values of p ; hence, (60) becomes

$$G_{VBOC2(p,n,\alpha)}(\omega) = 2G_C \quad (64)$$

Now, let us compute the individual expressions of $G_{VBOC2(p,n,\alpha)}(\omega)$ for values of $p = \{2,4\}$ based on (63) or (64).

First, for $p = \{2\}$ from (63) or (64) we obtain

$$G_{VBOC2(2,n,\alpha)}(\omega) = \frac{\begin{matrix} -8\cos\left[\frac{\omega T_c(1-\alpha)}{4}\right] - 4\cos\left[\frac{\omega T_c(1+\alpha)}{4}\right] \\ + 4\cos\left[\frac{\omega T_c(1-\alpha)}{2}\right] + 4\cos\left(\frac{\omega T_c}{2}\right) \\ - 4\cos\left[\frac{\omega T_c(3-\alpha)}{4}\right] + \cos\left(\frac{\omega T_c}{4}\right) + 7 \end{matrix}}{\omega^2 T_c} \quad (65)$$

Equation (65) is identical in Appendix A of Chap. 7 of [5].

Second and finally, for $p = \{4\}$ from (63) or (64) we obtain

$$G_{VBOC2(4,n,\alpha)}(\omega) = \frac{\begin{matrix} -16\cos\left[\frac{\omega T_c(1-\alpha)}{8}\right] - 12\cos\left[\frac{\omega T_c(1+\alpha)}{8}\right] \\ + 8\cos\left[\frac{\omega T_c(1-\alpha)}{4}\right] + 12\cos\left(\frac{\omega T_c}{4}\right) \\ + 4\cos\left[\frac{\omega T_c(1+\alpha)}{4}\right] - 12\cos\left[\frac{\omega T_c(3-\alpha)}{8}\right] \\ - 8\cos\left[\frac{\omega T_c(3+\alpha)}{8}\right] + 9\cos\left(\frac{\omega T_c}{2}\right) + 15 \\ - 8\cos\left[\frac{\omega T_c(5-\alpha)}{8}\right] - 4\cos\left[\frac{\omega T_c(5+\alpha)}{8}\right] \\ + 4\cos\left[\frac{\omega T_c(3-\alpha)}{4}\right] + 4\cos\left(\frac{\omega 3T_c}{4}\right) \\ - 4\cos\left[\frac{\omega T_c(7-\alpha)}{8}\right] \\ + \cos(\omega T_c) + 7\cos\left(\frac{\omega T_c}{2}\right) \end{matrix}}{\omega^2 T_c} \quad (66)$$

or

$$G_{VBOC2(4,n,\alpha)}(\omega) = \frac{\begin{matrix} -16\cos\left[\frac{\omega T_c(1-\alpha)}{8}\right] - 12\cos\left[\frac{\omega T_c(1+\alpha)}{8}\right] \\ + 8\cos\left[\frac{\omega T_c(1-\alpha)}{4}\right] + 12\cos\left(\frac{\omega T_c}{4}\right) \\ + 4\cos\left[\frac{\omega T_c(1+\alpha)}{4}\right] - 12\cos\left[\frac{\omega T_c(3-\alpha)}{8}\right] \\ - 8\cos\left[\frac{\omega T_c(3+\alpha)}{8}\right] + 16\cos\left(\frac{\omega T_c}{2}\right) + 15 \\ - 8\cos\left[\frac{\omega T_c(5-\alpha)}{8}\right] - 4\cos\left[\frac{\omega T_c(5+\alpha)}{8}\right] \\ + 4\cos\left[\frac{\omega T_c(3-\alpha)}{4}\right] + 4\cos\left(\frac{\omega 3T_c}{4}\right) \\ - 4\cos\left[\frac{\omega T_c(7-\alpha)}{8}\right] + \cos(\omega T_c) \end{matrix}}{\omega^2 T_c} \quad (67)$$

Equation (67) is identical in Appendix A of Chap. 7 of [5].

Equations (51) through (67) complete the proof of *theorem*

3.

Definition 7, theorem 3 and corollaries 9 and 10 are the most important *definition, theorem, and corollaries* of PSDs of the generalized, $VBOC2_{(m,n,\alpha)}(t)$, or $G_{VBOC2_{(m,n,\alpha)}}(f)$.

This concludes $VBOC2(\alpha, 1 - \alpha)$ generalized PSD definition and discussion; next, we continue with

$VBOC2(\alpha, 1 - \alpha)$ numerical results and one example as it pertains to the GPS M-code [3]-[5], [8]-[10].

3 Numerical, Theoretical Results

Next, let us consider a $VBOC2(\alpha, 1 - \alpha)$ example. In coming up with an example it is important to know that $VBOC2_{(1,1,\alpha)}(t) = VBOC1_{(1,1,\alpha)}(t)$; hence, this is already discussed in [1], [2]; i.e., $VBOC2_{(1,1,\alpha=0.5)}(t)$ is just as good as $VBOC1_{(1,1,\alpha=0.5)}(t)$ instead of $BOC_{(1,1)}(t)$ which is currently proposed in GPS III L1C or L2C signals [11]-[15].

3.1 $VBOC2(\alpha, 1 - \alpha)$ Examples

Now that we have developed this intellectual apparatus on the profound meaning of ACF $R_{VBOC2_{(m,n,\alpha)}}(\tau)$ and PSD $G_{VBOC2_{(m,n,\alpha)}}(f)$ functions let us consider a more specific example of $VBOC2_{(2,1,\alpha)}(t)$ and its corresponding ACF for $VBOC2_{(2,1,\alpha)}(t)$ or $R_{VBOC2_{(2,1,\alpha)}}(\tau)$.

Corollary 11: From *definition 1*, $VBOC2_{(2,1,\alpha)}(t)$ is simply

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm 2T_s) \quad (68)$$

and the relation between T_s and T_c is given by

$$4T_s = T_c \quad (68a)$$

hence,

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm 0.5T_c) \quad (69)$$

Corollary 12: From *definition 2* $VBOC2_{(m=2,n=1,\alpha)}(t)$

waveform is an odd function for all values of $-\infty < t < \infty$.

$$VBOC2_{(m=2,n=1,\alpha)}(t) = -VBOC2_{(m=2,n=1,\alpha)}(-t) \quad (70)$$

Corollary 13: From *definition 3* $VBOC2_{(m=2,n=1,\alpha)}(t)$ waveform is a periodic function with period T_s , for all values of m even and $t - \infty < t < \infty$.

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm T_s) \quad (71)$$

Corollary 14: From *definition 4*: The odd/even sub-carrier period of $VBOC2_{(m=2,n=1,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ or $+1/w_f$ and $-1/w_f$ or $-1/w_r$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ or $-1/w_r$ and $+1/w_r$ or $+1/w_f$ respectively that satisfy the following

$$w_r = 0.5T_s(1 \pm \alpha) = 0.5 \frac{T_c(1 \pm \alpha)}{4}, \quad (72)$$

$$w_f = 0.5T_s(1 \mp \alpha) = 0.5 \frac{T_c(1 \mp \alpha)}{4} \quad (73)$$

$$0 \leq w_r \leq 0.5T_c \text{ or } 0.5T_c \leq w_f \leq T_c; \quad 0 \leq \alpha \leq 1 \quad (73a)$$

Hence, the waveform $VBOC2_{(m=2,n=1,\alpha)}(t)$ can be obtained as

$$VBOC2_{(2,1,\alpha)}(t) = \sum_{q: \text{odd}}^{\infty} [p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)] + \sum_{q: \text{even}}^{\infty} [p_{w_f}(t_1, w_f) - p_{w_r}(t_2, w_r)] \quad (74)$$

where

$$t_1 = t - 0.5w_r - qT_s \quad (75)$$

$$t_2 = t - 0.5w_f - w_r - qT_s \quad (76)$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

It is straightforward to show that $VBOC2_{(m=2,n=1,\alpha)}(t)$ obeys *theorem 1* of [1]; i.e., for $VBOC2_{(m=2,n=1,\alpha)}(t)$ $0 \leq \alpha \leq 1$.

The ACF for $VBOC2_{(2,1,\alpha)}(t)$ is defined as $R_{VBOC2_{(2,1,\alpha)}}(\tau)$ given by *definition 25*: in Chap. 7 in [5] (full convolution for m even), which, for $0 \leq \alpha \leq 1/3$, is given by

$$R_{VBOC2_{(2,1,\alpha)}}(\tau) = \begin{cases} 1 - \frac{7|\tau|}{T_c}, & 0 \leq |\tau| \leq \frac{T_c(1-\alpha)}{4} \\ 2\alpha - 1 + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{4} \leq |\tau| \leq \frac{T_c(1+\alpha)}{4} \\ \alpha - 2 + \frac{5|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{4} \leq |\tau| \leq \frac{T_c}{2} \\ -\alpha + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{2} \leq |\tau| \leq \frac{T_c}{2} \\ 2 - \alpha - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{T_c(3-\alpha)}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{T_c(3-\alpha)}{4} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c = 4T_s \end{cases} \quad (77)$$

and for $1/3 \leq \alpha \leq 1$

$$R_{VBOC2_{(2,1,\alpha)}}(\tau) = \begin{cases} 1 - \frac{7|\tau|}{T_c}, & 0 \leq |\tau| \leq \frac{T_c(1-\alpha)}{4} \\ 2\alpha - 1 + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{4} \leq |\tau| \leq \frac{T_c(1-\alpha)}{2} \\ 1 - \frac{3|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{2} \leq |\tau| \leq \frac{T_c(1+\alpha)}{4} \\ -\alpha + \frac{|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{4} \leq |\tau| \leq \frac{T_c}{2} \\ 2 - \alpha - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{T_c(3-\alpha)}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{T_c(3-\alpha)}{4} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c = 4T_s \end{cases} \quad (78)$$

Corollary 15: Prove that the ACFs of $VBOC2_{(2,1,\alpha)}(t)$ given by (77) and (78) will result in the $BOC_{(2,1)}(t)$ for $\alpha = 0$ as indicated in *corollary 1* and $BOC_{(1,1)}(t)$ for $\alpha = 1$ as indicated in *corollary 2*.

Proof of corollary 15: The *proof of corollary 15* is straightforward. First, we substitute values of $\alpha = 0$ in (77)

and we get $R_{VBOC2_{(2n,n,\alpha=0)}}(\tau) = R_{BOC_{(2n,n)}}(\tau)$ as indicated in *corollary 1* of $VBOC2_{(2,1,\alpha)}(t)$.

$$R_{VBOC2_{(2n,n,\alpha=0)}}(\tau) = \begin{cases} 1 - \frac{7|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{4} = T_s \\ -2 + \frac{5|\tau|}{T_c}, & \frac{T_c}{4} \leq |\tau| \leq \frac{T_c}{2} \\ 2 - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{3T_c}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{3T_c}{4} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \\ R_{BOC_{(2n,n)}}(\tau) \end{cases} \quad (79)$$

Second, we substitute values of $\alpha = 1$ in (78) and we get $R_{VBOC2_{(2n,n,\alpha=1)}}(\tau) = R_{BOC_{(n,n)}}(\tau)$ as indicated in *corollary 2* of $VBOC2_{(2,1,\alpha)}(t)$.

$$R_{VBOC2_{(2n,n,\alpha=1)}}(\tau) = \begin{cases} 1 - \frac{3|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{2} \\ -1 + \frac{|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \\ R_{BOC_{(n,n)}}(\tau) \end{cases} \quad (80)$$

Equations (79) and (80) complete the *proof of corollary 15*.

Corollary 16: Finally, the PSD of $VBOC2_{(2,1,\alpha)}(t)$ or $G_{VBOC2_{(2,1,\alpha)}}(\omega)$ is given by

$$G_{VBOC2_{(2,1,\alpha)}}(\omega) = \frac{4 - 16\cos\left[\frac{\omega T_c(1-\alpha)}{4}\right] - 8\cos\left[\frac{\omega T_c(1+\alpha)}{4}\right] + 8\cos\left[\frac{\omega T_c(1-\alpha)}{2}\right] + 8\cos\left[\frac{\omega T_c}{2}\right] - 8\cos\left[\frac{\omega T_c(3-\alpha)}{4}\right] + 2\cos(\omega T_c)}{\omega^2 T_c} \quad (81)$$

After some simplifications it can be shown that

$$G_{VBOC2_{(2,1,\alpha=0)}}(\omega) = \frac{\frac{4}{\omega^2 T_c} \tan^2\left(\frac{\omega T_c}{8}\right) \sin^2\left(\frac{\omega T_c}{2}\right) = \frac{1}{\omega^2 T_s} \tan^2\left(\frac{\omega T_s}{2}\right) \sin^2(2\omega T_s)} \quad (82)$$

as indicated in *corollary 1* as depicted in Figs. 42 and 90 of [6], also

$$G_{VBOC2_{(2,1,\alpha=1)}}(f) = \frac{\frac{4T_c}{\left(\frac{\omega T_c}{2}\right)^2} \sin^4\left(\frac{\omega T_c}{4}\right) = T_c \text{sinc}^2(fT_c) \tan^2\left(\frac{\pi f T_c}{2}\right) = G_{BOC_{(1,1)}}(f)} \quad (83)$$

as indicated in *corollary 2* as depicted in Figs. 42, 90 of [6].

In Figure 1 the ACF of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $r_{VBOC2_{(2,1,\alpha=0.4)}}(\tau)$ is shown with solid green has lower secondary peaks than the ACF of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $r_{VBOC1_{(2,1,\alpha=0.5)}}(\tau)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $VBOC1_{(2,1,\alpha=0.5)}(t)$ or the proposed replacement of the GPS III M-code on both L1 and L2 frequencies [1], [2], [4], [5]. In Figure 1 also the PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $r_{VBOC2_{(2,1,\alpha=0.4)}}(\tau)$ is illustrated with solid green has lower

⁴ For ease of analysis we will assume that the $+$ for w_r and $-$ for w_f .

secondary peaks than the ACF of $BOC_{(2,1)}(t)$ or $r_{BOC_{(2,1)}}(\tau)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code [3]-[5], [8]-[10].

FIGURE 1: ACF of $VBOC2_{(2n,n,0.4)}(t)$ on Q channel vs. $VBOC1_{(2n,n,0.5)}(t)$ on I channel and $VBOC2_{(2n,n,0.4)}(t)$ vs. $BOC_{(2,n)}(t)$ on Q channel.

FIGURE 2: PSD of $VBOC2_{(2n,n,0.4)}(t)$ on Q channel vs. $VBOC1_{(2n,n,0.5)}(t)$ on I channel and $VBOC2_{(2n,n,0.4)}(t)$ vs. $BOC_{(2,n)}(t)$ on Q channel.

Similarly, in Figure 2 the PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $G_{VBOC2_{(2,1,\alpha=0.4)}}(f)$ is depicted with solid green is quasi-flatter and wider than the PSD of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $G_{VBOC1_{(2,1,\alpha=0.5)}}(f)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $VBOC1_{(2,1,\alpha=0.5)}(t)$ or the proposed replacement of the GPS III M-code on both L1 and L2 frequencies [1], [2], [4], [5].

In Figure 2 also the PSD of $VBOC2_{(2,1,\alpha=0.5)}(t)$ or $G_{VBOC2_{(2,1,\alpha=0.5)}}(f)$ is displayed with solid green is quasi-flatter and wider than the PSD of $BOC_{(2,1)}(t)$ or $G_{BOC_{(2,1)}}(f)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code [3]-[5], [8]-[10].

Interference with the current GPS L1 BPSK is avoided by having PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ orthogonal with the PSD of GPS L1 BPSK signal [5].

This completes the detailed discussion on $VBOC2(\alpha, 1 - \alpha)$ ACFs and PSDs which contains seven definitions, four theorems, and sixteen corollaries.

The $VBOC2(\alpha, 1 - \alpha)$ continuity theorems and optimization theorems are provided in [3].

$VBOC1(\alpha)$ [1], [2] and $VBOC2(\alpha, 1 - \alpha)$ [3] results are provided in the companion manuscript given here as a reference [4].

4 Conclusions

This paper is the first complete discussion on pure signal design for the first generation $VBOC2(\alpha, 1 - \alpha)$ generalized multidimensional geolocation modulation waveforms.

Contrast the results of this paper with previous signal design methodologies, this paper offers for the first time a complete pure signal design methodology subject to both signal design and optimization parameter α and generalized signal design and optimization parameter p .

Signal parameters α and p not only define the waveform $VBOC2(\alpha, 1 - \alpha)$ and generalized ACFs and PSDs but they also play a very important role in the optimization of $VBOC2(\alpha, 1 - \alpha)$ generalized ACFs and PSDs. The computational technique offers a unique and original description of the generalized ACFs and PSDs of $VBOC2(\alpha, 1 - \alpha)$ as functions of both α and p .

In the paper it is argued that the selection of $BOC(10,5)$ (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC2(\alpha, 1 - \alpha)$ for $\alpha = 0$ or $\alpha = 1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

The above is based on a discussion of $VBOC2(\alpha, 1 - \alpha)$ pure signal optimization in [1]: (1) the criteria for validating the closed form expression of the generalized ACF of $VBOC2(\alpha, 1 - \alpha)$ known as a set of *continuity theorems*; and (2) the criteria for selecting the optimum $0 \leq \alpha \leq 1$ based on a set of criteria known as *optimization theorems* regardless of generalized parameter p (or subcarrier frequency).

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6 References

- [1] I. Progri, "VBOC1(α) generalized multidimensional geolocation modulation waveforms," *J. Geoloc. GeoInf. Geoint.*, vol. 2015, pp. 19-31, Nov. 2015, DOI: <http://doi.org/10.18610/JG3.2015.082101>.
- [2] I. Progri, "VBOC1(α) ACF pure signal optimization," *J. Geoloc. GeoInf. Geoint.*, vol. 2015, pp. 32-43, Nov. 2015, DOI: <http://doi.org/10.18610/JG3.2015.082102>.
- [3] I. Progri, "VBOC2($\alpha, 1-\alpha$) ACF pure signal optimization," *J. Geoloc. GeoInf. Geoint.*, vol. 2015, pp. 57-69, Nov. 2015, DOI: <http://doi.org/10.18610/JG3.2015.082104>.
- [4] I. Progri, "VBOC1(α) and VBOC2($\alpha, 1-\alpha$) generalized multidimensional geolocation modulation waveforms—technical report," *J. Geoloc. GeoInf. Geoint.*, vol. 2015, pp. 70-82, Nov. 2015, DOI: <http://doi.org/10.18610/JG3.2015.082105>.
- [5] I. Progri, *Indoor Geolocation Systems—Theory and Applications. Vol. I*, 1st ed., New York, NY: Springer Science & Business Media (SBM), LLC, ~800 pp., ~2015 (not yet available in print).
- [6] I. Progri, "On generalized multi-dimensional geolocation modulation waveforms," in *Proc. IEEE/ION-PLANS 2012*, Myrtle Beach, SC, pp. 919-951, Apr. 2012.
- [7] I. Progri, M.C. Bromberg, W.R. Michalson, J. Wang, "A theoretical survey of the spreading modulation of the new GPS signals (L1C, L2C, and L5)," in *Proc. ION-NTM 2007*, San Diego, CA, pp. 561-569, Jan. 2007.
- [8] S.N. Kirillov, V.M. Berdnikov, "Multi-criteria synthesis of signals from satellite navigation systems," *Herald Ryazan State Radio Engineering University (RSREU)*, vol. 30, no. 4, 6 pp., 2009.
- [9] M.K. Yarlykov, "Correlation functions of single symbols of meander signals in new-generation satellite radio navigation systems," *J. Commun. Tech. Elect.*, vol. 54, no. 8, pp. 912-924, Aug. 2009.
- [10] J.W. Betz, "Binary offset carrier modulation for radionavigation," *Navigation*, vol. 48, no. 4, pp. 227-246, winter 2001-2002.
- [11] J. Betz, C.R. Cahn, P.A. Dafesh, C.J. Hegarty, K.W. Hudnut, A.J. Jones, R. Keegan, K. Kovach, L.S. Lenahan, H.H. Ma, J.J. Rushanan, T.A. Stansell, C.C. Wang, and S.K. Yi, "L1C signal design options," in *Proc. 2006 Nat. Tech. Mtg. ION*, Monterey, CA, pp. 685-697, Jan. 2006.
- [12] J. Betz, M.A. Blanco, C.R. Cahn, P.A. Dafesh, C.J. Hegarty, K.W. Hudnut, V. Kasemsri, R. Keegan, K. Kovach, L.S. Lenahan, H.H. Ma, J.J. Rushanan, D. Sklar, T.A. Stansell, C.C. Wang, S.K. Yi, "Description of the L1C signal," in *Proc. 19th Inter. Tech. Mtg. Sat. Div. ION (ION GNSS 2006)*, Fort Worth, TX, pp. 2080-2091, Sep. 2006.
- [13] T. Stansell, K.W. Hudnut, R.G. Keegan, "GPS L1C: enhanced performance, receiver design suggestions, and key contributions," in *Proc. 23rd Inter. Tech. Mtg. Sat. Div. ION (ION GNSS 2010)*, Portland, OR, pp. 2860-2873, Sep. 2010.
- [14] Anon., "GLONASS Navigational radio-signal in bands L1, L2," Russian Institute of Space Device Engineering, Moscow, Russia, 2008.
- [15] J.-A. Avila-Rodriguez, G.W. Hein, S. Wallner, J.-L. Issler, L. Ries, L. Lestarquit, A. de Latour, J. Godet, F. Bastide, T. Pratt, J. Owen, "The MBOC modulation: the final touch to the Galileo frequency and signal plan," *Navigation*, vol. 55, no. 1, pp. 14-28, spring 2008.
- [16] L. Lestarquit, G. Artaud, J.-L. Issler, "AltBOC for dummies or everything you always wanted to know about AltBOC," in *Proc. 21st Inter. Tech. Mtg. Sat. Div. ION (ION GNSS 2008)*, Savannah, GA, pp. 961-970, Sep. 2008.
- [17] F.M.G. Sousa, F.D. Nunes "New expressions for the autocorrelation function of BOC GNSS signals," *Navigation*, vol. 60, no. 1, pp. 1-9, spring 2013.
- [18] I. Progri, M.C. Bromberg, W.R. Michalson, "Maximum likelihood GPS parameter estimation," *Navigation*, vol. 52, no. 4, pp. 229-238, winter 2005-2006.
- [19] I. Progri, M.C. Bromberg, J. Wang, "Markov Chain, Monte Carlo global search and integration for Bayesian, GPS, parameter estimation," *Navigation*, vol. 56, no. 3, pp. 195-204, fall 2009.
- [20] W.R. Michalson, I. Progri, "Reconfigurable geolocation system," US Patent 7,079,025, July 2006.
- [21] I.F. Progri, "An assessment of indoor geolocation systems," *Ph.D. Dissertation*, Worcester Polytechnic Institute, May 2003.
- [22] I. Progri, J.M. Hill, and W.R. Michalson, "An investigation of the pseudolite's signal structure for indoor applications," in *Proc. ION-AM 2001*, Albuquerque, NM, pp. 453-462, June 2001.
- [23] I. Progri, W.R. Michalson, "The impact of proposed pseudolite's signal structure on the receiver's phase and code error," in *Proc. ION-AM 2001*, Albuquerque, NM, pp. 414-422, June 2001.

- [24] I. Progri, W.R. Michalson, “An alternative approach to multipath and near-far problem for indoor geolocation systems,” in *Proc. 14th Inter. Tech. Mtg. Sat. Div. ION (ION GPS 2001)*, Salt Lake City, UT, pp. 1434-1443, Sep. 2001.
- [25] I. Progri, W.R. Michalson, J. Wang, M.C. Bromberg, and R.J. Duckworth, “Requirements of a C-CDMA pseudolite indoor geolocation system,” in *Proc. ION-AM 2007*, Cambridge, MA, pp. 654-658, Apr. 2007.
- [26] I. Progri, W.R. Michalson, J. Wang, M.C. Bromberg, “Indoor geolocation using FCDMA pseudolites: signal structure and performance analysis,” *Navigation*, vol. 54, no. 3, pp. 242-256, fall 2007.
- [27] I. Progri, *Geolocation of RF Signals—Principles and Simulations*. 1st ed., New York, NY: Springer SBM, LLC, 330 pp., Jan. 2011, DOI: <http://doi.org/10.1007/978-1-4419-7952-0>.