Formal Specification and Model Checking of A* Algorithm

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Abstract—A* algorithm is formally specified in Maude and model checked with the Maude LTL model checker. We take into account a graph such that it is a DAG, a goal node is reachable from a start node and each edge is given a non-negative weight. If h is admissible, namely that $h(n)$ never overestimates the cost to the goal from n for all nodes n , then A* finds a shortest path. The condition, however, can be relaxed. Our model checking experiments make us conjecture that if there exists a shortest path such that for each node n in the path $h(n)$ plus the cost to n from the start node is less than the cost of any non-shortest path to the goal from the start, A* finds a shortest path.

Keywords—A* algorithm; Dijkstra algorithm; LTL; Maude; Model checking

I. INTRODUCTION

A* algorithm[1] is a generalized version of Dijkstra (shortest path finding) algorithm[2]. It uses an estimation of the distance between the goal node and each of the edge nodes to select the next node to be tackled. Dijkstra algorithm treats the estimation as 0.

A* is formally specified in Maude[3], a rewriting logic-based specification/programming language equipped with many facilities, among which are model checking ones (a reachability analyzer and an LTL model checker). We model check with the LTL model checker that A* terminates and finds a shortest path. Our model checking experiments say that A* always halts and if the estimation h is admissible, namely that $h(n)$ never overestimates the cost to the goal from n for all nodes n , then A* finds a shortest path. The condition, however, can be relaxed. Our model checking experiments make us conjecture that if there exists a shortest path such that for each node n in the path $h(n)$ plus the cost to n from the start node is less than the cost of any non-shortest path to the goal from the start, A* finds a shortest path.

The contribution of the work described in the present paper is to demonstrate how A* is formally specified in Maude and model checked with the Maude LTL model checker and to find a relaxed sufficient condition, through model checking experiments, that A* finds a shortest path.

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Saberi, Groote and Keshishzadeh [4] have formally specified in the mCRL2 language a simple path planning algorithm that makes multiple robots reach a destination on a discretized planar surface, described properties in μ -calculus and conducted model checking experiments that the algorithm enjoys the properties. The desired properties they have taken into account are deadlock-freeness, collision-freeness and reachability. Their motivation to conduct the research is to demonstrate how useful model checking is to verify that the collective behavior of multiple robots satisfy some desired properties. Maragos, Kleftouris and Ziogou [5] have formally modeled a path finding planning in Colored Petri Nets (CP-Nets). Given a discretized planar space where there is one robot located in one position, there are some obstacles located in some positions and there is a goal position, the planning is to find a path along which the robot reaches the goal. Their CP-Nets model is executable with the Design/CPN tool. Their main motivation to conduct the research is to demonstrate how powerful CP-Nets are for simulating such a planning. Alphan, Smith, Belta and Rus [6] have proposed a method that automatically plans optimal paths for a group of robots that satisfy LTL properties. They formally modeled as a timed automaton a system in which multiple robots asynchronously move to some vertexes from the current vertexes on a graph if there are direct edges from the latter to the former. They use a bi-simulation relation between a timed automaton that is inherently infinite and a finite-state transition system so that they can use their earlier algorithm dedicated to a single-robot system.

The rest of the paper is organized as follows: § II Preliminaries, § III A* Algorithm, § IV Formal Specification, § V Model Checking, and § VI Conclusion.

II. PRELIMINARIES

A Kripke structure K is $\langle S, I, T, P, L \rangle$, where S is a set of states, $I \subseteq S$ is the set of initial states, $T \subseteq S \times S$ is a total binary relation over S , P is a set of atomic propositions and L is a labeling function whose type is $S \rightarrow 2^P$. Each element $(s, s') \in T$ is called a state transition from s to s' and T may be called the state transitions (with respect to K). For a state $s \in S$, $L(s)$ is the set of atomic propositions that hold in s . A path π is an infinite sequence $s_0, \dots, s_i, s_{i+1}, \dots$ of states such that $s_i \in S$ and

$(s_i, s_{i+1}) \in T$ for each i . Let π^i be $s_i, s_{i+1}, \dots$ and $\pi(i)$ be s_i . Let P be the set of all paths. π is called a computation if $\pi(0) \in I$. Let C be the set of all computations.

The syntax of a formula φ in LTL for K is $\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bigcirc \varphi \mid \varphi \mathcal{U} \varphi$, where $p \in P$. Let $\mathcal{F}$ be the set of all formulas in LTL for K . An arbitrary path $\pi \in \mathcal{P}$ of K and an arbitrary LTL formula $\varphi \in \mathcal{F}$ of K , $K, \pi \models \varphi$ is inductively defined as $K, \pi \models \top$, $K, \pi \models p$ iff $p \in L(\pi(0))$, $K, \pi \models \neg\varphi_1$ iff $K, \pi \not\models \varphi_1$, $K, \pi \models \varphi_1 \wedge \varphi_2$ iff $K, \pi \models \varphi_1$ and $K, \pi \models \varphi_2$, $K, \pi \models \bigcirc \varphi_1$ iff $K, \pi^1 \models \varphi_1$, and $K, \pi \models \varphi_1 \mathcal{U} \varphi_2$ iff there exists a natural number $j < i$, $K, \pi^j \models \varphi_2$ and for all natural numbers $j < i$, $K, \pi^j \models \varphi_1$, where φ_1 and φ_2 are LTL formulas. Then, $K \models \varphi$ iff $K, \pi \models \varphi$ for each computation $\pi \in \mathcal{C}$ of K . The temporal connectives $\bigcirc$ and $\mathcal{U}$ are called the next connective and the until connective, respectively. The other logical and temporal connectives are defined as usual as follows: $\perp \triangleq \neg\top$, $\varphi_1 \vee \varphi_2 \triangleq \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \Rightarrow \varphi_2 \triangleq \neg\varphi_1 \vee \varphi_2$, $\diamond\varphi \triangleq \top \mathcal{U} \varphi$, and $\square\varphi \triangleq \neg(\diamond\neg\varphi)$. The temporal connectives $\diamond$ and $\square$ are called the eventually connective and the always connective, respectively.

There are multiple possible ways to express states. We express a state as a braced associative-commutative (AC) collection of name-value pairs. AC collections are called soups, and name-value pairs are called observable components. That is, a state is expressed as a braced soup of observable components. The juxtaposition operator is used as the constructor of soups. Let oc_1, oc_2, oc_3 be observable components, and then $oc_1 \ oc_2 \ oc_3$ is the soup of those three observable components. A state is expressed as $\{oc_1 \ oc_2 \ oc_3\}$. There are multiple possible ways to specify state transitions. We specify them as rewrite rules. Concretely, we use Maude [3], a programming/specification language based on rewriting logic. Maude makes it possible to specify complex systems flexibly and is also equipped with model checking facilities (a reachability analyzer and an LTL model checker). A conditional rewrite rule (or just a rule) is in the form $\text{crl } [lb] : l \Rightarrow r \text{ if } \dots \wedge c_i \wedge \dots$, where lb is the label given to the rule and c_i is part of the condition, which may be an equation $lc_i = rc_i$. The negation of $lc_i = rc_i$ could be written as $(lc_i \neq rc_i) = \text{true}$, where $= \text{true}$ could be omitted. If the condition $\dots \wedge c_i \wedge \dots$ holds under some substitution σ , $\sigma(l)$ can be replaced with $\sigma(r)$.

The search command tries to find a state reachable from t such that the state matches p and satisfies c :

```
search [1] in M : t => * p such that c .
```

where M is a specification of the S and T parts of K . t typically represents an initial state of K .

Let $init$ be the only initial state of K and φ be an LTL formula. Then, the Maude LTL model checker checks that K satisfies φ by reducing `modelCheck (init,  $\varphi$ )`.

Figure 1. A DAG (1)

III. A* ALGORITHM

Let us consider the DAG shown in Fig. 1 in which node $n0$ is the start node and node $n3$ is the goal node. Let us use the estimation $h(n)$ for each node n defined as follows: $h(n1) = 2$, $h(n3) = 0$ and $h(n) = 1$ for any other node n . Let $w(n, n')$ be the weight of the edge from node n to node n' . For example, $w(n2, n1) = 2$. For each node n , two pieces $p(n)$ and $d(n)$ of information are maintained, where $p(n)$ is the shortest path found so far to node n from the start node at any given moment and $d(n)$ is its distance. One more piece oq of information is maintained, which is an ordered queue of node & natural number pairs, where such pairs are stored in increasing order based on their second elements. When a pair $\langle n, d \rangle$ of node n and natural number d is enqueued into oq that contains some other pairs whose second elements are d , $\langle n, d \rangle$ is stored following all those pairs. When $\langle n, d \rangle$ is enqueued into oq that contains $\langle n, d' \rangle$, if $d' < d$, then $\langle n, d \rangle$ is stored at the designated place in oq and $\langle n, d' \rangle$ is deleted and otherwise $\langle n, d \rangle$ is not stored in oq . For example, let oq consist of $\langle n0, 1 \rangle$, $\langle n1, 1 \rangle$, $\langle n3, 2 \rangle$ and $\langle n2, 3 \rangle$ in this order, denoted $\langle n0, 1 \rangle \mid \langle n1, 1 \rangle \mid \langle n3, 2 \rangle \mid \langle n2, 3 \rangle$, and if $\langle n3, 1 \rangle$ is put into oq , oq is $\langle n0, 1 \rangle \mid \langle n1, 1 \rangle \mid \langle n3, 1 \rangle \mid \langle n2, 3 \rangle$. We use `empq` to represent the empty queue and suppose that `empq` is an identity of the queue constructor $\mid$, namely that $q \mid \text{empq} = q$ and $\text{empq} \mid q = q$ for all queues q . We also suppose that a single element e is treated as a singleton queue that only consists of e . For example, $\langle n3, 1 \rangle$ is treated as the singleton ordered queue that only consists of $\langle n3, 1 \rangle$. Node n of each $\langle n, d \rangle$ in oq at any moment is called an edge node at that moment in this paper.

Initially, $p(n0) = n0$ (which is the path that only consists of $n0$), $p(n) = \varepsilon$ (the empty path) for any other node n , $d(n0) = 0$, $d(n) = \infty$ for any other node n and $oq = \langle n0, 1 \rangle$ (which is the ordered queue that only consists of $\langle n0, 1 \rangle$, where $d(n0) + h(n0) = 1$).

While oq is not empty, the following is repeated. Let $\langle n, d \rangle$ be the top element of oq . If node n is the goal node, $p(n)$ is the path to be found from the start node to the goal node. Otherwise, for each direct successor node n' of node n , if $d(n) + w(n, n') < d(n')$, $d(n')$ is set to $d(n) + w(n, n')$, $p(n')$ is set to the path obtained by adding n' to $p(n)$ at the end and $\langle n', d(n) + w(n, n') + h(n') \rangle$ is enqueued into oq . The top element is deleted from oq .

At some moment,

- $p(n0) = n0 \ \& \ d(n0) = 0$
- $p(n1) = (n0 \rightarrow n1) \ \& \ d(n1) = 4$
- $p(n2) = (n0 \rightarrow n2) \ \& \ d(n2) = 1$
- $p(n3) = (n0 \rightarrow n3) \ \& \ d(n3) = 6$
- $p(n) = \varepsilon \ \& \ d(n) = \infty$ for any other node n
- $oq = \langle n2, 2 \rangle \mid \langle n1, 6 \rangle \mid \langle n3, 6 \rangle$

$\langle n2, 2 \rangle$ is the top element of oq . $n1$ and $n5$ are the direct successor nodes of $n2$. Since $d(n2) + w(n2, n1) = 3$, $d(n1) = 4$ and $3 < 4$, $d(n1)$ is set to 3, $p(n1)$ is set to $(n0 \rightarrow n2 \rightarrow n1)$ and $\langle n1, 5 \rangle$, where $d(n2) + w(n2, n1) + h(n1) = 5$, is enqueued into oq . Since $d(n2) + w(n2, n5) = 2$, $d(n5) = \infty$ and $2 < \infty$, $d(n5)$ is set to 2, $p(n5)$ is set to $(n0 \rightarrow n2 \rightarrow n5)$ and $\langle n5, 3 \rangle$, where $d(n2) + w(n2, n5) + h(n5) = 3$, is enqueued into oq . The top element is deleted from oq . At this moment,

- $p(n0) = n0 \ \& \ d(n0) = 0$
- $p(n1) = (n0 \rightarrow n2 \rightarrow n1) \ \& \ d(n1) = 3$
- $p(n2) = (n0 \rightarrow n2) \ \& \ d(n2) = 1$
- $p(n3) = (n0 \rightarrow n3) \ \& \ d(n3) = 6$
- $p(n4) = \varepsilon \ \& \ d(n4) = \infty$
- $p(n5) = (n0 \rightarrow n2 \rightarrow n5) \ \& \ d(n5) = 2$
- $oq = \langle n5, 3 \rangle \mid \langle n1, 5 \rangle \mid \langle n3, 6 \rangle$

$\langle n5, 3 \rangle$ is the top element of oq . $n1$ and $n3$ are the direct successor nodes of $n5$. Since $d(n5) + w(n5, n1) = 3$, $d(n1) = 3$ and $3 \not< 3$, nothing changes. Since $d(n5) + w(n5, n3) = 5$, $d(n3) = 6$ and $5 < 6$, $d(n3)$ is set to 5, $p(n3)$ is set to $(n0 \rightarrow n2 \rightarrow n5 \rightarrow n3)$ and $\langle n3, 5 \rangle$, where $d(n5) + w(n5, n3) + h(n3) = 5$, is enqueued into oq . The top element is deleted from oq . At this moment,

- $p(n0) = n0 \ \& \ d(n0) = 0$
- $p(n1) = (n0 \rightarrow n2 \rightarrow n1) \ \& \ d(n1) = 3$
- $p(n2) = (n0 \rightarrow n2) \ \& \ d(n2) = 1$
- $p(n3) = (n0 \rightarrow n2 \rightarrow n5 \rightarrow n3) \ \& \ d(n3) = 5$
- $p(n4) = \varepsilon \ \& \ d(n4) = \infty$
- $p(n5) = (n0 \rightarrow n2 \rightarrow n5) \ \& \ d(n5) = 2$
- $oq = \langle n1, 5 \rangle \mid \langle n3, 5 \rangle$

$\langle n1, 5 \rangle$ is the top element of oq . $n4$ is the direct successor node of $n1$. Since $d(n1) + w(n1, n4) = 4$, $d(n4) = \infty$ and $4 < \infty$, $d(n4)$ is set to 4, $p(n4)$ is set to $(n0 \rightarrow n2 \rightarrow n1 \rightarrow n4)$ and $\langle n4, 5 \rangle$, where $d(n1) + w(n1, n4) + h(n4) = 5$, is enqueued into oq . The top element is deleted from oq . At this moment,

- $p(n0) = n0 \ \& \ d(n0) = 0$
- $p(n1) = (n0 \rightarrow n2 \rightarrow n1) \ \& \ d(n1) = 3$
- $p(n2) = (n0 \rightarrow n2) \ \& \ d(n2) = 1$
- $p(n3) = (n0 \rightarrow n2 \rightarrow n5 \rightarrow n3) \ \& \ d(n3) = 5$
- $p(n4) = (n0 \rightarrow n2 \rightarrow n1 \rightarrow n4) \ \& \ d(n4) = 4$
- $p(n5) = (n0 \rightarrow n2 \rightarrow n5) \ \& \ d(n5) = 2$
- $oq = \langle n3, 5 \rangle \mid \langle n4, 5 \rangle$

$\langle n3, 5 \rangle$ is the top element of oq . Since $n3$ is the goal node, we have found the path $n0 \rightarrow n2 \rightarrow n5 \rightarrow n3$ whose distance (or cost) is 5. The path is one of the three shortest paths from $n0$ to $n3$ in DAG (1) shown in Fig. 1.

In this paper, we take into account a graph such that it is a DAG, a goal node is reachable from a start node and each edge is given a non-negative weight.

IV. FORMAL SPECIFICATION

Let K_{A^*} be the Kripke structure formalizing A^* that tackles a DAG in which there are N nodes.

The four kinds of observable components are used:

- $(node[ni] : d, ps_1, ps_2, p) - ni$ is a node ID, d is a natural number or ∞ that is the distance of the path found so far from a start node to the node ni , ps_1 & ps_2 are soups of the direct successor node IDs of the node ni that have not yet been tackled & that have already been tackled and p is the path found so far from a start node to the node ni ; if d is ∞ , no path from a start node to the node ni has yet been found;
- $(oq : q) - q$ is an ordered queue of node ID & natural number pairs;
- $(path : npp) - npp$ is a pair of a natural number and a path; when a path from a start node to a goal node is found, the pair of the distance of the path and the path is stored in it;
- $(gstat : gs) - gs$ is either $nFin$ or Fin ; a path from a start node to a goal node has been found if gs is Fin .

Each state in S_{A^*} is expressed as $\{obs\}$, where obs is a soup of those observable components such that there is one $gstat$ observable component, there is one $path$ observable component, there is one oq observable component and there are N node observable components.

I_{A^*} consists of one state. When DAG (1) shown in Fig. 1 is tackled, the initial state is expressed as follows:

```
{(gstat: nFin)
(oq: (< n0, 0 >)) (path: (< 0, nil >))
(node[n0]: 0, (< n1, 4 > < n2, 1 > < n3, 6 >),
empty, n0)
(node[n1]: oo, (< n4, 1 >), empty, nil)
(node[n2]: oo, (< n1, 2 > < n5, 1 >), empty, nil)
(node[n3]: oo, empty, empty, nil)
(node[n4]: oo, (< n3, 1 >), empty, nil)
(node[n5]: oo, (< n1, 1 > < n3, 3 >), empty, nil)}
```

where $\langle n0, 0 \rangle$ in the oq observable component represents the singleton ordered queue that only consists of $\langle n0, 0 \rangle$, nil is the empty path, $empty$ is the empty soup, $\langle n1, 4 \rangle \mid \langle n2, 1 \rangle \mid \langle n3, 6 \rangle$ is the soup that consists of $\langle n1, 4 \rangle$, $\langle n2, 1 \rangle$ and $\langle n3, 6 \rangle$, and oo is ∞ .

T_{A^*} is specified as six rewrite rules. Let OCs be a Maude variable of observable component soups, NI & NI' be Maude variables of node IDs, D , W , D'' & D''' be Maude variables of natural numbers, D' be a Maude variable of natural numbers or ∞ , Q be a Maude variable of ordered queues, $NNPs1$, $NNPs2$, $NNPs1'$ & $NNPs2'$ be Maude variables of node ID & natural number pair soups, and L , L' & L'' be Maude variables of paths.

The first one is as follows:

```
rl [A*-stutter] : {(gstat: fin) OCs}
=> {(gstat: fin) OCs} .
```

We need to use rule `stutter` to make T_{A^*} total.

The second one is as follows:

```
cr1 [A*-goal] :
{(gstat: nFin) (oq: (< NI,D''' > | Q))
 (path: (< D'',L'' >))
 (node[NI]: D,NNPs1,NNPs2,L) OCs}
=>
{(gstat: fin) (oq: (< NI,D''' > | Q))
 (path: (< D,L >))
 (node[NI]: D,NNPs1,NNPs2,L) OCs}
if goal?(NI) .
```

`goal?(NI)` holds if `NI` is the goal node ID. When the first element `NI` of the top element $\langle NI, D''' \rangle$ of the ordered queue stored in the `oq` observable component is the goal node ID, then the path `L` has been found and $\langle D, L \rangle$, where `D` is its cost, is stored in the path observable component.

The third one is as follows:

```
cr1 [A*-srch1] :
{(gstat: nFin) (oq: (< NI,D'' > | Q))
 (node[NI]: D,< NI',W > NNPs1,NNPs2,L)
 (node[NI']: D',NNPs1',NNPs2',L') OCs}
=>
{(gstat: nFin)
 (oq: enq(< NI,D'' > | Q,
          < NI',(D + W + h(NI')) >))
 (node[NI]: D,NNPs1,< NI',W > NNPs2,L)
 (node[NI']: D + W,NNPs1',NNPs2',L -> NI')
 OCs}
if D + W < D' /\ NNPs1 != empty .
```

Rule `srch1` says that if the `gstate` observable component is `nFin` (meaning that the path has not been found), `NI` is the node ID found in the top element pair of the ordered queue stored in the `oq` observable component, `NI'` is a direct successor node ID such that it has not been tackled and the weight of the edge between `NI` and `NI'` is `W`, `D + W` is less than `D'` and `NNPs1` is not empty, then $\langle NI', (D + W + h(NI')) \rangle$ is put into the ordered queue stored in the `oq` observable component, $\langle NI', W \rangle$ is moved to the second soup from the first soup in the `node[NI]` observable component (meaning that `NI'` has been tackled), the path to `NI'` from the start node is updated as `L -> NI'` obtained by adding `NI'` to `L` at the end and the distance of the path is updated as `D + W`.

The fourth one is as follows:

```
cr1 [A*-srch2] :
{(gstat: nFin) (oq: (< NI,D'' > | Q))
 (node[NI]: D,< NI',W > NNPs1,NNPs2,L)
 (node[NI']: D',NNPs1',NNPs2',L') OCs}
=>
{(gstat: nFin)
 (oq: enq(Q,< NI',(D + W + h(NI')) >))
```

```
(node[NI]: D,NNPs1,< NI',W > NNPs2,L)
(node[NI']: D + W,NNPs1',NNPs2',L -> NI')
OCs}
if D + W < D' /\ NNPs1 = empty .
```

The only difference between the situations dealt with by rule `srch1` and rule `srch2` is whether `NNPs1` is empty. If `NNPs1` is empty, the top element pair $\langle NI, D'' \rangle$ is deleted from the ordered queue stored in the `oq` observable component.

The fifth one is as follows:

```
cr1 [A*-srch3] :
{(gstat: nFin) (oq: (< NI,D'' > | Q))
 (node[NI]: D,< NI',W > NNPs1,NNPs2,L)
 (node[NI']: D',NNPs1',NNPs2',L') OCs}
=>
{(gstat: nFin) (oq: (< NI,D'' > | Q))
 (node[NI]: D,NNPs1,< NI',W > NNPs2,L)
 (node[NI']: D',NNPs1',NNPs2',L') OCs}
if not (D + W < D') /\ NNPs1 != empty .
```

The only difference between the situations dealt with by rule `srch1` and rule `srch3` is whether `D + W < D'` holds. If `D + W < D'` does not, nothing changes except that $\langle NI', W \rangle$ is moved to the second soup from the first soup in the `node[NI]` observable component.

The sixth one `A*-srch4` deals with the case in which `D + W < D'` does not hold and `NNPs1` is empty. If so, $\langle NI', W \rangle$ is moved to the second soup from the first soup in the `node[NI]` observable component and the top element pair $\langle NI, D'' \rangle$ is deleted from the ordered queue stored in the `oq` observable component.

V. MODEL CHECKING

Let us consider a straightforward algorithm to find all paths from a start node to a goal node for a given graph. The algorithm is formalized as part of K_{all} because we do not need to use P_{all} and L_{all} .

The two kinds of observable components are used:

- $(node[ni]: nstat, n, ps_1, ps_2, ps) - ni$ is a node ID, $nstat$ is a node status (which is one of `notYet`, `visited` and `done`), n is the number of the incoming edges that have not been tackled, ps_1 & ps_2 are soups of the direct successor node IDs of the node ni that have not yet been tackled & that have already been tackled and ps is a soup of (d, p) -pairs, where p is a path from a start node to this node that has been found so far and d is the distance; if $nstat$ is `notYet`, the node has not been tackled, if it is `visited`, the node has been visited (partially tackled) and if it is `done`, the node has been fully tackled;
- $(gstat: gs) - gs$ is either `nFin` or `Fin`; all paths from a start node to a goal node have been found if gs is `Fin`.

I_{all} consists of one state. When DAG (1) shown in Fig. 1 is tackled, the initial state is expressed as follows:

```

{ (gstat2: nFin)
  (node[n0]: done, 0, (< n1, 4 > < n2, 1 > < n3, 6 >),
    empty, < 0, n0 >)
  (node[n1]: notYet, 3, (< n4, 1 >), empty, empty)
  (node[n2]: notYet, 1, (< n1, 2 > < n5, 1 >), empty,
    empty)
  (node[n3]: notYet, 3, empty, empty, empty)
  (node[n4]: notYet, 1, (< n3, 1 >), empty, empty)
  (node[n5]: notYet, 1, (< n1, 1 > < n3, 3 >), empty,
    empty) }

```

T_{all} is specified as five rewrite rules. In addition to the Maude variables above-mentioned, let NLs & NLs' be Maude variables of soups of (natural number, path)-pairs, N' is a Maude variable of natural numbers and NS' is a Maude variable of node statuses. The first rule $All-stutter$ is essentially the same as the one of T_{A*} . The second one is as follows:

```

crl [All-done] : { (gstat2: nFin)
  (node[NI]: done, 0, NNPs1, NNPs2, NLs) OCs }
=> { (gstat2: fin)
  (node[NI]: done, 0, NNPs1, NNPs2, NLs) OCs }
if goal?(NI) .

```

The rule says that if all paths to the goal node from the start node have been found, the value of the $gstat2$ observable component is set to fin .

The last three rules are as follows:

```

rl [All-srch1] :
{ (gstat2: nFin)
  (node[NI]: visited, 0, NNPs1, NNPs2, NLs) OCs }
=>
{ (gstat2: nFin)
  (node[NI]: done, 0, NNPs1, NNPs2, NLs) OCs } .

rl [All-srch2] :
{ (gstat2: nFin)
  (node[NI]: done, 0, < NI', W > NNPs1, NNPs2, NLs)
  (node[NI']: notYet, s(N'), NNPs1', NNPs2', NLs') OCs }
=>
{ (gstat2: nFin)
  (node[NI]: done, 0, NNPs1, < NI', W > NNPs2, NLs)
  (node[NI']: visited, N', NNPs1', NNPs2',
    add(NLs, NI', W)) OCs } .

crl [All-srch3] :
{ (gstat2: nFin)
  (node[NI]: done, 0, < NI', W > NNPs1, NNPs2, NLs)
  (node[NI']: NS', s(N'), NNPs1', NNPs2', NLs') OCs }
=>
{ (gstat2: nFin)
  (node[NI]: done, 0, NNPs1, < NI', W > NNPs2, NLs)
  (node[NI']: NS', N', NNPs1', NNPs2',
    NLs' add(NLs, NI', W)) OCs }
if NS' /= notYet .

```

where s of $s(N')$ is the successor function of natural numbers.

If the first and second values of the node $[NI]$ observable component are $visited$ and 0 , the node NI has been fully tackled. If that is the case, the rule $All-srch1$ changes the first value into $done$.

If the node NI has been fully tackled and has a direct successor NI' that has not been treated and the first value of the node $[NI']$ observable value is $notYet$, the first

and second values of the node $[NI']$ observable value are changed into $visited$ and the number N' obtained by decrementing the original value $s(N')$; moreover, NI' and W (which is the weight of the edge from NI to NI') are added to each path in the soup NLs of the node $[NI']$ observable value. This is done by the rule $All-srch2$.

The rule $All-srch3$ does almost the same thing as the rule $All-srch2$. The difference is that the rule $All-srch3$ deals with the case where the first value of the node $[NI']$ observable value is not $notYet$, while the rule $All-srch2$ deals with the case where it is $notYet$.

The search command, where $ALLPF$ is the specification of the straightforward algorithm and $init$ is the initial state for $ALLPF$ and the DAG shown in Fig 1,

```

search [1] in ALLPF : init
=>* { (node[n3]: done, 0, empty, NNPs2, NLs) OCs } .

```

finds all paths to the node $n3$ from the node $n0$, which are assigned to NLs :

```

NLs --> < 5, n0 -> n2 -> n5 -> n3 >
< 5, n0 -> n2 -> n1 -> n4 -> n3 >
< 5, n0 -> n2 -> n5 -> n1 -> n4 -> n3 >
< 6, n0 -> n3 > < 6, n0 -> n1 -> n4 -> n3 >

```

among which there are three shortest paths:

```

< 5, n0 -> n2 -> n5 -> n3 >
< 5, n0 -> n2 -> n1 -> n4 -> n3 >
< 5, n0 -> n2 -> n5 -> n1 -> n4 -> n3 >

```

Let $sPaths$ refer to the soup of the three shortest paths. The extraction of shortest paths from all paths to a goal node from a start node is done by a program written in Maude.

To model check that K_{A*} satisfies some desired properties, we define (or specify) P_{A*} and L_{A*} . P_{A*} has two atomic propositions: fin and $isSPath$. L_{A*} is specified as follows:

```

eq { (gstat: fin) OCs } |= fin = true .
eq { (path: (< D, L >)) OCs } |= isSPath
  = (< D, L > \in sPaths) .
eq { OCs } |= PROP = false [otherwise] .

```

The three equations say that fin holds a state s iff s contains $(gstat: fin)$ and $isSPath$ holds for a state s iff s contains $(path: (< D, L >))$ and $sPaths$ contains $< D, L >$ as an element. Let two LTL formulas $halt$ and $correct$ be defined as $<> fin$ and $[] (fin -> <> isSPath)$, where $<>$ is $\Diamond$, $[]$ is $\Box$ and $->$ is $\Rightarrow$.

Let h be as follows: $h(n1) = 2$, $h(n3) = 0$ and $h(n) = 1$ for any other node n , and $init$ be the initial state for the specification $ASTAR$ of K_{A*} and the DAG shown in Fig 1. We model check that K_{A*} satisfies $halt$ and $correct$ as follows:

```

red modelCheck(init, halt) .
red modelCheck(init, correct) .

```

Figure 2. A DAG (2)

No counterexample is found for both model checking experiments. Because h is admissible, namely that it never overestimates the actual cost to the goal $n3$ from any node, K_{A^*} satisfies `correct`.

Let us use another h : $h(n1) = 3$, $h(n5) = 4$, $h(n3) = 0$ and $h(n) = 1$ for any other node n . This h is not admissible because the actual cost to the goal $n3$ from the node $n1$ is 2 and the actual cost to the goal $n3$ from the node $n5$ is 3. The first model checking experiment does not find any counterexamples, while the second one does. This is because h is not admissible. For this case, K_{A^*} finds $n0 \rightarrow n3$, which is not the shortest one.

Let us use yet another h : $h(n1) = 3$, $h(n3) = 0$ and $h(n) = 1$ for any other node n . This h is not admissible either because the actual cost to the goal $n3$ from the node $n1$ is 2. The two model checking experiments, however, do not find any counterexamples, meaning that K_{A^*} still satisfies `correct`. In this case, K_{A^*} finds $n0 \rightarrow n2 \rightarrow n5 \rightarrow n3$. This is because for each node n in the path, $h(n)$ never overestimates the cost to the goal $n3$. This case may make us conjecture that if there exists a shortest path such that for each node n in the path $h(n)$ never overestimates the cost to the goal from n , then A^* finds a shortest path.

Let us consider the DAG shown in Fig. 2. Let us suppose that h is defined as follows: $h(n) = 0$ for all nodes n , namely that A^* is equivalent to the Dijkstra algorithm. The two model checking experiments do not find any counterexamples.

Let us suppose that h is defined as follows: $h(n) = 10$ for all nodes n . This h is not admissible. Moreover, there exists no shortest path such that for each node n in the path $h(n)$ never overestimates the cost to the goal from n . The two model checking experiments, however, do not find any counterexamples. Thus, the sufficient condition that A^* finds a shortest path can be relaxed.

A relaxed sufficient condition would be that there exists a shortest path such that for each node n in the path $h(n)$ plus the cost to n from the start node is less than the cost of any non-shortest path to the goal from the start. Let us suppose that $h(n2) = 5$, $h(n4) = 0$, $h(n5) = 3$, $h(n6) = 4$, $h(n7) = 0$, $h(n8) = 2$, $h(n10) = 0$ and $h(n) = 10$ for

any other node n . This h is not admissible but does satisfy the relaxed sufficient condition. The two model checking experiments do not find any counterexamples. Let us modify h a bit as follows: $h(n8) = 2$ and for any other node n $h(n)$ is the same as the last version. This modified h does not satisfy the relaxed sufficient condition. The first model checking experiment does not find any counterexamples, while the second one finds a counterexample, where A^* finds $n0 \rightarrow n4 \rightarrow n10$ that is not a shortest path. The two cases support that the relaxed sufficient condition is likely to make sense.

VI. CONCLUSION

We have reported on a case study in which A^* algorithm is formally specified in Maude and model checked with the Maude LTL model checker. Two properties have been taken into account: the termination property and the correctness property. The former says whether A^* halts, while the latter says whether A^* finds a shortest path. If the estimation h is admissible, our model checking experiments say that A^* satisfies both properties. Our model checking experiments have also suggested that the condition can be relaxed. We have then conjectured that if there exists a shortest path such that for each node n in the path $h(n)$ plus the cost to n from the start node is less than the cost of any non-shortest path to the goal from the start, A^* finds a shortest path.

Among future directions are as follows. Many path finding algorithms have been proposed. One piece of our future work is to formally specify such algorithms in Maude and model check them with the Maude LTL model checker, coming up with a framework in which such algorithms can be systematically analyzed. Another piece of our future work is to apply Maude and the Maude LTL model checker to path planning for multiple robots or vehicles, which could contribute to a near future autonomous vehicle world.

REFERENCES

- [1] P. E. Hart, et al., "A formal basis for the heuristic determination of minimum cost paths," *IEEE Trans. Syst. Sci. Cybern.*, vol. 4, pp. 100–107, 1968.
- [2] E. W. Dijkstra, "A note on two problems in connexion with graphs," *Numerische Mathematik*, vol. 1, pp. 269–271, 1959.
- [3] M. Clavel, et al., *All About Maude*, ser. Lecture Notes in Computer Science. Springer, 2007, vol. 4350.
- [4] A. K. Saberi, et al., "Analysis of path planning algorithms: a formal verification-based approach," in *ECAL 2013*, 2013, pp. 232–239.
- [5] N. Maragos, et al., "A formal and executable model for path finding," in *SEEFM 2003*, 2003, pp. 129–139.
- [6] A. Ulusoy, et al., "Optimality and robustness in multi-robot path planning with temporal logic constraints," *I. J. Robotics Res.*, vol. 32, pp. 889–911, 2013.