HOW NOT TO FIND AN INVERSE FUNCTION

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Abstract

An example is given to show that the standard procedure for finding the inverse function of a given one-to-one function f can lead a careless student to an incorrect answer for the domain of f^{-1} . We indicate the mathematical reason that this kind of inattentive application of the standard procedure may lead to such errors. We illustrate how to tweak the standard procedure in order to produce a more complete method that invariably leads to the correct answer to such questions. The discussion also includes some historical and pedagogical comments.

1. Introduction

Some advice that has often been given for designing a possible theory or an explanation is also appropriate for designing a list of instructions: it should be as simple as possible, but not simpler. Our purpose here is not merely to paraphrase Occam's Razor, but rather to examine a case in point, namely, the standard procedure (or "list of instructions") that is taught,

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typically in high school, for finding the inverse function f^{-1} when one has been given a one-to-one real-valued function f of a real variable.

2. Discussion

In introductory classes, the standard procedure for finding f^{-1} begins by assuming that a one-to-one function f has been given by a formula (or “rule”) $y = f(x)$ which holds for a domain D , which is typically an interval, of real numbers. (If no D has been specified, then D is tacitly assumed to be the *natural domain* determined by the formula $f(x)$, that is, the set of all real numbers r such that $f(r)$ is a real number). Rather than continuing with a list of all the steps in the standard procedure, let us examine how the standard procedure plays out in a typical example, for instance when one is given the formula $f(x) = x^3 + 4$.

One is instructed to begin by describing this given function by $y = f(x)$, that is, by $y = x^3 + 4$. In this example, a student would be expected to realize that $D = (-\infty, \infty)$, the set of all real numbers. A sufficiently cautious student would also realize, possibly using a graphing calculator, that the range of f is also the interval $(-\infty, \infty)$. Next, one is instructed to describe f^{-1} by interchanging x and y in the above formula $y = x^3 + 4$. The result is $x = y^3 + 4$. One is told to solve *this* equation for y and that the resulting formula for y would be $f^{-1}(x)$. Following that advice, one would get $f^{-1}(x) = y = (x - 4)^{\frac{1}{3}}$, with (natural) domain $(-\infty, \infty)$ and range $(-\infty, \infty)$. In this example, all is well since, as one would expect, the domain of f^{-1} turns out to be the range of f and the range of f^{-1} turns out to be the domain of f . However, we will turn next to an example where a similar use of the standard procedure leads to a different sort of situation.

In this next example, one is given $y = f(x) = \sqrt{x+8}$. As we are considering real-valued functions of a real variable, the natural domain of f is the set of all real numbers r such that $r+8 \geq 0$, that is, the interval $D = [-8, \infty)$. One finds, for instance via a graphing calculator, that the range of f is $[0, \infty)$. Then, following the standard procedure, we would get that $y = f^{-1}(x)$ can be found by solving for y in the equation $x = \sqrt{y+8}$. To solve this equation, we square both sides, finding $x^2 = y+8$, or equivalently, $y = x^2 - 8$. Thus, $f^{-1}(x) = x^2 - 8$. What is the domain of f^{-1} in this example? Students may experience some difficulty in answering this question. The *natural* domain of $x^2 - 8$ is $(-\infty, \infty)$, but that is *not* the domain of f^{-1} , since the correct answer for the domain of f^{-1} is the range of f , namely, $[0, \infty)$.

What went wrong in the second example? Why did a “natural” outgrowth of the standard methodology not produce the correct answer for the domain of f^{-1} in that example? How, if at all, should an instructor change how he/she teaches such material? We proceed to give our answers/opinions concerning these issues.

In the second example, when we passed from the equation $x = \sqrt{y+8}$ to the equation $x^2 = y+8$, we did not find an equivalent equation. In *other* words, although $x = \sqrt{y+8}$ implies that $x^2 = y+8$, the converse is false. In *other* words, $x^2 = y+8$ does not imply that $x = \sqrt{y+8}$. For instance, by taking $x = -3$ and $y = 1$, we see that $x^2 = 9 = 1+8 = y+8$ but $x = -3 \neq 3 = \sqrt{9} = \sqrt{1+8} = \sqrt{y+8}$. (Notice that no such difficulty arose in the first example, since the equation $x = y^3 + 4$ is equivalent to the equation $y = (x-4)^{\frac{1}{3}}$).

Let us pursue the above analysis of the second example more deeply. For that example, we found (correctly) that $f^{-1}(x) = x^2 - 8$; the natural domain of the function given by the expression $x^2 - 8$ is $(-\infty, \infty)$; and that $(-\infty, \infty)$ is not the domain of f^{-1} , which is “only” $[0, \infty)$.

Where did the “other” real numbers in $(-\infty, \infty)$ go? Why did $(-\infty, 0)$ disappear from the correct answer to finding the domain of f^{-1} ?

The answer to the last question comes from the following two facts: we are dealing with domains and ranges that are subsets of the set of real numbers; and we are using a definition of “function” that has been standard since early in the 20th century. Such was not always the case, at least not for undergraduate students. When I was an undergraduate (approximately 60 years ago), students were expected to deal with what were called “many-valued functions,” a prime example being given by $z = x^2$ leading to the “two-valued function” described by $x = \pm\sqrt{z}$. While such an “equation” could be used to introduce future mathematicians, engineers and physicists to concepts such as the “sheets” of a Riemann surface, it hardly seemed appropriate in mid-century, when “New Math” revolutionized curricula in response to Sputnik, to burden undergraduates, let alone school children, with such subtleties. Accordingly, it was (quite rightly, in my opinion), decided that “many-valued functions” would be banished from school curricula and that all functions considered (at least through the study of the typical calculus sequence) would be (single-valued) real-valued functions of a real variable. (It is not an overstatement to say that the curricular changes that were brought about by “New Math” were revolutionary. In particular, I can attest that “many-valued functions” were part of a typical school experience in the late 1950s, as I recall my class being told by our high school algebra teacher that $\sqrt{4} = \pm 2$).

We will next explain that the numbers in $(-\infty, 0)$ did not “disappear” from the domain of f^{-1} in the second example that was discussed above.

We will also see that these numbers do have a function-theoretic role to play in relation to the function f . Recall that f was given by $y = f(x) = \sqrt{x+8}$, with domain $[-8, \infty)$ and range $[0, \infty)$. In finding a formula for $f^{-1}(x)$, we passed from the equation $x = \sqrt{y+8}$ to the inequivalent equation $x^2 = y+8$. What the equation $x^2 = y+8$ is equivalent to is “Either $x = \sqrt{y+8}$ or $x = -\sqrt{y+8}$ ”. In fact, let us see what happens when we consider the function g defined by $g(x) = -\sqrt{x+8}$. Observe that g has natural domain $[-8, \infty)$ and range $(-\infty, 0]$. The standard procedure leads easily to $g^{-1}(x) = x^2 - 8$. Moreover, the domain of g^{-1} is the range of g , namely, $(-\infty, 0]$, and the range of g^{-1} is the domain of g , namely, $[-8, \infty)$. In particular, the numbers in $(-\infty, 0)$ have “reappeared”, as they (when united with the singleton set $\{0\}$) constitute the domain of g^{-1} .

One should say a word about the one-point discrepancy between the sets $(-\infty, 0)$ and $(-\infty, 0]$. Recall that the equation $x^2 = y+8$ is equivalent to “Either $x = \sqrt{y+8}$ or $x = -\sqrt{y+8}$ ”, that is, to $x = \pm\sqrt{y+8}$. Notice that this “many-valued” expression is ambiguous except when $y = -8$, that is, it is ambiguous except when $x = 0$. In describing the domain for f^{-1} as $[0, \infty)$, we needed to include 0 in order to account for the fact that 0 is in the range of f . Similarly, in describing the domain for g^{-1} as $(-\infty, 0]$, we needed to include 0 in order to account for the fact that 0 is in the range of g . However, from the perspective of the function f , the numbers in $(-\infty, 0)$ are the redundancies (or, if you will, the ambiguities) that can be ignored in describing the function f^{-1} (and its domain).

Recall that, while presenting the second example, we wrote that the “*natural* domain of $x^2 - 8$ is $(-\infty, \infty)$, but that is *not* the domain of f^{-1} ”. One can, quite reasonably, ask about the function h defined by the expression

$h(x) = x^2 - 8$. This function has (natural) domain $(-\infty, \infty)$ and range $[-8, \infty)$. Moreover, the function h is not one-to-one. In particular, h is not an inverse function and h does not have an inverse function. All the more so, $h \neq f^{-1}$.

We have somewhat pilloried the above function h . However, we did not mean to disparage its authenticity as a function. In fact, the failure of h to be one-to-one can play a role in a classroom explanation of how the modern function-theoretic attitudes can accommodate the goals of yesteryear's approach via "sheets." Recall that h was given by $h(x) = x^2 - 8$, with domain $(-\infty, \infty)$ and range $[-8, \infty)$. Consider two functions, h_1 and h_2 , defined as follows: $h_1(x) = x^2 - 8$, with domain $(-\infty, 0]$ and range $[-8, \infty)$; and $h_2(x) = x^2 - 8$, with domain $[0, \infty)$ and range $[-8, \infty)$. In other words, h_1 is the restriction of h (from its natural domain) to the domain $(-\infty, 0]$, and h_2 is the restriction of h to the domain $[0, \infty)$. The introduction of h_1 and h_2 is appropriate because (unlike h) both h_1 and h_2 are one-to-one functions (and hence have inverse functions), and both h_1 and h_2 have the same range as h . Applying the standard procedure, one finds easily that h_1^{-1} can be described by $h_1^{-1}(x) = -\sqrt{x+8}$ with domain $[-8, \infty)$ and range $(-\infty, 0]$; and that h_2^{-1} can be described by $h_2^{-1}(x) = \sqrt{x+8}$ with domain $[-8, \infty)$ and range $[0, \infty)$. The upshot is that by using the formula that defined $h(x)$ to "split" the definition of h into the definition of two one-to-one functions h_1 and h_2 whose domains overlap only at $x = 0$, we have recovered some of the formerly perplexing data. Indeed, even though h does not have an inverse function, we do have the functional identities $h_1^{-1} = g$ and $h_2^{-1} = f$. Equivalently, one has the functional identities $f^{-1} = h_2$ and $g^{-1} = h_1$. Thus, by reformulating the "many-valued" expression $y = \pm\sqrt{x+8}$ in terms of the two functions h_1 and h_2 (with their natural

domains being the same as that of the expression $\sqrt{x+8}$), we see that f^{-1} is indeed the function h_2 and that the formerly “missing” numbers in $(-\infty, 0)$ play a natural function-theoretic role by forming nearly all the domain of h_1 .

The above explanations have been given from a number of points of view and, we hope, will be useful to instructors. However, we can not, and should not, expect the typical student to rapidly digest those explanations upon first meeting the concept of an inverse function. What seems to be called for, if one wishes to build a more foolproof “standard procedure” for finding the inverse function of a given one-to-one function f , is a clear final step in the list of instructions (after one can obtain an expression for $f^{-1}(x)$) stipulating that the domain of f^{-1} is the range of f . With that instruction in hand, a student should not be tempted to consider irrelevancies such as the above function h (with its overly large domain). One may ask whether such an editing of some standard textbooks would be appropriate. In reply, I will simply describe what can be found in the three textbooks that I have used during the last five years while teaching some below-calculus courses at two state-supported colleges and universities in South Carolina.

If an introductory class has been given a one-to-one function f by means of a formula (or “rule”) $f(x)$ that holds for all elements x in a domain D , virtually all textbooks at that level agree on the first four steps of a standard procedure for finding f^{-1} , as follows. First, write $y = f(x)$. Second, interchange x and y in the last equation, thus obtaining $x = f(y)$. Third, solve the equation $x = f(y)$ for y in terms of x , thus obtaining an expression $(y =) e(x)$. (Notice that the third step can be carried out, at least in principle, because f is known to be one-to-one.) Fourth, set $f^{-1}(x) := e(x)$. Of course, in order for this formula or rule to lead to a definition of f^{-1} as a function, one also needs to specify the domain of f^{-1} . Thus, it seems clear that the

standard procedure needs at least one more step. Our extensive discussion of the above second example showed that one cannot, in general, take the domain of f^{-1} to be the natural domain of the expression $e(x)$. Since *that* should not be the fifth step of the general procedure, let us survey what sort of fifth step is offered by the three textbooks that I mentioned.

Some material on pages 722-724 of the textbook [3] is titled “Finding the equation of the inverse of a function,” and the above four steps are given at the top of page 723. No fifth step is given, and the worked examples on page 723 which are given to illustrate the four-step procedure do not suggest that a fifth step is needed. Perhaps one’s criticism of [3] should not be severe. After all, [3] is primarily an algebra text that, in our view, chose to somewhat de-emphasize functions by discussing the “natural domain” of an algebraic expression (rather than of a function). But I was, frankly, disappointed that [3] was content to find “the equation of the inverse of a function” without actually specifying the domain of that inverse function.

I had been required to use [3] in teaching a course at a technical college. Later, when I was teaching a basic freshman course for non-STEM majors at a branch campus of a state university, I was happy to be allowed to choose the text for that course. I chose [4]. As was true for [3], it is also the case that [4] is a textbook on what is commonly known as “Intermediate Algebra.” Unlike [3] however, [4] does give a fifth step for the standard procedure for finding f^{-1} . On pages 643-646 of [4], the worked examples indicate that the fifth step is to show that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. (This fifth step, called “Check,” is carried out in full only in Examples 5 and 7 on page 642-644). I find that fifth step to be unacceptable for two reasons. First, the set containing the elements x that are appropriate in the first equation is conceptually (and in many examples, actually) distinct from the set containing the elements x that are appropriate in the second equation. In my opinion, this lack of clarity about domains and ranges de-emphasizes the importance of the concept of a function. In some examples, a competent student would be hard-pressed to know which elements x to test when

carrying out this “Check.” My second objection to the fifth step in [4] is that it is too time-consuming, even if one clarifies it by further stipulating that x is to be taken from the range of f in the first equation and from the domain of f in the second equation. With that stipulation, the fifth step in [4] is nothing more than a restatement of the assertion that f^{-1} is the inverse function of f . It would be quicker (and sufficient) to do what I recommended two paragraphs ago, namely, to simply have a fifth step that stipulates that the domain of f^{-1} is the range of f .

It is natural to wonder if a suitable fifth step is given in a higher-level textbook, such as [1], which I used in teaching a Precalculus course at the above-mentioned technical college. Although [1] organizes the rules for the general procedure (for finding f^{-1}) differently, it does include a final step that goes beyond simply finding a formula for $f^{-1}(x)$: see page 248 of [1]. Unfortunately, that final step in [1] is essentially the same as the fifth step in [4], and so it suffers from the two criticisms that were mentioned above in the discussion of [4]. To the credit of its author, [1, page 253 ff] does contain a section titled, “Finding the inverse of a domain-restricted function.” In that section, the text studies the inverse function of the function f , where $f(x) = x^2 - 1$ and the domain of f is $[0, \infty)$. Unfortunately, in that example, the text does not explicitly identify the domain or the range of f^{-1} ; but, to be fair, that information can be inferred from the graph of f^{-1} in Figure 1.72 on page 253 of [1].

With the recent experience of teaching from a small sample of three textbooks, I was led to ask whether any text below the level of calculus has listed an effective and succinct fifth step for the standard procedure of finding an inverse function. On looking at the Precalculus text [2] that John C. Peterson and I wrote some 25 years ago, I found that the instructions for finding f^{-1} and the relevant worked examples on pages 125-126 of [2] were succinct, clear and adequate (and that the worked examples were somewhat

more comprehensive than in the other texts that have been mentioned here). As the publication of [2] does not seem to have caused the community of mathematics textbook authors to follow its example, I decided to write this note and to make my case for the best formulation of the elusive fifth step. This note has already stated twice how, in my opinion, that fifth step should be formulated. In the final two paragraphs, I will make that suggestion again, this time setting the advice in terms of some general pedagogic principles.

While the points of view that were expressed above concerning some historical matters may resonate with some readers, we must address the more urgent matter of how to teach students to find the domain of f^{-1} in examples such as the second example given above (where $y = f(x) = \sqrt{x+8}$). First, do no harm (in other words, do not lie). Second, as any parent knows, an age-appropriate answer should be given. Third, as any communicator knows, there is a natural limit to the number of facts or principles that an audience can be expected to take away from any educational encounter. With these three precepts in hand, I suggest that one should solve problems such as the above two examples as follows. After finding a formula for $f^{-1}(x)$ by the four-step standard procedure, simply remind the class that the domain of f^{-1} is the already-known range of f (and that the range of f^{-1} is the already-known domain of f). Moreover, if a student in an introductory class notices that (in some examples) the natural domain of the expression for $f^{-1}(x)$ is a larger set than the actual domain of f^{-1} , I suggest that the instructor point out that a step in the analysis was not reversible (just as I noted in the second example that $x^2 = y + 8$ does not imply $x = \sqrt{y+8}$).

If an instructor will follow the advice in the preceding paragraph, I believe that some good will accrue. First, in the spirit of the title of this note, he/she will have tweaked (or clarified) the standard procedure to aid students in invariably getting the correct answer to a certain kind of question. Second,

the instructor will be practicing the three precepts in the preceding paragraph, which, in my opinion, should be guiding principles for all instructors. Finally, in the spirit of the curricular reform of the past two or three decades, the clarity provided by such function-oriented instruction in introductory courses of mathematics may help society to recognize that mathematics can, and should, be a “pump and not a filter”.

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