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The Determinants over Associative Rings: a Definition, Properties, New Formulas and a Computational Complexity

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We give a new definition for the determinants over an associative ring $\mathbf{Q}$ and study their properties. In particular, we obtain a new family of polynomial identities (computational formulas) for these determinants that contain up to $n!$ free variables.

Keywords: determinants, associative rings, noncommutative variables, the polarization theorem, polynomial identities.

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Introduction

Recently the author in [1,2] has found a new family of formulas with free variables (polynomial identities) for the determinant over the commutative ring K . These formulas have allowed me to give a new definition for determinants (an e-determinant $edet(A)$) over a wide class of rings [1,3] and to establish their basic properties (without proof). These definitions (for a noncommutative case) are closely related to a definition of the symmetrized Barvinok determinant $sdet(A)$ [4].

Here the complete proof of statements from [3] for $edet(A)$ over the associative ring $\mathbf{Q}$ is given. With the help of known theorem of polarization [5,6] to obtain a new family of polynomial identities for $edet(A)$ that contain up to $n!$ free variables. The basic properties $edet(A)$ are proved. These properties coincide with many well-known properties of a determinant over a commutative ring. The necessary and sufficient conditions of equality $edet(A) = 0$ are found, in particular. We also estimate a computational complexity of the obtained formulas for determinants. Some prospects of the basic results are specified.

1. Notations and definitions

Let K be a commutative ring, $\mathbf{K}$ a noncommutative associative ring, $\mathbf{Q}$ a noncommutative ring with associative n -powers (one-monomial associativeness), and $\mathbf{Q}$ be an associative ring; let also each ring be with division by integers. Let $A = (a_{ij})$ be an $n \times n$ matrix with elements from the ring K . Let S_n be the set of all permutations $\sigma = (\sigma(1), \dots, \sigma(n))$ of the set $\{1, \dots, n\}$, $\tau(\sigma)$ be the number of inversions in σ . Let $S_n^{(e)}$ and $S_n^{(o)}$ be the subsets of even and odd permutations in S_n , respectively. We call the sequence of elements of $a_{1\sigma(1)}, \dots, a_{n\sigma(n)}$ the diagonal $l(\sigma)$ of matrix A , the sequence of elements $a_{i_1\sigma(i_1)}, \dots, a_{i_k\sigma(i_k)}$, $1 \leq i_1 < \dots < i_k \leq n$, the subdiagonal l

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of length k of matrix A . We denote by $L_k^{(e)}(L_k^{(o)})$ the set of all subdiagonals of length k of $L_n^{(e)}(L_n^{(o)})$, $k = 1, \dots, n$.

Definition 1. A classic definition for the determinant over the commutative ring K :

$$\det(A) := \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} a_{1\sigma(1)} \dots a_{n\sigma(n)} = \frac{1}{n!} \sum_{\mu \in S_n} \sum_{\sigma \in S_n} (-1)^{\tau(\mu) + \tau(\sigma)} a_{\mu(1)\sigma(1)} \dots a_{\mu(n)\sigma(n)}. \quad (1)$$

One of natural generalizations of a determinant (1) on noncommutative rings has been made in 2000 by A. Barvinok who has successfully applied it for construction of an algorithm to calculate permanents [4]. This definition of determinant differs from the well-known determinant of Dieudonné over a division ring, the quasideterminant [7] and other well-known determinants over noncommutative associative rings [8].

Definition 2. The definition of symmetrized Barvinok determinant $s\det(A)$ over the ring $\mathbf{K}$, [4]:

$$s\det(A) := \frac{1}{n!} \sum_{\mu \in S_n} \sum_{\sigma \in S_n} (-1)^{\tau(\mu) + \tau(\sigma)} a_{\mu(1)\sigma(1)} \dots a_{\mu(n)\sigma(n)} = \quad (2)$$

$$= \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} \text{Sym}(a_{1\sigma(1)} \times a_{2\sigma(2)} \dots \times a_{n\sigma(n)}), \quad (3)$$

then the symmetrization operator Sym on the product elements $x_1, x_2, \dots, x_n \in \mathbf{K}$ is defined to be the following sum

$$\text{Sym}(x_1 x_2 \dots x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}. \quad (4)$$

The classical polarization theorem of recovering a polyadditive symmetric function from its values on a diagonal is fundamental in the algebraic theory of polynomials. This theorem is often applied in the theory of rings, the linear and polylinear algebra, the theory of functions, the theory of mixed volumes and in other areas of mathematics. We use it in the following simple and general enough form.

Theorem 1 (The polarization theorem, Egorychev, [6]). *Let H be the additive commutative semigroup $H^{(n)} = H \times \dots \times H$. The function $f(x) = f(x_1, \dots, x_n): H^{(n)} \rightarrow \mathbb{C}$, is symmetric and polyadditive if and only if for any γ from H it admits the following representation (the polarization formula):*

$$\begin{aligned} f(x_1, \dots, x_n) = & \left\{ (-1)^n f(\gamma, \dots, \gamma) + (-1)^{n-1} \sum_{i=1}^n f(\gamma + x_i, \dots, \gamma + x_i) + \right. \\ & + (-1)^{n-2} \sum_{i < j} f(\gamma + x_i + x_j, \dots, \gamma + x_i + x_j) + \dots + \\ & + \sum_{1 \leq j_1 < \dots < j_k \leq n} f\left(\gamma + \sum_{l=1}^k x_{j_l}, \dots, \gamma + \sum_{l=1}^k x_{j_l}\right) + \dots + \\ & \left. f\left(\gamma + \sum_{i=1}^n x_i, \dots, \gamma + \sum_{i=1}^n x_i\right) \right\} / n!. \end{aligned} \quad (5)$$

2. Determinants over an associative ring

We give new

Definition 3. The definition of e -determinant $edet(A)$ over the associative ring $\mathbf{Q}$ (Egorychev, [1], [3]):

$$edet(A) := \frac{1}{n!} \left\{ \left(\sum_{l \in L_n^{(e)}} su^n(l) - \sum_{l \in L_n^{(o)}} su^n(l) \right) - \left(\sum_{l \in L_{n-1}^{(e)}} su^n(l) - \sum_{l \in L_{n-1}^{(o)}} su^n(l) \right) \right\}, \quad (6)$$

where the function $su(l)$ on the sequence l of elements $x_1, x_2, \dots, x \in \mathbf{Q}$ is defined to be the sum of its elements.

In the following theorem we employ the polarization theorem to obtain a new family of polynomial identities for $edet(A)$ that contain up to $n!$ free variables.

Theorem 2 ([3]). *If $A = (a_{ij})$ is an $n \times n$ matrix over the ring $\mathbf{Q}$, then the following formulas are valid*

$$edet(A) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} \left\{ (-1)^n \gamma_\sigma^n + \sum_{k=1}^n \left((-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma_\sigma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n \right) \right\}, \quad (7)$$

where $\{\gamma_\sigma\}_{\sigma \in S_n}$ is the set of $n!$ free variables $\gamma_\sigma \in \mathbf{Q}$.

Proof. If the ring $\mathbf{Q}$ is commutative then the validity of the formula (7) has been established by the author in [2]. If $\mathbf{Q} = \mathbf{K}$ then the formula (7) for $edet(A)$ has been mentioned (without proof) in [3]. Its validity directly follows from several simple statements of Lemmas 3–5. $\square$

Lemma 3. *The formula (7) is valid for the determinant $sdet(A)$ over the ring $\mathbf{K}$ in (3).*

Proof. As the symmetrization operator (4) is a symmetric and polyadditive function of its variables, then by means of the polarization formula for each diagonal product $a_{1\sigma(1)} \dots a_{n\sigma(n)}$, $\sigma \in S_n$, of elements from $\mathbf{K}$ we obtain

$$\begin{aligned} \text{Sym}(a_{1\sigma(1)} \times a_{2\sigma(2)} \dots \times a_{n\sigma(n)}) &= \\ &= \frac{1}{n!} \left\{ (-1)^n \gamma_\sigma^n + \sum_{k=1}^n \left((-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma_\sigma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n \right) \right\}, \quad (8) \end{aligned}$$

where γ_σ is an arbitrary variable from $\mathbf{K}$. To complete the proof it is enough to replace under the sign of the sum (3) each summand by (8). $\square$

Lemma 4. *The following formula is valid for the determinant $sdet(A)$ over the ring $\mathbf{K}$:*

$$\begin{aligned} sdet(A) &= \frac{1}{n!} \left\{ \left(\sum_{l \in L_n^{(e)}} (\gamma + su(l))^n - \sum_{l \in L_n^{(o)}} (\gamma + su(l))^n \right) - \right. \\ &\quad \left. - \left(\sum_{l \in L_{n-1}^{(e)}} (\gamma + su(l))^n - \sum_{l \in L_{n-1}^{(o)}} (\gamma + su(l))^n \right) \right\}, \quad (9) \end{aligned}$$

for any $\gamma \in \mathbf{K}$. In particular for $\gamma = 0$

$$sdet(A) = \frac{1}{n!} \left\{ \left(\sum_{l \in L_n^{(e)}} su^n(l) - \sum_{l \in L_n^{(o)}} su^n(l) \right) - \left(\sum_{l \in L_{n-1}^{(e)}} su^n(l) - \sum_{l \in L_{n-1}^{(o)}} su^n(l) \right) \right\}. \quad (10)$$

Proof. Put in (7) $\gamma_\sigma = \gamma \in \mathbf{K}$ for each $\sigma \in S_n$. Then we have

$$\begin{aligned} sdet(A) &= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} \left\{ (-1)^n \gamma^n + \sum_{k=1}^n \left((-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n \right) \right\} = \\ &= \frac{(-1)^n}{n!} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} \gamma^n + \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \left\{ \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n \right\} = \end{aligned}$$

$$= \frac{(-1)^n}{n!} \left\{ \sum_{\sigma \in S_n^{(e)}} \gamma^n - \sum_{\sigma \in S_n^{(o)}} \gamma^n \right\} + \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \left\{ \sum_{\sigma \in S_n^{(e)}} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n - \sum_{\sigma \in S_n^{(e)}} \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\gamma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n \right\}. \quad (11)$$

It is easy to see that the first term in the right part of (11) is equal to zero, as for any $n \geq 2$, obviously $|S_n^{(e)}| = |S_n^{(o)}|$. Similarly, if we fix any k , $k = 1, \dots, n-2$ and any set elements $a_{j_1 \sigma(j_1)}, a_{j_2 \sigma(j_2)}, \dots, a_{j_s \sigma(j_s)}$ of matrix A then, obviously, this set of elements contains exactly $(n-k)!/2$ even and $(n-k)!/2$ odd permutations from the set S_n . Thus, under these assumptions any term $\left(\gamma + \sum_{s=1}^k a_{j_s \sigma(j_s)} \right)^n$ appears in the sum in the right part of the equality (11) the equal number of times with different signs, and therefore their sum is equal to zero. The remaining terms at $k = n-1$ and $k = n$ coincide with the expression in the right part of the equality (9) for $s \det(A)$. $\square$

The following assertion follows from the equality of expressions (6) and (10) for $\text{edet}(A)$ and $\det(A)$.

Lemma 5. *The following formula is valid for the determinant $\text{edet}(A)$ over the ring $\mathbf{K}$:*

$$\text{edet}(A) = \det(A). \quad (12)$$

Therefore for $\text{edet}(A)$ over $\mathbf{K}$ the formula (7) and its corollaries are valid.

Theorem 6 ([3]). *The properties $\text{edet}(A)$ over the associative ring $\mathbf{Q}$:*

- (a) $\text{edet}(A)$ is a polyadditive function of rows and columns of matrix A ;
- (b) $\text{edet}(A)$ is an antisymmetric function of rows and columns of matrix A ;
- (c) $\text{edet}(A) = 0$, if the matrix A contains zero row or zero column;
- (d) $\text{edet}(A) = 0$, if the matrix A contains two equal rows (columns);
- (e) $\text{edet}(A) = \text{edet}(A^T)$;
- (f) $\text{edet}(I_n) = e$, if the ring $\mathbf{Q}$ contains the unit e , and I_n is the identity $n \times n$ matrix.

In general case the Laplace formulas and the formula $\text{edet}(AB) = \text{edet}(A) \cdot \text{edet}(B)$ are not valid.

Proof. If the ring $\mathbf{Q}$ is commutative, then the validity of the properties (a) – (f) has been established by the author in [2]. If $\mathbf{Q} = \mathbf{K}$, then we use the equality $\text{edet}(A) = s \det(A)$ over $\mathbf{K}$ (Lemma 5). The rest of the proof of these properties goes along the well-known scheme for the determinant over the commutative ring $\mathbf{K}$. $\square$

Corollary 7. *If $A = (a_{ij})$ is an $n \times n$ matrix over $\mathbf{Q}$ then for $t = 1, 2, \dots, n-1$ following identities are valid:*

$$\sum_{l \in L_n^{(e)}} su^t(l) - \sum_{l \in L_n^{(o)}} su^t(l) = \sum_{l \in L_{n-1}^{(e)}} su^t(l) - \sum_{l \in L_{n-1}^{(o)}} su^t(l), \quad (13)$$

and $\text{edet}(A) = 0$ iff the following identity holds:

$$\sum_{l \in L_n^{(e)}} su^n(l) - \sum_{l \in L_n^{(o)}} su^n(l) = \sum_{l \in L_{n-1}^{(e)}} su^n(l) - \sum_{l \in L_{n-1}^{(o)}} su^n(l). \quad (14)$$

The coefficients at each power γ^t , $t = 1, \dots, n$ of the polynomial in the right part of (9) are equal to zero, as coefficients of this polynomial identity with respect to a free variable γ . Now to prove (13) it is enough to count up these coefficients by differentiation by γ of the right part of (13) or, equivalently, using the binomial formula (with noncommutative variables) for each term in the right part of (9).

Remark 8. In [2] it is proved, that the equality (14) can be considered as a (new) criterion for linear independence of rows (columns) of the matrix A over the commutative ring K . If the rows (the columns) of the matrix A over the noncommutative ring $\mathbf{K}$ are linearly dependent, then due to theorem 2 $\text{edet}(A) = 0$, i. e. equality (14) holds. However, it is obvious, that the converse is not always true.

Lemma 9 (The computational complexity of $\text{edet}(A)$ over the ring $\mathbf{K}$). *The computation $\text{edet}(A)$ by formulas (7) and (10) for large n requires (at most) of order $(2n-1) \times (n+1)!$ additions and $(n+1)! \times \ln n$ multiplications.*

Proof. In [9] an (optimal) algorithm is given that allows for large n to compute 2^n (starting with 2) by $\ln n$ operations of multiplication. In turn, it is easy to count that the sum (7) contains only $(n+1)!$ summands (n -powers). $\square$

Remark 10. It is clear that the sum (2) for $\text{sdet}(A)$ contains $(n!)^2$ terms. Thus the traditional approach of computation $\text{sdet}(A)$, that uses computation of each term of (determinant) sum (2) requires of order $(2n-1) \times (n!)^2$ additions and $(n-1) \times (n!)^2$ multiplications.

3. Conclusion

Here we make several short remarks explaining the basic results of this article and some prospects of their use.

Remark 11. (a) The formulas (7)–(14) are obtained using all characteristic properties of the e -determinant (compare [10]);

(b) Formula (7) for $n!$ free variables $\{\gamma_\sigma\}_{\sigma \in S_n}$ generates $2^{n!}$ various polynomial identities if we put each $\gamma_\sigma = 0$ or $\gamma_\sigma \neq 0$. Each of these formulas can be taken as the definition for $\text{edet}(A)$, it requires its own number of arithmetical operations when calculated and according to Corollary 7 generates a new set of identities;

(c) Formula (7) and its special cases employ (besides divisions by $n!$) only operations $+$, $-$ and raising to power n . Therefore the formula (7) can be used as a definition of the e -determinant over noncommutative ring Q with associative n -powers, keeping the properties (a)–(f) of Theorem 6.

It is of interest to obtain similar results for Shur functions, the mixed discriminants and many other matrix functions of planar and space matrices related to determinants. Our results may find applications in the theory of permanents [11], the tensor algebra and its applications, the theory of n -Lie algebras [12, 13] and the theory of associative rings [14] and of noncommutative skew fields [15], the differential geometry [16] and others.

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Детерминанты над ассоциативными кольцами: определение, свойства, новые формулы и сложность вычисления

Георгий П. Егорычев

Дано новое определение детерминантов над ассоциативным кольцом $\mathbf{Q}$, и изучены их свойства. В частности, для них получено семейство полиномиальных тождеств (формул вычисления), содержащих до $n!$ свободных параметров.

Ключевые слова: детеминанты, ассоциативные кольца, некоммутативные переменные, теорема поляризации, полиномиальные тождества.