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Minimal Algebras of Unary Multioperations

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A matrix impression of algebras of unary multioperations of a finite rank and the list of the identities which are carried out in such algebras are gained. These results are used for the proof of the main result: descriptions of the minimal algebras of unary multioperations of a finite rank. As a result the list of all such minimal algebras for small ranks is received.

Keywords: multioperation, algebra, minimal algebra, matrix, operation, substitution.
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Introduction

Algebras of unary multioperations which are considered in this paper are finite algebras. Description of minimal algebras is important to study the structure of these algebras [1]. A description of all algebras of unary multioperations of rank 3 was obtained in [2]. The main result of this paper was announced in [3]. We note that algebras of unary multioperations are used for the study of the superclones and hence the clones [4].

Let $B(A)$ be the set of all subsets of A. A mapping from A into $B(A)$ is called *unary multioperation* on A. The set of all unary multioperations on A will be denoted by M_A^1.

Multioperation f on finite set $A = \{a_0, \dots, a_{k-1}\}$ can be represented as mapping

$$f : \{2^0, 2^1, \dots, 2^{k-1}\} \to \{0, 1, \dots, 2^k - 1\},$$

which is obtained from f by coding $a_i \to 2^i; \varnothing \to 0; \{a_{i_1}, \dots, a_{i_s}\} \to 2^{i_1} + \dots + 2^{i_s}$.

And multioperation f is represented by vector $(\alpha_0, \dots, \alpha_{k-1})$, where $f(a_i) = \alpha_i$, using the coding.

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Let $S \subseteq M_A^1$. Algebra $F =< S; *, \cap, \mu, \varepsilon, \theta, \pi >$ with operations of substitution $(f * g)$, intersection $(f \cap g)$, reversibility (μf) and nullary operations ε, θ, π is called *algebra of unary multioperations* on A:

$(f * g)(a) = \{b|$ there exists $c \in g(a)$ such that $b \in f(c)\}$;

$(f \cap g)(a) = f(a) \cap g(a)$;

$(\mu f)(a) = \{b | a \in f(b)\}$;

$\varepsilon(a) = \{a\}$;

$\theta(a) = \varnothing$;

$\pi(a) = A$.

The power of set A is called *rank* of algebra. Further we believe that rank is finite and equal $k \geqslant 2$.

We note some simple properties of operations of algebra of unary multioperations:
$f*(g*h) = (f*g)*h,\ f \cap (g \cap h) = (f \cap g) \cap h,\ f \cap g = g \cap f,\ \mu(\mu f) = f,\ \mu(f \cap g) = \mu f \cap \mu g,$
$\mu(f*g) = \mu g * \mu f,\ f*\varepsilon = \varepsilon * f = f,\ \theta * f = f * \theta = \theta,\ f \cap \pi = f,\ f \cap \theta = \theta,\ \mu\varepsilon = \varepsilon,\ \mu\theta = \theta,\ \mu\pi = \pi.$

There is the following matrix representation of algebras of unary multioperations.

Let $B =< \{0, 1\}; *, + >$ be two-element Boolean algebra. Boolean matrices are binary matrices on the elements which define the Boolean operations.

For unary multioperation f on A we define Boolean square matrix $M_f = (\alpha_{ij})$ of order k as follows: $\alpha_{ij} = 1$ if $a_i \in f(a_j)$ else $\alpha_{ij} = 0$.

Operations of algebra of unary of multioperations are represented by matrix operations in the following way:

$M_{f*g} = M_f * M_g$ is matrix multiplication;

$M_{f \cap g} = M_f \circ M_g$ is element-wise matrix multiplication;

$M_{\mu f} = M_f^T$ is transposition of matrix;

$M_\varepsilon = E$ is diagonal matrix;

$M_\theta = O$ is null matrix;

$M_\pi = P$ is unit matrix.

For example, unary multioperation in vector form $f = (3, 7, 1)$ is represented by matrix

$$M_f = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The main result

The smallest algebra which not equal trivial algebra consisting of only multioperations π, θ, ε is called *minimal algebra* of unary multioperations. It is obvious that necessary and sufficient condition for minimality of algebra of unary multioperations is the generating of any its multioperation which not equal π, θ, ε. The following theorem describes the multioperations generating minimal algebras of unary multioperations.

Theorem 1. *Multioperation f on A which not equal π, θ, ε generates minimal algebra of unary multioperations of rank k if and only if it satisfies one of the following conditions:*
1) $f \cap \varepsilon = \varepsilon,\ \mu f = f,\ f^2 = f$;
2) $f \cap \varepsilon = \varepsilon,\ \mu f = f,\ f^2 = \pi$;
3) $f \cap \varepsilon = \mu f \cap f = \varepsilon,\ f * \mu f = \mu f \cap f = \pi,\ f^2 = f$;
4) $f \cap \varepsilon = \mu f \cap f = \varepsilon,\ f * \mu f = \mu f \cap f = \pi,\ f^2 = \pi$;

5) $f \cap \varepsilon = \theta,\ \mu f = f,\ f^2 = \pi$*;*
6) $f \cap \varepsilon = \theta,\ \mu f = f^{p-1},\ f^p = \varepsilon$, *where* p *is simple divisor of* k*;*
7) There exists not empty set $B \subsetneq A$ *such that*
either $f(a) = B$ *for all* $a \in A$*,*
or $f(b) = \{b\}$ *for all* $b \in B$ *and* $f(a) = \varnothing$ *for all* $a \in A \setminus B$*,*
or $f(b) = A$ *for all* $b \in B$ *and* $f(a) = \varnothing$ *for all* $a \in A \setminus B$*,*
or $f(b) = B$ *for all* $b \in B$ *and* $f(a) = \varnothing$ *for all* $a \in A \setminus B$*.*

Proof. The fact that the algebras generated by the multioperations f with these properties will be minimal follows from the fact that

if conditions 1), 2), 5) are fulfilled then algebras consists of four elements $\pi, \theta, \varepsilon, f$;

if conditions 3), 4) are fulfilled then consists of five elements $\pi, \theta, \varepsilon, f, \mu f$;

if condition 6) is fulfilled then consists of $p+2$ elements $\pi, \theta, \varepsilon, f, f^2, \dots, f^{p-1}$;

if condition 7) is fulfilled in case of one-element set A then consists of six elements

$$\pi, \theta, \varepsilon, (0, \dots, 0, \underset{i}{2^i}, 0, \dots, 0), (0, \dots, 0, \underset{i}{2^{k-1}}, 0, \dots, 0), (2^i, \dots, 2^i),$$

else consists of seven elements

$$\pi, \theta, \varepsilon, (2^{i_1} + \dots + 2^{i_s}, \dots, 2^{i_1} + \dots + 2^{i_s}), (0, \dots, 0, \underset{i_1}{2^{i_1}}, 0, \dots, 0, \underset{i_s}{2^{i_s}}, 0, \dots, 0),$$

$$(0, \dots, 0, \underset{i_1}{2^{k-1}}, 0, \dots, 0, \underset{i_s}{2^{k-1}}, 0, \dots, 0), (0, \dots, 0, \underset{i_1}{2^{i_1} + \dots + 2^{i_s}}, 0, \dots, 0, \underset{i_s}{2^{i_1} + \dots + 2^{i_s}}, 0, \dots, 0)$$

(here we specify that for the last three components of the non-zero elements are in positions $i_1, \dots, i_s$). In addition each multioperation other than π, θ, ε generates all elements of its algebra.

We now show that any f generating a minimal algebra of unary multioperations will satisfy one of the seven conditions of the theorem.

We consider the possible cases:

1. $f \cap \varepsilon = \varepsilon$. It is clear that $\langle f^2 \rangle \subseteq \langle f \rangle$ and since f generates minimal algebra then it holds either $\langle f^2 \rangle = \langle f \rangle$ or $\langle f^2 \rangle = \{\pi, \theta, \varepsilon\}$. Since $f \cap \varepsilon = \varepsilon$ then units of matrix M_f stored in matrix M_{f^2}. Hence in first case $f^2 = f$, since else $f \notin \langle f^2 \rangle$, and in second case it is obvious that $f^2 = \pi$.

1.1. If $\mu f = f$ then first case corresponds condition 1) of the theorem, and second case — condition 2).

1.2. Let $\mu f \neq f$. By the properties of algebra operations multioperation $g = f \cap \mu f$ has properties $g \cap \varepsilon = \varepsilon,\ g = \mu g$. It is clear that $\langle g \rangle \subseteq \langle f \rangle$ and since f generates minimal algebra then it holds either $\langle g \rangle = \langle f \rangle$ or $\langle g \rangle = \{\pi, \theta, \varepsilon\}$. By $g \cap \varepsilon = \varepsilon,\ g = \mu g$ in first case we obtain $\langle g \rangle = \{\pi, \theta, \varepsilon, g\} = \langle f \rangle$ that is impossible in view of $f \neq g$. From the second case implies $f \cap \mu f = g = \varepsilon$. Similarly we obtain that multioperation $h = f * \mu f$ has properties $h \cap \varepsilon = \varepsilon,\ h = \mu h$. Since $\langle h \rangle \subseteq \langle f \rangle$ and f generates minimal algebra then it holds either $\langle h \rangle = \langle f \rangle$ or $\langle h \rangle = \{\pi, \theta, \varepsilon\}$. As above, the first case is impossible, and in the second case we have $f * \mu f = h = \pi$. Equality $\mu f * f = \pi$ is obtained analogously. In case $f^2 = f$ we obtain condition 3) of the theorem, and in case $f^2 = \pi$ – condition 4).

2. $f \cap \varepsilon = \theta$. Consideration of the case is divided into two subcases.

2.1. $\mu f = f$. In this case $f^2 \cap \varepsilon = \varepsilon$ since null rows are absent in matrix M_{f^2} else algebra $\langle f \rangle$ contains a subalgebra satisfying condition 7) of the Theorem. Since $\langle f^2 \rangle \subseteq \langle f \rangle$ and f generates minimal algebra then it holds either $\langle f^2 \rangle = \langle f \rangle$ or $\langle f^2 \rangle = \{\pi, \theta, \varepsilon\}$. The first case is impossible since according to paragraph 1 would have received $\langle f^2 \rangle = \{\pi, \theta, \varepsilon, f^2\}$ or $\langle f^2 \rangle = \{\pi, \theta, \varepsilon, f^2, \mu f^2\}$, but $f \neq f^2$ and $f \neq \mu f^2$ because of $f \cap \varepsilon = \theta$ and $f^2 \cap \varepsilon = \varepsilon$, $\mu f^2 \cap \varepsilon = \varepsilon$. In the second case we have $f^2 = \pi$ or $f^2 = \varepsilon$. The first version corresponds condition 5) of the

Theorem and the second version — condition 6) where $p = 2$.

2.2. $\mu f \neq f$. By the properties of algebra operations multioperation $g = f \cap \mu f$ has properties $g \cap \varepsilon = \theta$, $g = \mu g$. Since $\langle g \rangle \subseteq \langle f \rangle$ and f generates minimal algebra then it holds either $\langle g \rangle = \langle f \rangle$ or $\langle g \rangle = \{\pi, \theta, \varepsilon\}$. In the first case since $g \cap \varepsilon = \theta$, $g = \mu g$ we have $\langle g \rangle = \{\pi, \theta, \varepsilon, g\} = \langle f \rangle$, it is impossible because of $f \neq g$. In the second case since $g \cap \varepsilon = \theta$ then $g = \theta$. Hence $f \cap \mu f = \theta$. Thus units in matrix M_f no more $\dfrac{k^2 - k}{2}$.

Multioperation $h = f * \mu f$ has properties $h \cap \varepsilon = \varepsilon$, $h = \mu h$. Since $\langle h \rangle \subseteq \langle f \rangle$ and f generates minimal algebra then it holds either $\langle h \rangle = \langle f \rangle$ or $\langle h \rangle = \{\pi, \theta, \varepsilon\}$. The first case is impossible because of $f \neq h$, and in the second case we have $f * \mu f = h = \pi$ or $f * \mu f = h = \varepsilon$. But $f * \mu f = h = \pi$ is also impossible since because of $f \cap \varepsilon = \theta$ matrix M_f must have units more $\dfrac{k^2 - k}{2}$. We have $f * \mu f = \varepsilon$. Equality $\mu f * f = \varepsilon$ is obtained analogously. From these equalities it follows that each row and each column of the matrix M_f has one unit, and it means that multioperation f is a permutation. Degrees of this permutation $f, \dots, f^p$ respect to the operations $*, \mu, \varepsilon$ form a cyclic group which has no proper subgroups for simple p which is a divisor of k. Also it holds $f^p = \varepsilon$ and $\mu f = f^{p-1}$. Since $\mu f \neq f$ then $p \geqslant 3$. This case corresponds condition 6) of the theorem for $p \geqslant 3$.

3. $f \cap \varepsilon = (0, \dots, 0, \underset{i_1}{2^{i_1}}, 0, \dots, 0, \underset{i_s}{2^{i_s}}, 0, \dots, 0)$. We consider the cases $s = 1$ and $s \geqslant 2$.

3.1. $f \cap \varepsilon = (0, \dots, 0, \underset{i}{2^i}, 0, \dots, 0)$. In this case algebra have minimal subalgebra which contains three elements $(0, \dots, 0, \underset{i}{2^i}, 0, \dots, 0), (0, \dots, 0, \underset{i}{2^{k-1}}, 0, \dots, 0), (2^i, \dots, 2^i)$ in addition to π, θ, ε, and it means that algebra is minimal only if f is equal one of these multioperations. It corresponds condition 7) of the theorem for one-element set $B = \{a_i\}$.

3.2. $f \cap \varepsilon = (0, \dots, 0, \underset{i_1}{2^{i_1}}, 0, \dots, 0, \underset{i_s}{2^{i_s}}, 0, \dots, 0)$. In this case algebra have minimal subalgebra which contains four elements $(2^{i_1} + \dots + 2^{i_s}, \dots, 2^{i_1} + \dots + 2^{i_s}), (0, \dots, 0, \underset{i_1}{2^{i_1}}, 0, \dots, 0, \underset{i_s}{2^{i_s}}, 0, \dots, 0)$, $(0, \dots, 0, \underset{i_1}{2^{k-1}}, 0, \dots, 0, \underset{i_s}{2^{k-1}}, 0, \dots, 0)$, $(0, \dots, 0, \underset{i_1}{2^{i_1} + \dots + 2^{i_s}}, 0, \dots, 0, \underset{i_s}{2^{i_1} + \dots + 2^{i_s}}, 0, \dots, 0)$ in addition to π, θ, ε, and it means that algebra is minimal only if f is equal one of these multioperations. It corresponds condition 7) of the theorem for set $B = \{a_{i_1}, \dots, a_{i_s}\}$.

These arguments concludes the proof of the theorem. □

Using this theorem one can find all minimal algebras for small ranks. We will do it for rank $k = 2, 3, 4$. Also we will indicate type of multioperation which generating a minimal algebra of unary multioperations according to the number of properties in the theorem.

Minimal algebras of unary multioperations of rank 2 (total 4)

Type 1: does not exist
Type 2: does not exist
Type 3: (1,3)
Type 4: does not exist
Type 5: does not exist
Type 6: (2,1)
Type 7: (1,1), (2,2)

Minimal algebras of unary multioperations of rank 3 (total 18)

Type 1: (1,6,6), (5,2,5), (3,3,4)
Type 2: (7,3,5), (3,7,6), (5,6,7)
Type 3: (1,3,7), (7,2,6), (5,7,4)
Type 4: (3,6,5)
Type 5: (6,5,3)
Type 6: (2,4,1)
Type 7: (1,1,1), (2,2,2), (4,4,4), (3,3,3), (5,5,5), (6,6,6)

Minimal algebras of unary multioperations of rank 4 (total 86)

Type 1: (1,14,14,14), (13,2,13,13), (11,11,4,11), (7,7,7,8), (1,2,12,12), (1,10,4,10), (1,6,6,8), (9,2,4,9), (5,2,5,8), (3,3,4,8), (3,3,12,12), (5,10,5,10), (9,6,6,9).

Type 2: (11,7,14,13), (13,14,7,11), (7,11,13,14), (15,3,5,9), (15,7,7,9), (15,3,13,13), (15,11,5,11), (3,15,6,10), (3,15,14,14), (11,15,6,11), (7,15,7,10), (5,6,15,12), (5,14,15,14), (6,6,15,13), (7,7,15,12),(9,10,12,15), (9,14,14,15), (13,10,13,15), (11,11,12,15), (15,15,7,11), (15,7,15,13), (15,11,13,15), (7,15,15,14), (11,15,14,15), (13,14,15,15).
Type 3: (1,3,7,15), (3,2,7,15), (5,7,4,15), (1,7,5,15), (7,2,6,15), (7,6,4,15), (1,3,5,15), (1,7, 7,15), (7,2,7,15), (3,2,6,15), (7,7,4,15), (5,6,4,15), (15,2,6,10), (15,2,6,14), (15,2,14,10), (15,2,14,14), (15,6,4,12), (15,6,4,14), (15,14,4,12), (15,14,4,14), (5,15,4,12), (5,15,4,13), (13,15,4,12), (13,15,4,13).
Type 4: does not exist.
Type 5: (11,13,11,7), (6,13,11,6), (10,13,10,7), (12,12,11,7), (14,13,3,3), (14,5,11,5), (14,9,9,7).
Type 6: (2,1,8,4), (4,8,1,2), (8,4,2,1).
Type 7: (1,1,1,1), (2,2,2,2), (4,4,4,4), (8,8,8,8), (3,3,3,3), (5,5,5,5), (6,6,6,6), (7,7,7,7), (9,9,9,9), (10,10,10,10), (11,11,11,11), (12,12,12,12), (13,13,13,13), (14,14,14,14).

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Минимальные алгебры унарных мультиопераций

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Получено матричное представление алгебр унарных мультиопераций конечного ранга и список тождеств, выполняемых в таких алгебрах. Эти результаты используются для доказательства основного результата: описания минимальных алгебр унарных мультиопераций конечного ранга. Как следствие, получен список всех таких минимальных алгебр для небольших рангов.

Ключевые слова: мультиоперация, алгебра, минимальная алгебра, матрица, операция, подстановка.