Nārāyaṇa's Generalisation of *Mātrā-vṛtta-prastāra* and the Generalised Virahāṅka-Fibonacci Representation of Numbers

Raja Sridharan*, R Sridharan** and M D Srinivas***

(Received 15 September 2014)

Abstract

In his *Vṛttajātisamuccaya* (c. 600 AD), Virahāṅka discussed the *prastāra* or enumeration of *mātrā-vṛttas* or moric metres, metres of value n consisting of long and short syllables, *laghu* and *guru*, assigning value 1 to *laghu* and 2 to *guru*, the value of the metre being equal to the sum of the values of its constituent syllables. Virahāṅka noted that the *saṅkhyāṅkas* or number of rows in the *prastāra* is given by a sequence of numbers, which were rediscovered much later in the 13th century by Fibonacci in a different context. Similar problems of enumeration were considered in the context of *tāla-prastāra* in music by Śārṅgadeva in his *Saṅgītaratnākara* (c. 1250 AD).

A general mathematical treatment of most of the combinatorial problems considered in the earlier literature was given by Nārāyaṇa Paṇḍita in his work *Gaṇitakaumudī* (c.1356 AD). One of the problems discussed by Nārāyaṇa is the *prastāra* of a general class of *mātrā-vṛttas* where, apart from the syllabic units *laghu* and *guru* (of values 1, 2), there could be other syllabic units (such as *pluta*, etc.) of values 3, 4, ..., q . This is also a general form of *tāla-prastāra*, but it does not subsume the specific *tāla-prastāra* considered by Śārṅgadeva in *Saṅgītaratnākara*, where the *tāla*-units have values 1, 2, 4 and 6.

The questions of *naṣṭa* and *uddiṣṭa*, which consist of finding the row number associated with a generalised *mātrā-vṛtta* and its converse, in relation to this *prastāra*, have been dealt with by Nārāyaṇa Paṇḍita by means of a tabular form called *unmeru*. In this paper, we shall show that these *naṣṭa* and *uddiṣṭa* processes are indeed based upon a certain representation of each number uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers, i.e., *saṅkhyāṅkas* associated with the generalised *mātrā-vṛtta-prastāra* of Nārāyaṇa.

Key words: Generalised *mātrā-vṛttas*, *Prastāra* or enumeration, Generalised Virahāṅka-Fibonacci numbers, *Naṣṭa*, *Uddiṣṭa*, *Unmeru*, Virahāṅka-Fibonacci representation of numbers.

1. INTRODUCTION

Indian combinatorics goes as far back as Piṅgala (c. 300 BC), who introduced six combinatorial tools (*pratyayas*) to study Sanskrit prosody. Among these is *prastāra*, an enumeration rule for generating all possible metric patterns, *naṣṭa* and *uddiṣṭa* the processes of finding any metrical pattern given its row number and its converse, in the *prastāra*. The technique of

pratyayas introduced by Piṅgala provided a model for many enumeration problems discussed by later Indian mathematicians (Sridharan, 2005).

Calling the shorter syllable *laghu* (L) and the longer syllable *guru* (G), Piṅgala enumerated the 2^n metres of length n consisting of Ls and Gs. In view of the fact that it takes twice as much time to utter a long syllable (G) as it takes to utter the short syllable (L), Indian prosodists also

*School of Mathematics, Tata Institute of Fundamental Research, Mumbai

**Chennai Mathematical Institute, Chennai

***Center for Policy Studies, 6 Balaiah Avenue, Luz Mylapore, Chennai 600004; Email: mdsrinivas50@gmail.com

considered the enumeration of *mātrā-vṛttas*, metres of value n consisting of long and short syllables, assigning value 1 to L and 2 to G, the value of the metre being equal to the sum of the values of its constituent syllables. In his *Vṛttajāṭisamuccaya* (c. 600 AD), Virahāṅka showed that the *saṅkhyāṅkas*, or number of rows in the *prastāra*, is given by a sequence of numbers, which were rediscovered much later in the 13th century by Fibonacci in a different context (Sridharan, 2006).

In his treatise on music *Saṅgītaratnākara*, Śārṅgadeva (c. 1250 AD) discussed the enumeration of both *tānas* (musical phrases) and *tālas* (musical rhythms). The *tāna-prastāra* considered by Śārṅgadeva is essentially a procedure for enumerating the $n!$ permutations of the symbols 1, 2, ..., n . The *tāla-prastāra* is a generalisation of the *mātrā-vṛtta-prastāra*, where Śārṅgadeva considers all possible rhythmic forms composed of the units *druta*, *laghu*, *guru* and *pluta* with values 1, 2, 4 and 6 respectively (Raja Sridharan et al., 2010).

Prior to the seminal work *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita (c. 1356 AD), most of the work in combinatorics had its basis in practical problems. Nārāyaṇa Paṇḍita was the first to treat these problems from a general mathematical viewpoint, generalising and unifying many of the earlier results. Specifically, Nārāyaṇa considered *prastāras* of syllabic metres with more than two types of syllables generalising the work of Piṅgala, *prastāras* of permutations with repetitions generalising the work of Śārṅgadeva on *svara-prastāra*, and also *prastāras* of combinations which were briefly considered earlier by Varāhamihira (c. 550 AD) and Bhaṭṭotpala (c. 950 AD) (Raja Sridharan et al., 2012). In this paper we shall study Nārāyaṇa Paṇḍita's generalisation of the *mātrā-vṛtta-prastāra*.

In Chapter XIII of *Gaṇitakaumudī* on *Ankapāśa* (net of numbers), Nārāyaṇa Paṇḍita gives an abstract mathematical formulation encompassing most of the combinatorial problems

considered in earlier literature. After listing the various *pratyayas*, Nārāyaṇa defines a number of *pañktis* (sequences) and *merus* (tabular figures) that are employed in various combinatorial problems. Nārāyaṇa then considers different kinds of *prastāras* which generalise those considered in prosody and music. He formulates the problem as one of enumerating the various possibilities which arise when there are p slots or places (*sthānas*) in which the q digits 1, 2, ..., q , are placed, subject to various conditions.

One of the problems discussed by Nārāyaṇa is the *prastāra* of a general class of *mātrā-vṛttas* where, apart from the syllabic units *laghu* and *guru* (of values 1, 2), there could be other syllabic units (such as *pluta*, etc.) of values 3, 4, ..., q . This is also a general form of *tāla-prastāra*, but it does not subsume the specific *tāla-prastāra* considered by Śārṅgadeva in *Saṅgītaratnākara*, where the *tāla*-units have values 1, 2, 4 and 6. Before discussing the *prastāras* considered by Nārāyaṇa, we first give a brief description of *mātrā-vṛttas* and their *prastāra*.

2. MĀTRĀ-VṚTTAS AND THEIR PRASTĀRA

In the case of *varṇa-vṛttas* or syllabic metres, it is the number of syllables which are fixed and the *prastāra* or the enumeration consists of all possible metrical forms which are characterised by a sequence of *laghus* and *gurus* having the same number of syllables. In the case of *mātrā-vṛttas* or *moric* metres, it is the total value of the metrical form that is fixed, when we assign values of 1 *mātrā* for each *laghu* (L) and 2 *mātrās* for each *guru* (G).

The rule for the construction of the *prastāra* for *mātrā-vṛttas* is stated explicitly in the *Vṛttajāṭisamuccaya* of Virahāṅka (c. 600 AD) and is as follows:

- The first row consists of all Gs if the total value is even and an L followed by all Gs if the total value is odd.

- Given any row in the *prastāra*, to generate the next row, scan from the left to identify the first G and place an L below that. The elements to the right are brought down as they are.
- The remaining *mātrās* to the left are filled by Gs, and an L is added at the beginning if need be to keep the total number of *mātrās* the same.
- Go on till the last row is reached which is formed by only Ls.

For instance, the *prastāra* of *vṛttas* of 5-*mātrās* is shown in Table 1:

Table 1: *Prastāra* of *vṛttas* of 5-*mātrās*

1			L	G	G
2			G	L	G
3		L	L	L	G
4			G	G	L
5		L	L	G	L
6		L	G	L	L
7		G	L	L	L
8	L	L	L	L	L

Virahāṅka also gave the rule for the *saṅkhyā* or the number of rows S_n in the *prastāra* of *vṛttas* of value n , which is the following:

$$S_n = S_{n-1} + S_{n-2}$$

Taking $S_0 = 1$, and since $S_1 = 1$, and $S_2 = 2$, the above relation generates the Virahāṅka sequence (later rediscovered by Fibonacci): 1, 1, 2, 3, 5, 8, 13, 21, ...

The other *pratyayas* such as *naṣṭa*, *uddiṣṭa* and *lagakriyā* (the last one being a method for finding the number of metres in the *prastāra* with a given number of *laghus* or *gurus*) for *mātrā-vṛttas* are worked out in later texts such as *Prākṛtapaiṅgala* (of unknown authorship composed around 12th century), *Vāṇībhūṣaṇa* of Dāmodara (c. 1550) and the commentary of Nārāyaṇabhaṭṭa (c.1555) on *Vṛttaratnākara*. In fact, it can be shown that the *naṣṭa* and *uddiṣṭa*

processes for the *mātrā-vṛttas* are indeed based on a certain representation of each number uniquely as a sum of non-consecutive Virahāṅka-Fibonacci numbers (Sridharan 2006). The *lagakriyā* process for these *vṛttas* also leads to an interesting relation between binomial coefficients and the Virahāṅka numbers.

3. NĀRĀYAṆA'S GENERALISATION OF MĀTRĀ-VṚTTA-PRASTĀRA

Nārāyaṇa presents a generalisation of the *mātrā-vṛtta*, which has, apart from *laghu* and *guru* with values 1, 2, other elements also (like *pluta* etc.) which have values 3, 4, ..., q . Instead of using *laghus* and *gurus*, Nārāyaṇa considers sequences of variable length p (the number of places (*sthānas*)) formed from the digits (*aṅkas*) 1, 2, ..., q where q is the highest or the last digit (*antimāṅka*), such that their total value or sum (*yoga*) n is fixed. In other words, each row of the *prastāra* is an ordered partition of n in terms of 1, 2, ..., q .

Nārāyaṇa refers to this case as

नियतयोग-अनियतस्थान-नियतान्तिमाङ्क-भेदानयनम्

To generate all the variations (all possible sequences) composed of digits 1, 2, ..., q , where both the total value or sum (n) of the sequence, as also the highest digit (q) that is allowed, are fixed, while the number of places (p) is variable.

If $q = 2$, we get the usual *mātrā-vṛtta-prastāra* formed with *laghus* and *gurus* which are now denoted by 1, 2 respectively. For instance, the 5-*mātrā-prastāra* displayed earlier will appear as shown in Table 2, in Nārāyaṇa's notation.

In verses 79-80 of Chapter XIII of *Gaṇitakaumudī* on *Aṅkapāśa* (net of numbers), Nārāyaṇa Paṇḍita states the rule for the enumeration of generalised *mātrā-vṛttas* as follows:¹

¹ The verses of *Gaṇitakaumudī* have been cited from Kusuba T., *Combinatorics and Magic-squares in India: A Study of Nārāyaṇa Paṇḍita's Gaṇita-kaumudī*, Chapters 13-14, Ph.D. Dissertation, Brown University (Unpublished), 1993.

Table 2: *Prastāra* with $n = 5$ and $q = 2$

1			1	2	2
2			2	1	2
3		1	1	1	2
4			2	2	1
5		1	1	2	1
6		1	2	1	1
7		2	1	1	1
8	1	1	1	1	1

अन्तिमाङ्कं लिखेदादौ वामे वाङ्मैक्यपूरणम् ।
न्यस्याल्पमादान्महतोऽधस्ताच्छेषं यथोपरि ॥
अङ्कैक्यपूरणं वामे यावत्सर्वैकको भवेत् ।
प्रस्तारोऽयं समाख्यातो भरतज्ञैः पुरातनैः ॥

Write the final number at the beginning and fill out the sum of the numbers on the left. After one has put the smaller number below the first larger number, the rest [of the numbers are brought down] as above; the process [is repeated] till a row all of whose entries are 1s is obtained. This is the enumeration as declared by the ancients well-versed in Bharata.

Nārāyaṇa's rule of *prastāra* is essentially the following:

- In the first row of the *prastāra*, write from the right to left as many last digits (q) as possible, such that their sum does not exceed n . Follow this procedure the same way with the penultimate digit ($q - 1$), etc., till the sum of all the digits is n .
- To go from any row of the *prastāra* to the next, scanning from the left, identify the first digit (say m) which is larger than 1. Below that, place the digit less than that by 1 (i.e., $m - 1$). To the right of it, bring down the digits as they are from the row above.
- To the left, start from the highest digit q and so on (as in the first row) till the total sum of all the digits reaches the value n .
- Repeat the process till the last row of the *prastāra* is reached, all of whose entries are 1.

This clearly, is a generalisation of the rule of *prastāra* for the usual *mātrā-vṛttas* given by Virahāṅka and others. Here, each *prastāra* is characterised by both the total value n and the last digit q . To illustrate this rule, Nārāyaṇa gives two examples which are displayed in Tables 3, 4. The first one is the *prastāra* of total value $n = 7$ and highest digit $q = 3$. This is nothing but 7-*mātrā-prastāra* with *laghu*, *guru* and *pluta* being the syllabic elements. The second one is the *prastāra* of total value $n = 7$ and highest digit $q = 7$.

Table 3: *Prastāra* with total value $n = 7$ and the highest digit $q = 3$

1	133	12	2122	23	2131	34	3211
2	223	13	11122	24	11131	35	12211
3	1123	14	1312	25	1321	36	21211
4	313	15	2212	26	2221	37	111211
5	1213	16	11212	27	11221	38	13111
6	2113	17	3112	28	3121	39	22111
7	11113	18	12112	29	12121	40	112111
8	232	19	21112	30	21121	41	31111
9	1132	20	111112	31	111121	42	121111
10	322	21	331	32	2311	43	211111
11	1222	22	1231	33	11311	44	1111111

Table 4: *Prastāra* with total value $n = 7$ and the highest digit $q = 7$

1	7	17	52	33	61	49	511
2	16	18	142	34	151	50	1411
3	25	19	232	35	241	51	2311
4	115	20	1132	36	1141	52	11311
5	34	21	322	37	331	53	3211
6	124	22	1222	38	1231	54	12211
7	214	23	2122	39	2131	55	21211
8	1114	24	11122	40	11131	56	111211
9	43	25	412	41	421	57	4111
10	133	26	1312	42	1321	58	13111
11	223	27	2212	43	2221	59	22111
12	1123	28	11212	44	11221	60	112111
13	313	29	3112	45	3121	61	31111
14	1213	30	12112	46	12121	62	121111
15	2113	31	21112	47	21121	63	211111
16	11113	32	111112	48	111121	64	1111111

4. THE SĀMASIKĪ-PAṆKTI AND THE SĀṆKHYĀ OR THE NUMBER OF ROWS IN THE PRASTĀRA

In order to discuss the various *pratyayas* of the generalised *mātrā-vṛtta-prastāra*, Nārāyaṇa

makes use of various sequences (*pañktis*) and tabular forms (*merus*) which he introduces right at the beginning of the chapter on combinatorics (*aṅkapāśa*). Of these, the most important is the *sāmāsikī-pañkti* (additive sequence), which is a generalisation of the Virahāṅka-Fibonacci sequence, and gives the *saṅkhyā* or the total number of rows in the *prastāra*. In verses 13-14 of the Chapter on *Aṅkapāśa*, Nārāyaṇa defines the *sāmāsikī-pañkti* as follows:

एकाङ्कौ विन्यस्य प्रथमं तत्संयुतिं पुरो विलिखेत् ।
उत्क्रमतोऽन्तिमतुल्यस्थानयुतिं तत्पुरस्ताच्च ॥
अन्तिमतुल्यस्थानाभावे तत्संयुतिं पुरस्ताच्च ।
एवं सैकसमासस्थाना सामासिकीयं स्यात् ॥

Having first put down two ones, then write their sum in front. And in their front write the sum in reverse order of those numbers which occupy the places equal in number to the last digit. In the absent places equal in number to the last digit, write in front the sum of them [that are present]. In this way is formed the additive sequence whose number of places is equal to the sum plus one.

The *sāmāsikī-pañkti* when the last digit is q is thus defined by the relations

$$S_0^q = S_1^q = 1$$

$$S_n^q = S_{n-1}^q + S_{n-2}^q + \dots + S_0^q \text{ when } 1 < n \leq q.$$

$$S_n^q = S_{n-1}^q + S_{n-2}^q + \dots + S_{n-q}^q \text{ when } n > q.$$

Clearly, when $q = 2$, S_n^2 is nothing but the Virahāṅka-Fibonacci sequence defined earlier. As we shall see below, S_n^q is the *saṅkhyā* or the total number of rows in the *prastāra* of total value or sum n with the highest digit being q .

In verses 15-16, Nārāyaṇa also defines the associated *pātāla-pañkti*, the elements of which give the number of times any digit occurs in the *prastāra*, as follows:

सामासिकाख्यपङ्क्तेश्च विरमङ्को भिदाम् मितिः ।
उत्क्रमात्तदुपान्त्याङ्काद्भेदा एकान्तकादयः ॥
उत्क्रमादन्तिमयोगसमपातालपङ्क्तिजाः ।
अङ्का एकादयस्तेस्युस्तत्संयोगेऽङ्कपातनम् ॥

Below the *sāmāsikī-pañkti*, write 0 followed by 1. Then, the number above the last number is added to the sum in reverse order of those numbers which occupy the places equal in number to the last digit. In this way, all the places in front are to be filled. In the absence of places equal in number to the last digit, the sum of those present is written.

The *pātāla-pañkti* when the last digit is q is thus defined by the relations

$$P_0^q = P_1^q = 1$$

$$P_r^q = S_{r-1}^q + P_{r-2}^q + \dots + P_0^q \text{ when } 1 < r \leq q.$$

$$P_n^q = S_{n-1}^q + P_{n-1}^q + P_{n-2}^q \dots + P_{n-q}^q \text{ when } n > q.$$

In verses 78-79, Nārāyaṇa notes the significance of the *sāmāsikī-pañkti* and the *pātāla-pañkti* as follows:

सामासिकाख्यपङ्क्तेश्च विरमङ्को भिदाम् मितिः ।
उत्क्रमात्तदुपान्त्याङ्काद्भेदा एकान्तकादयः ॥
उत्क्रमादन्तिमयोगसमपातालपङ्क्तिजाः ।
अङ्का एकादयस्तेस्युस्तत्संयोगेऽङ्कपातनम् ॥

The end of the *sāmāsikā* sequence gives the number of possible variations (metrical forms of generalised *mātrā-vṛttas*, or the number of rows in the *prastāra*). The numbers of the sequence taken in the reverse order, from the penultimate element are the number of variations ending in 1, etc. The numbers of the *pātāla* sequence, which correspond to the same last digit and the sum, when taken in reverse order give the (number of occurrences) of 1, etc. The sum of all these is the occurrence of all the digits.

Thus, the *saṅkhyā* or the number of rows in the *prastāra* is given by the last element of the *sāmāsikā* sequence and the penultimate and other elements in the reverse order give the number of rows in the *prastāra* which end in digits 1, 2, etc. Similarly the numbers in the *pātāla* sequence, taken in the reverse order give the number of times the different digits 1, 2, etc. occur in the *prastāra*. The sum of all the elements of the *pātāla* sequence,

gives the total number of all the digits occurring in the *prastāra*. We display below two examples of the *sāmāsikī-pankti* and *pātāla-pankti* given by Nārāyaṇa.

Table 5, displays the *sāmāsikī-pankti* and *pātāla-pankti* for the *prastāra* with total value 7 and the last digit 3. This is the generalised *mātrā-vṛtta*, where we include apart from the syllabic units *laghu* and *guru* (of values 1, 2), a third syllabic unit *pluta*, which has value 3. Recalling the *prastāra* that we displayed earlier in Table 3, we can easily verify that 44 is the *saṅkhyā* or the number of rows in the *prastāra*; and that the last 24 rows end in 1, the previous 13 rows end in 2 and the first 7 rows end in 3. We can also verify that the digit 1 appears 118 times in the *prastāra*, 2 appears 56 times, and 3 appears 26 times. The total number of digits appearing in the *prastāra* is $26+56+118=200$.

Table 5: The *Sāmāsikī-pankti* and *pātāla-pankti* for $n = 7$ and $q = 3$

1	1	2	4	7	13	24	44
0	1	2	5	12	26	56	118

Table 6: The *Sāmāsikī-pankti* and *pātāla-pankti* for $n = 7$ and $q = 7$

1	1	2	4	8	16	32	64
0	1	2	5	12	28	64	144

Unlike in the case of *varṇa-vṛtta-prastāra* where the number of syllables is fixed to start with, in a *mātrā-vṛtta-prastāra* where only the total *mātrā* of the metric form is held fixed, the number of syllables in different rows of the *prastāra* can be different. This is clearly seen for example in the example of 5-*mātrā-prastāra* that was displayed in Table 1. In the same way, in the generalised *mātrā-vṛtta-prastāra* as defined by Nārāyaṇa, the lengths of the sequences of digits that constitute different rows of a *prastāra* (characterised by a given sum n and last digit q) can be different as can be readily seen from the

examples of *prastāra* displayed in Tables 3, 4. That is why, Nārāyaṇa refers to this case as *aniyata-sthāna*, where the number of places (the number of elements in each row of the *prastāra*) is not fixed.

Thus, for a generalised *mātrā-vṛtta-prastāra*, we need to answer the question as to how many rows in the *prastāra* (where the total sum is n and the last digit is q), have length p . Nārāyaṇa answers this question by means of the elements of the so called needle sequence (*sūcī-pankti*) or the arrow-head sequence (*nārācīkā-pankti*), which are also put together in a fish-tabular figure (*matsya-meru*). The elements $U_{p,q}(r)$ of the needle sequence, as defined by Nārāyaṇa, are essentially multinomial coefficients, the coefficients of various powers x^r in the expansion of $(1 + x + x^2 + \dots + x^{q-1})^p$. Nārāyaṇa also gives an interesting expression for the elements of the *sāmāsikī-pankti* or the generalised Virahāṅka-Fibonacci sequence of *saṅkhyāṅkas* S_n^q , as sums of suitable elements of the *sūcī-pankti* $U_{p,q}(r)$. We have summarised Nārāyaṇa's discussion of *sūcī-pankti* and its application to the study of the generalised *mātrā-vṛtta-prastāra* in the Appendix.

5. THE UNMERU AND THE NAṢṬA AND UDDIṢṬA PROCESSES

In verses 84-86 of the Chapter on *Ankapāśa*, Nārāyaṇa introduces the following *unmeru*, which helps in carrying out the *naṣṭa* and *uddiṣṭa* processes in a generalised *mātrā-vṛtta-prastāra*.

एकादोकोत्तराः कोष्ठत्रेण्यः सैकैक्यसम्मिताः ।
अधस्थायां तिर्यग्गायां कोष्ठपङ्क्तौ लिखेत् क्रमात् ॥
पङ्क्तिं सामासिकामन्यासु पूर्वसूत्रक्रमेण च ।
चयपङ्क्तिस्तिर्यग्गासूत्रासु ताः स्युः स्वभावतः ॥
अन्तिमाङ्काधिका येऽङ्का येषु कोष्ठेषु संस्थिताः ।
कार्यं प्रलोपनं तेषामुन्मेषुरयमिति ॥

The number of cells in each row, starting with one and increasing by one at each step, are equal to the total sum plus one. In the bottom row, the elements of the

additive sequence (*sāmāsikī-paṅkti*) are written in order. In the others, the arithmetical sequence (*caya-paṅkti* 1, 2, ..., q , ...) is written in the reverse (from right to left). In the upper horizontal rows they appear naturally. In those cells where numbers larger than the last digit is to be written, an omission is made. This figure is called the lofty tabular figure (*unmeru*).

The bottom row of *unmeru* has a number of entries which is one more than total sum (n) and subsequent rows have one number less at each step. The bottom row is filled with the *sāmāsikī-paṅkti* or the generalised Virahāṅka sequence of *saṅkhyāṅkas*. In the subsequent rows the numbers 1, 2, ..., q , are written from the right end. In Table 7 and Table 8, we display two examples of *unmeru* given by Nārāyaṇa.

Table 7: *Unmeru* for $n = 7$ and $q = 3$

1							
2	1						
3	2	1					
	3	2	1				
		3	2	1			
			3	2	1		
				3	2	1	
1	1	2	4	7	13	24	44

Table 8: *Unmeru* for $n = 7$ and $q = 7$

1							
2	1						
3	2	1					
4	3	2	1				
5	4	3	2	1			
6	5	4	3	2	1		
7	6	5	4	3	2	1	
1	1	2	4	8	16	32	64

In verses 87-89, Nārāyaṇa presents a rather cryptic description how the *naṣṭa* process is carried out with the help of *unmeru*:

सामासिकान्त्याङ्कान्नष्टाङ्कं पातयेदथ विशेषात् ।
आद्यादाङ्कं तस्मात्प्रपातयेदावदेव नो पतति ॥
तिर्यक्स्थोर्ध्वस्थितयोः पङ्क्त्योः संयुक्तकोष्ठाङ्कः ।
स स्यान्नष्टादाङ्को लब्धास्ते संख्याङ्काः श्रेण्यः ॥
विधिरयमुन्मेषवशाद् एवं कार्यो मुहूर्नष्टे ।

Subtract the *naṣṭāṅka* (row number for which the variation or the corresponding sequence of digits is to be determined) from the last number of the *sāmāsikī-paṅkti* (the sequence of *saṅkhyāṅkas* or the generalised Virahāṅka sequence). Then, subtract from the result the next number and so on till no subtraction is possible. The number in the cell which is the intersection of the row and column associated (with the subtracted number) is the first digit of the variation. The rest of the elements of the row are similarly obtained, by repeating the process with *unmeru*.

The method is essentially as follows: First, subtract the given row number from the last entry of the bottom row of *unmeru* which consists of the sequence of *saṅkhyāṅkas* or the generalised Virahāṅka sequence. Then, from what remains, see if the penultimate number can be subtracted and so on. In this way, amongst the numbers of the bottom row, identify those which are *patita* (numbers which have been subtracted) and *apatita* (numbers which have not been subtracted). Then, starting from right, the digit just above the first *apatita* number is the last element of the desired sequence. Now, go one cell above the top entry of that column and move left till the intersection with the column above the next *apatita* number is reached. That digit is the penultimate element of the desired sequence of digits. And so on.

Nārāyaṇa's Example 1: To Find the 36th row of the 7-mātrā-prastāra with highest digit 3

$$44 - 36 = 8, 8 - 7 = 1, \text{ and } 1 - 1 = 0$$

Thus the second 1 and 7 are *patita* and the rest are *apatita-Saṅkhyāṅkas*. Note the entry 1 in the first row above the last *apatita* 24. In the row above the topmost entry of the column of that *apatita*, move left till you reach the column of the

next *apatita* 13. Note the corresponding entry 1. And so on. Thus the sequence of digits in 36th row of the *prastāra* is 21211.

Table 9: Example of *naṣṭa* process when $n = 7$ and $q = 3$

1							
2	1						
3	2	1					
	3	2	1				
		3	2	1			
			3	2	1		
				3	2	1	
1	1	2	4	7	13	24	44

Nārāyaṇa's Example 2: To Find the 36th row of the 7-*mātrā-prastāra* with highest digit 7

$64 - 36 = 28$, $28 - 16 = 12$, $12 - 8 = 4$ and $4 - 4 = 0$

Thus 16, 8 and 4 are *patita* and the rest are *apatita-saṅkhyāṅkas*. We find that the desired sequence in the 36th row of the *prastāra* is 1141

Table 10: Example of *naṣṭa* process when $n = 7$ and $q = 7$

1							
2	1						
3	2	1					
4	3	2	1				
5	4	3	2	1			
6	5	4	3	2	1		
7	6	5	4	3	2	1	
1	1	2	4	8	16	32	64

Nārāyaṇa's description of the *uddiṣṭa* process, presented in verse 90, is even more cryptic than his description of *naṣṭa* process:

उन्मेरुवशात्तटे लभ्यन्तेऽङ्कः प्रपातितैर्यैः ।
सङ्ख्याङ्कानङ्कानपास्य सोदितभेदमितः ॥

Those numbers in the (bottom row of) *unmeru* which were subtracted in the *naṣṭa* process, if these are subtracted from

the number (*saṅkhyāṅka*) of the enumeration, one gets the row-number of a given variation.

The *uddiṣṭa* process is indeed the reverse of the *naṣṭa* process as will be clear from the following example:

Nārāyaṇa's Example: To find the row where 2212 appears in the 7-*mātrā-prastāra* with highest digit 3:

From the right, we first identify the *apatita* 13 above which 2 appears. Then we move left in the row above the topmost entry of the column of that *apatita* till we get 1 and note the *apatita* number 7. And so on.

Row number = $44 - \text{sum of the } patitas = 44 - (24+4+1) = 15$

Table 11: Example of *uddiṣṭa* process when $n = 7$ and $q = 3$

1							
2	1						
3	2	1					
	3	2	1				
		3	2	1			
			3	2	1		
				3	2	1	
1	1	2	4	7	13	24	44

6 ALTERNATE METHOD FOR THE NAṢṬA AND UDDIṢṬA PROCESSES

We here describe an alternative method for the *naṣṭa* and *uddiṣṭa* processes, by making use of a correspondence between the various digits 1, 2, ..., q , and associated sequences of *patita* and *apatita saṅkhyāṅkas*. This method is similar to the one discussed in works such as the commentary of Nārāyaṇabhaṭṭa on *Vṛttaratnākara* for the usual *mātrā-vṛttas* (with only *laghus* and *gurus*). Such a method was also used by, Śārṅgadeva for the *uddiṣṭa* process in the case of *tāla-prastāra* in *Saṅgītaratnākara* (Raja Sridharan et al., 2010).

Table 12: Signature of different digits as sequences of a and p

	S_{r-q}	$S_{r-(q-1)}$		S_{r-4}	S_{r-3}	S_{r-2}	S_{r-1}	S_r
1							a	(a)
2						a	p	(a)
3					a	p	p	(a)
4				a	p	p	p	(a)
q	a	p		p	p	p	p	(a)

Write down the sequence of *saṅkhyāṅkas* or the generalised Virahāṅka sequence for the *prastāra*. For the given row-number, identify *patita* and *apatita-saṅkhyāṅkas* and place the mark *a* or *p* below each *saṅkhyāṅka*. From the right, scan the sequence of these marks and identify the corresponding element of the sequence of digits, successively, via their signatures as given in Table 12.

We illustrate the method with the following examples which are displayed in Tables 13, 14.

Table 13: To Find the 36th row of the 7-mātrā-prastāra with highest digit 3

1	1	2	4	7	13	24	44
a	p	a	a	p	a	a	
2	1	2	1	1			

Table 14: To Find the 36th row of the 7-mātrā-prastāra with highest digit 7

1	1	2	4	8	16	32	64
a	a	a	p	p	p	a	
1	1	4	1				

We can also similarly devise an alternative method for *uddiṣṭa* process by using the above correspondence between the various digits and associated sequences of *patita* and *apatita saṅkhyāṅkas*. The procedure is as follows.

Write the sequence of *saṅkhyāṅkas* (the generalised Virahāṅka sequence) such that one

number is written above 1, two numbers above 2, and so on and q numbers are written above q . Omit the number above 1, and also the first number above each of the digits 2, 3, ..., q , and sum the rest. Subtract the sum from the last entry of the sequence of *saṅkhyāṅkas* (which is nothing but the number of rows in the given *prastāra*), and the result is the row-number associated with the given sequence of digits. We illustrate the above process in Tables 15-18.

Table 15: To Find the row where 2212 appears in the 7-mātrā-prastāra with highest digit 3

Row-number = $44 - (1 + 4 + 24) = 15$

1	1	2	4	7	13	24	44
2	2	1	2				

Table 16: To Find the row where 322 appears in the 7-mātrā-prastāra with highest digit 3

Row-number = $44 - (1 + 2 + 7 + 24) = 10$

1	1	2	4	7	13	24	44
3	2	2					

Table 17: To Find the row where 21211 appears in the 7-mātrā-prastāra with highest digit 3

Row-number = $44 - (1 + 7) = 36$

1	1	2	4	7	13	24	44
2	1	2	1	1			

Table 18: To Find the row where 1141 appears in the 7-mātrā-prastāra with highest digit 7

Row-number = $64 - (4 + 8 + 16) = 36$

1	1	2	4	8	16	32	64
1	1	4	1				

One important feature that we notice in the above examples of *naṣṭa* and *uddiṣṭa* processes, is that they are based on certain representations of numbers as sums of the generalised Virahāṅka numbers $\{S_q^n\}$.

For the case when $q = 3$, we have the representations

$$\begin{aligned} 8 &= 1 + 7 \\ 29 &= 1 + 4 + 24 \\ 34 &= 1 + 2 + 7 + 24 \end{aligned}$$

For the case when $q = 7$, we have the representation

$$28 = 4 + 8 + 16$$

These are particular instances of a general result that every number can be decomposed uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers. We shall discuss this, and other mathematical aspects of the generalised *mātrā-vṛtta-prastāra*, in the next section.

7. THE GENERALISED VIRAHĀṆKA-FIBONACCI REPRESENTATION OF NUMBERS

While discussing the mathematics underlying the generalised *mātrā-vṛtta-prastāra* of Nārāyaṇa, we shall, for simplicity consider the case $q = 3$, i.e., there are only three digits, 1, 2, 3, in each row of the *prastāra*.

7.1 An inductive construction of the *prastāra* and the *saṅkhyā* rule

We consider the symbols 1, 2, 3, having the same value as the numbers that these digits represent. We write down all rows consisting of these symbols, such that the total value of each row is n , the value of a row being defined to be the sum of the values of the constituent symbols in the row. A *prastāra* of metres of value n is a table enumerating all such ordered partitions (allowing repetitions) of n in terms of the numbers 1, 2, 3.

The *prastāra* of rows of total value 0 consists of one row with no symbols. The *prastāra* of rows of total value 1 consists of a single row with the entry 1. The *prastāra* of rows of value 2 is given in Table 19. This is obtained by appending the symbol 2 to the *prastāra* of rows of value 0 and 1 to the *prastāra* of rows of total value 1.

Table 19: *Prastāra* of value 2

	2
1	1

The *prastāra* of rows of total value 3 is given in Table 20. This is obtained by appending the symbol 3 to the *prastāra* of rows of value 0 and then appending the symbol 2 to the *prastāra* of rows of value 1 and finally appending 1 to the *prastāra* of rows of total value 2.

Table 20: *Prastāra* of value 3

		3
	1	2
	2	1
1	1	1

More generally, we can construct the *prastāra* of rows of total value $n > 3$ inductively as follows: Write down the *prastāra* of rows of value $n - 3$ and append 3, then write down the *prastāra* of rows of value $n - 2$ and append 2, and finally write down the *prastāra* of rows of value $n - 1$ with 1 appended at the end. It follows by induction on n that this is indeed the *prastāra* obtained by following Nārāyaṇa's rule for generalised *mātrā-vṛttas* total value n , and highest digit $q = 3$.

It follows that if S_n^3 is the number of rows in the *prastāra* of metres of value n , then $S_0^3 = 1$, $S_1^3 = 1$, $S_2^3 = 2$ and

$$S_n^3 = S_{n-1}^3 + S_{n-2}^3 + S_{n-3}^3, \quad n \geq 3.$$

We shall now see how this also follows from an argument that uses generating functions.

Let us assume that a, b, c are non-commuting variables and consider the formal expression

$$1 + (a + b + c) + (a + b + c)^2 + \dots$$

This formal sum will consist of monomials $a_1 a_2 \dots a_n$, where each a_i is a, b or c . Further, because of non-commutativity we cannot permute the various constituent symbols of the monomials. Now suppose that we assign values 1, 2, 3 to the symbols a, b, c , and define the total value of a monomial to be the sum of the values of its constituent symbols. If we now specialize $a = x, b = x^2, c = x^3$ in the formal sum, we obtain the following power series

$$1 + (x + x^2 + x^3) + (x + x^2 + x^3)^2 + \dots = [1 - (x + x^2 + x^3)]^{-1}$$

Now, the co-efficient of the power x^n in the above power series is really the *saṅkhyāṅka* S_n^3 , as it is equal to the number of monomials of total value n in a, b, c , when we assign the values 1, 2, 3 to the symbols a, b, c . Hence we have the equality

$$1 + (x + x^2 + x^3) + (x + x^2 + x^3)^2 + \dots = S_0^3 + S_1^3 x + S_2^3 x^2 + \dots$$

Therefore we obtain

$$[1 - (x + x^2 + x^3)]^{-1} [S_0^3 + S_1^3 x + S_2^3 x^2 + \dots] = 1$$

By equating the coefficients of x^n , we obtain the relation

$$S_n^3 = S_{n-1}^3 + S_{n-2}^3 + S_{n-3}^3, \quad n \geq 3$$

Remark: In the case of the usual *mātrā-vṛttas*, with *laghus* and *gurus* only, with values 1, 2 respectively, the *saṅkhyāṅkas* are the usual Virahāṅka-Fibonacci numbers $\{S_n^2\}$, which satisfy

$$S_0^2 = 1, S_1^2 = 1, \text{ and } S_n^2 = S_{n-1}^2 + S_{n-2}^2, \text{ for } n \geq 2$$

These relations for S_n^2 can be obtained following a similar argument as above as a consequence of the identity

$$|1 - (x + x^2)|^{-1} |S_0^2 + S_1^2 x + S_2^2 x^2 + \dots| = 1$$

7.2 Representation of Numbers as Sums of Generalised Virahāṅka- Fibonacci Numbers

We first recall the following result for Virahāṅka-Fibonacci numbers $\{S_n^2\}$, which was proved in an earlier investigation (Sridharan, 2006) on the mathematical theory of the usual *mātrā-vṛtta-prastāra*:

Proposition 1: Any natural number n is either equal to a Virahāṅka-Fibonacci number S_k^2 or can be uniquely expressed as a sum of non-consecutive Virahāṅka-Fibonacci numbers $\{S_i^2\}$.

Now let us consider the case of the generalised Virahāṅka-Fibonacci numbers S_n^3 . We have $S_1^3 = 1, S_2^3 = 2, S_3^3 = 4$ and $S_n^3 = S_{n-1}^3 + S_{n-2}^3 + S_{n-3}^3, n > 3$, and we have the following:

Proposition 2: Any natural number n is either a generalised Virahāṅka-Fibonacci number S_k^3 or can be uniquely expressed uniquely as a sum of generalised Virahāṅka-Fibonacci numbers S_i^3 , no three of which are consecutive.

To prove this we first need the following Lemma:

Lemma 1: For all $i > 0$, we have

$$2S_i^3 \geq S_{i+1}^3,$$

with equality holding only when $i = 1, 2$.

Proof: We have to show that $2S_i^3 > S_i^3 + S_{i-1}^3 + S_{i-2}^3$. It is thus enough to show that $S_i^3 > S_{i-1}^3 + S_{i-2}^3$. But this follows from the fact that $S_i^3 = S_{i-1}^3 + S_{i-2}^3 + S_{i-3}^3 > S_{i-1}^3 + S_{i-2}^3$.

Remark: We note that the *saṅkhyāṅkas* of the *varṇa-vṛttas* are $S_i = 2^i$ which satisfy $2S_i = S_{i+1}$. The *saṅkhyāṅkas* of ordinary *mātrā-vṛttas*, the Virahāṅka-Fibonacci numbers S_i^2 , also satisfy $2S_i^2 > S_{i+1}^2$.

Now, let n be a natural number. We define the canonical decomposition of n as follows: We choose the largest i_1 such that $S_{i_1}^3 \leq n$. Then, we choose the largest i_2 such that $S_{i_2}^3 \leq n - S_{i_1}^3$.

Continuing in this manner, we obtain a decomposition

$$n = S_{i_1}^3 + S_{i_2}^3 + \dots S_{i_k}^3,$$

which we refer to as the *canonical decomposition* of n .

Since $2S_i^3 > S_{i+1}^3$, every S_i^3 occurs only once in the canonical decomposition. For, otherwise, if S_i^3 were to occur more than once in the canonical decomposition of n , we would have at a certain stage of the canonical decompositions, i as the largest number such that

$$S_i^3 \leq n - (S_{i_1}^3 + S_{i_2}^3 + \dots S_{i_k}^3) = m,$$

and that i also satisfies $S_i^3 \leq m - S_i^3$. In other words, $2S_i^3 \leq m$, which implies that $S_{i+1}^3 \leq m$, contradicting the fact that i is the largest number such that $S_i^3 \leq m$.

A similar argument can also be made to show that three consecutive generalised Virahāṅka-Fibonacci numbers $\{S_i^3\}$ do not appear in any canonical decomposition.

Lemma 2: Let

$$m = S_a^3 + S_c^3 + \dots + S_l^3 + S_p^3 + S_t^3,$$

where, $a < c < \dots < l < p < t$ are natural numbers with no three of $a, c, \dots, l, p$, and t being consecutive. Then

$$m < S_{t+1}^3.$$

Proof: Under the hypothesis of the lemma, we have to prove that

$$S_a^3 + S_c^3 + \dots + S_l^3 + S_p^3 + S_t^3 < S_{t+1}^3.$$

Note that since $p < t$, we have $p \leq t-1$. It is enough to prove that

$$S_a^3 + S_c^3 + \dots + S_l^3 + S_p^3 < S_{t+1}^3 - S_t^3 = S_{t-1}^3 + S_{t-2}^3.$$

If $p \leq t-2$, we have, by induction

$$S_a^3 + S_c^3 + \dots + S_l^3 + S_p^3 < S_{p+1}^3 \leq S_{t-1}^3 \leq S_{t-1}^3 + S_{t-2}^3,$$

thus proving the result

If $p > t-2$; since $p \leq t-1$, we have $p = t-1$; and we thus have to prove that

$$S_a^3 + S_c^3 + \dots + S_l^3 + S_{t-1}^3 + S_t^3 < S_{t+1}^3.$$

Now since no three consecutive $\{S_i^3\}$ occur in the decomposition, we have $l < t-2$, and by induction

$$S_a^3 + S_c^3 + \dots + S_l^3 < S_{t+1}^3 < S_{t-2}^3.$$

Hence,

$$\begin{aligned} S_a^3 + S_c^3 + \dots + S_l^3 + S_{t-1}^3 + S_t^3 &< S_{t-2}^3 + S_{t-1}^3 + S_t^3 \\ &= S_{t+1}^3, \end{aligned}$$

proving the lemma.

Remark: The above lemma also holds for the *saṅkhyāṅkas* of the *varṇa-vṛttas* $S_i = 2^i$, and also for the *saṅkhyāṅkas* of ordinary *mātrā-vṛttas*, the Virahāṅka-Fibonacci numbers $\{S_i^2\}$, where in the latter case we only need to impose the condition that the decomposition does not contain consecutive Virahāṅka-Fibonacci numbers.

Now we state a proposition, which follows immediately from the above lemma.

Proposition 3: Let m be a natural number and

$$\begin{aligned} m &= S_{i_1}^3 + S_{i_2}^3 + \dots S_{i_l}^3 \quad \text{with } i_1 < i_2 < \dots < i_l \\ &= S_{j_1}^3 + S_{j_2}^3 + \dots S_{j_r}^3 \quad \text{with } j_1 < j_2 < \dots < j_r \end{aligned}$$

be two decompositions of m such that both the decompositions do not contain three consecutive *saṅkhyāṅkas* $\{S_k^3\}$. Then the two decompositions are the same. In particular, any decomposition of m having no three consecutive *saṅkhyāṅkas* $\{S_k^3\}$ is the canonical decomposition.

If we associate the $(n+1)$ -tuple $(a_0, a_1, a_2, \dots, a_n)$, where each a_i is either 0 or 1, with the number $a_0 S_0^3 + a_1 S_1^3 + a_2 S_2^3 + \dots + a_n S_n^3$, then we can formulate the result regarding the existence and uniqueness of the generalised Virahāṅka-Fibonacci decomposition as follows:

Proposition 4: There exists a bijection between numbers $0, 1, 2, \dots, S_n^3 - 1$, and the $(n + 1)$ -tuples $(a_0, a_1, a_2, \dots, a_n)$, where each a_i is either 0 or 1, satisfying the following conditions:

- (i) $a_0 = 0, a_n = 0$.
- (ii) No three consecutive a_i 's are 1.

The bijection is obtained by associating each number k (such that $0 \leq k \leq S_n^3 - 1$) with its canonical decomposition.

Remark: We can also prove that the above assignment is a bijection by showing that the cardinality of the set of all $(n + 1)$ -tuples satisfying (i) and (ii) is S_n^3 which can be done via a simple argument using induction.

In the above discussion, for the sake of simplicity, we have considered the case of the generalised *mātrā-vṛtta-prastāra* of 3 digits, 1, 2, 3, or, to use Nārāyaṇa's terminology, the case where the maximum digit $q = 3$. All the results which were demonstrated above for $q = 3$, can be easily generalised to the case where q is any positive integer greater than or equal to 2. We shall here just state (without proof) the basic result on the representation of any natural number uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers $\{S_n^q\}$:

Proposition 5: Let $q \geq 2$ be an integer. Then, every natural number is either a generalised Virahāṅka-Fibonacci number S_k^q for some $k > 0$, or can be uniquely expressed as a sum of generalised Virahāṅka-Fibonacci numbers $\{S_1^q, S_2^q, S_3^q, \dots\}$, under the condition that the sum does not contain any set of q consecutive generalised Virahāṅka-Fibonacci numbers.

7.3 The Mathematical Basis of the Naṣṭa and Uddiṣṭa Processes

We again restrict ourselves, for the sake of simplicity, to the case of the generalised *mātrā-vṛtta-prastāra* of the three digits, 1, 2, 3. We propose to enumerate the rows of the *prastāra* of

value n , using symbols 1, 2, 3, starting from the last row (consisting of n ones), numbering the last row as 0, the penultimate row as 1, and so on. If the rank of the row under our enumeration is t , then the numbering of the row in the usual enumeration (starting from the top row and taking its row-number as 1) is $S_n^3 - t$.

The process of *naṣṭa* involves writing down the row explicitly (without the aid of the *prastāra*) given the row-number associated with it.

In a *prastāra* of value n :

- i The last symbol of the m -th row is 1 if and only if $m < S_{n-1}^3$, and the penultimate symbol of the m -th row is the last symbol of the m -th row in the *prastāra* of value $n - 1$ (where the last row consists of $n - 1$ ones).
- ii The last symbol of the m -th row is 2 if and only if $S_{n-1}^3 \leq m < S_{n-1}^3 + S_{n-2}^3$, and the penultimate symbol of the m -th row is the last symbol of the $(m - S_{n-1}^3)$ -th row in the *prastāra* of value $n - 2$ (where the last row consists of $n - 2$ ones).
- iii The last symbol of the m -th row is 3 if and only if $S_{n-1}^3 + S_{n-2}^3 \leq m < S_{n-1}^3 + S_{n-2}^3 + S_{n-3}^3$, and the penultimate symbol of the m -th row is the last symbol of the $(m - S_{n-1}^3 - S_{n-2}^3)$ -th row in the *prastāra* of value $n - 3$ (where the last row consists of $n - 3$ ones).

Proceeding as above by induction, we can obtain each symbol of the m -th row of the *prastāra* of value n .

Given any integer, $0 < m < S_{n-1}^3$, we have the canonical decomposition of m :

$$m = a_0 S_0^3 + a_1 S_1^3 + a_2 S_2^3 + \dots + a_n S_n^3,$$

where, $a_0 = 0, a_n = 0$, and the rest of the a_i could be 0 or 1, as discussed above. To this canonical decomposition, we associate the string $(a_0, a_1, a_2, \dots, a_n)$ of zeroes and ones, with $a_0 = 0, a_n = 0$. Summarising the discussion above, we have:

Proposition 6: In a *prastāra* of value n :

- i If the string associated with m ends with (0, 0), the last symbol of the m -th row is 1, and the penultimate symbol is given by the truncated binary sequence with the last zero removed in the *prastāra* of value $n - 1$.
- ii If the string associated with m ends with (0, 1, 0), then the last symbol of the m -th row is 2 and the penultimate symbol is given by the truncated binary sequence with the end portion (1, 0) removed in the *prastāra* of value $n - 2$.
- iii If the string associated with m ends with (0, 1, 1, 0), the last symbol of the m -th row is 3 and the penultimate symbol is given by the truncated binary sequence with the end portion (1, 1, 0) removed in the *prastāra* of value $n - 3$.

The above proposition gives a way of writing down the m -th row of a *prastāra* of value n , given the canonical decomposition of m . Let the canonical decomposition of m be given by

$$m = a_0 S_0^3 + a_1 S_1^3 + a_2 S_2^3 + \dots + a_n S_n^3,$$

with $a_0 = 0$, $a_n = 0$, and the rest of the $a_i = 0$ or 1. We call the S_i^3 where the $a_i = 0$ as an *apatita-saṅkhyāṅka*, and the S_i^3 where the $a_i = 1$ as a *patita-saṅkhyāṅka*. The sequence of *patitas* and *apatitas* corresponding to m determine the sequence of entries from 1, 2, 3, comprising the m -th row. The above proposition actually gives the signature of 1, 2, 3, in terms of the *apatitas* (a) and the *patitas* (p) as shown in Table 21 (this is a particular case of Table 12).

Table 21: Signature of the digits 1; 2; 3 in terms of a and p

	S_{r-3}^3	S_{r-2}^3	S_{r-1}^3	S_r^3
1			a	(a)
2		a	p	(a)
3	a	p	p	(a)

We now briefly consider the converse process *uddiṣṭa*, where the objective is to determine the row-number (in the enumeration given above) given the sequence of symbols that occur in that particular row. Suppose that the row-number that we wish to determine is m and we now describe a process to determine that.

Suppose that the last symbol in the given row is 1, then $m < S_{n-1}^3$ and therefore S_{n-1}^3 does not occur in the decomposition of m . If we delete the last symbol 1, we obtain a row in the *prastāra* of value $n - 1$, where the row number t can be determined by following the same process by induction, and we have $m = t$.

Suppose that the last symbol of the m -th row is 2, then S_{n-1}^3 occurs in the decomposition of m . If we delete the last symbol 2, we obtain a row of value $n - 2$, whose row-number t in the *prastāra* of value $n - 2$, can be determined by following the same process by induction. We then set $m = S_{n-1}^3 + t$.

If the last symbol of the m -th row is 3, then S_{n-1}^3 and S_{n-2}^3 both occur in the canonical decomposition of m . If we delete the last symbol 3, we obtain a row of value $n - 3$, whose row number t in the *prastāra* of value $n - 3$, can be determined following the same process by induction. We then set $m = S_{n-1}^3 + S_{n-2}^3 + t$.

The above process is exactly the same as the one discussed in the previous section, where one, two and three *saṅkhyāṅkas* were written above each of the symbols 1, 2, 3, and only the second *saṅkhyāṅka* above 2, and the second and the third *saṅkhyāṅkas* above 3, were summed together. The total so obtained is the row-number in the alternate enumeration (starting with row-number 0 from the bottom of the *prastāra*) that we have used in this section. The total subtracted from S_n^3 gives the row-number in the usual enumeration which starts with row number 1 from the top of the *prastāra*.

The above discussion can be generalised to the case where the largest entry in each row of the *prastāra* is any natural number q . Clearly, the essential mathematical result that is at the heart of the *naṣṭa* and *uddiṣṭa* processes is the canonical decomposition of any positive integer uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers $\{S_i^n\}$.

Appendix

The *sūcī-pankti*, *matsya-meru* and the number of rows of *prastāra* with *p-sthānas*

In verse 21 of the Chapter on *Aṅkapāśa*, Nārāyaṇa defines the *sūcī-pankti* (needle sequence) or the *nārācīkā-pankti* (arrow-head sequence) as follows:

अन्तिममितवैशेषस्थानाङ्कमिताश्च ताः पृथक् स्थाप्याः ।
तासां यातः सूचीपङ्क्तिर्नारचिका वा स्यात् ॥

Vaiśeṣikī-panktis measured by the last digit, which are equal in number to the number of places are kept separately. Their product is the *sūcī-pankti* (needle sequence) or the *nārācīkā-pankti* (arrow-head sequence)

If p is the number of places and q is the final digit, then the arrow-head sequence is defined by multiplying the *vaiśeṣikī-pankti*, which is of the form 1, 1, 1, . . . , 1 (1s repeated q times), multiplied by itself p times, by the door-junction (*kapāṭa-sandhi*) method, which is nothing but the algebraic method of multiplying keeping in mind the different place-values (see Table 22).

Hence, the $(r+1)$ -th element of the sequence is a sum of multinomial co-efficients:

$$\begin{aligned} U_{p,q}(r) &= \text{Coefficient of } x^r \text{ in } (1 + x + x^2 + \dots + x^{q-1})^p \\ &= \text{Coefficient of } x^{p+r} \text{ in } (x + x^2 + \dots + x^q)^p \end{aligned}$$

In particular, for $q = 2$, $U_{p,2}(r) = C(p, r)$, i.e., in this case the elements of the arrow-head sequence are nothing but the binomial coefficients.

Nārāyaṇa gives an example of how the needle-sequence is obtained for the case when p

$= q = 3$, which is shown in Table 22. Clearly the elements of this sequence are nothing but the coefficients of different powers of x in the expansion of $(1 + x + x^2)^3$.

If we look at the above definition of the $U_{p,q}(r)$, that it is the coefficient of x^{p+r} in $(x + x^2 + \dots + x^q)^p$, we can easily see that this is also the number of sequences of length p , formed out of the digits 1, 2, . . . , q , such that the sum of the sequence is $p + r$. In other words, the number of rows of length p in the generalised *mātrā-vṛtta-prastāra* characterised by the total sum n and last digit q , is $U_{p,q}(n - p)$, where clearly we need to have $\lfloor \frac{n}{q} \rfloor \leq p \leq n$, (where $\lfloor \frac{n}{q} \rfloor$ denotes the integral part of $\frac{n}{q}$), in order that the sum of a sequence of length p composed of the digits 1, 2, . . . , q , may have the sum n . We shall now see that Nārāyaṇa also states essentially the same result while identifying the number of rows of different lengths in the *prastāra* with the elements of the so called *matsya-meru* (fish tabular figure).

Table 22: Needle sequence for $p = q = 3$ by door-junction method

			1	1	1
			1	1	1
			1	1	1
		1	1	1	
	1	1	1		
	1	2	3	2	1
		1	1	1	
	1	2	3	2	1
	1	2	3	2	1
1	2	3	2	1	
1	3	6	7	6	3
					1

In verses 40-43 of the chapter on *Aṅkapāśa*, Nārāyaṇa defines the tabular figure called the *matsya-meru* associated with a given sum n , last digit q , and also the number of places p , as follows:

रूपादिनिरेकान्तिमाङ्कवृद्धिकोष्ठपङ्क्तयः ।
स्थानमेकाधिकं यावत्तन्मिताः कोष्ठपङ्क्तयः ॥
मुक्ता स्वोर्ध्वादिमं कोष्ठं द्वितीयस्याधराधराः ।

पङ्क्तयः तिर्यग्गाः कार्या अङ्केक्येन समा अथ॥
 अङ्कं विन्यस्य ह्याद्यायां पङ्क्तौ रूपं च विन्यसेत्।
 उत्क्रमादन्तिमसमस्वोर्ध्वकोष्ठाङ्कसंयुतिम्॥
 अधो लिखेदन्तिमाङ्कसमाभावोभवेद्दि।
 तथा यथा सम्भवाङ्कयोगः कार्यः क्रमेण च॥

Rows of cells are made beginning with one cell and increasing each time by the last digit minus one. The number of rows is measured by the number of places plus one. Leaving the first cell above, horizontal rows equal in number to the sum of numbers are to be made below the second. One should place one in the first cell and, in each row below, write the sum in the reverse order of entries in the row above equal in number to the last digit. If there are not as many numbers as the last digit, one should write the sum of the available digits.

When we consider the *prastāra* which is defined by the sum n and the last digit q , the number of rows in *matsya-meru* is equal to $n + 1$. The first row has a single cell which has entry 1. The successive rows have their first cell moved to the right and each of them have $(q - 1)$ cells more than the previous row. This later rule is followed till the number of rows is equal to the total number of places $p + 1$. What is really meant by this condition is not clear, though from the example given below we can see that the number of columns is not increased after this stage. The cells of the successive rows are filled as follows. Each cell is filled by the sum of the entries in q cells of the row above which are just to the left of the given cell.

After defining the *matsya-meru*, Nārāyaṇa displays the form of the *meru* shown in Table 23, when the sum $n = 3$, the last digit $q = 3$ and the number of places $p = 3$.

While working out this example, Nārāyaṇa also notes that

अस्मिन् मत्स्यमेरौ नाराच्यः खण्डनारच्यः पङ्क्तयः सम्भवन्ति।

Table 23: *Matsyameru* for $n = 3$, $q = 3$ and $p = 3$

1									
	1	1	1						
		1	2	3	2	1			
			1	3	6	7	6	3	1
				1	4	10	16		
					1	5	15		
						1	6		
							1		

In this *matsya-meru*, we will have [the rows given by] the arrow-head and partial arrow-head sequences.

In fact, excluding the first row, the succeeding rows of the *matsya-meru* are filled with elements of the arrow-head sequences $U_{1,q}(r)$, $U_{2,q}(r)$, and so on. The process of generating the *meru* is actually based on the recurrence relation $U_{p,q}(r) = U_{p-1,q}(r-1) + U_{p-1,q}(r-2) + \dots + U_{p-1,q}(r-q)$, which can be derived easily from the definition of $U_{p,q}(r)$.

Later in the same Chapter on *Aṅkapāśa*, while discussing the generalised *mātrā-vṛtta-prastāra*, Nārāyaṇa seems to characterise the *matsya-meru* in terms of just the sum n and last digit q without any reference to some fixed number of places p . All the defining features of the *matsya-meru* as described above are retained, except that both the number of rows and columns of the *meru* are now fixed at $n + 1$. This is clear from the example of *matsya-meru*, displayed in Table 24, which is merely characterised by the sum $n = 3$ and the last digit $q = 3$. This form of the *meru* is presented by Nārāyaṇa while discussing an example following verse 79 in the Chapter on *Aṅkapāśa*.

In verse 80 of the Chapter on *Aṅkapāśa*, Nārāyaṇa indicates how the *matsya-meru* can be used in order to obtain the number of rows in the *prastāra* which have a given length:

अवसानस्थिता मत्स्यमेरोरुर्ध्वाङ्कपङ्क्तिः।
 ततः स्थानविभेदाः स्युः सर्वाङ्केष्वसमाहताः॥

Table 24: *Matsyameru* for $n = 3$; $q = 3$

1							
	1	1	1				
		1	2	3	2	1	
			1	3	6	7	6
				1	4	10	16
					1	5	15
						1	6
							1

The column (vertical row) at the end of the *matsya-meru*, from that are known the (number of) variations (or rows of the *prastāra* for each number of places (*sthāna*)...

In other words, the last column of the *matsya-meru* gives the number of rows in the *prastāra* with different lengths, or number of places. For the case where $n = q = 3$, the last column of the *matsya-meru* (shown in Table 24) is given by 6, 16, 15, 6, 1. These indeed are the number of rows in the *prastāra* which are of length 3, 4, 5, 6 and 7 respectively. The sum of this last column of the *meru* is equal to 44 which is indeed S_7^3 , the *saṅkhyā* or the total number of rows in the *prastāra* characterised by $n = q = 3$. In fact, the various column sums of the above *matsya-meru* are 1, 1, 2, 4, 7, 13, 24 and 44, which are nothing but the generalised Virahāṅka sequence $\{S_i^3\}$.

The last property mentioned above is indeed a particular case of a general relation between the elements of the *nārācika-paṅkti* $U_{p,q}(r)$ (which are also entries of *matsya-meru*) and the *sāmāsikī-paṅkti* or the generalised Virahāṅka sequence $\{S_n^q\}$, which is noted by Nārāyaṇa in verses 43-44 of the Chapter on *Aṅkapāśa*, while discussing the sums of the rows and columns of *matsya-meru*.

नाराच्यस्तिर्यगाः स्थानसम्मितास्तदुतिः पृथक्।
गुणोत्तरा भवेत्पङ्क्तिरूर्ध्वा अङ्कैक्यसम्मिताः ॥
पृथक्तदूर्ध्वकोष्ठाङ्कयोगात् सामासिका भवेत्।

The elements of the arrowhead sequence in each row are measured by the number of places. The sum of each of these (rows) separately becomes a geometric sequence (*guṇottara-paṅkti*). The columns are measured by the sum. The columns summed separately become equal to the elements of the *sāmāsikī-paṅkti*.

Here, Nārāyaṇa first mentions that the sum of the *sūcī-paṅkti* (for a given value of q) form a geometrical sequence. This is essentially the relation

$$\sum_{r=0}^{p(q-1)} U_{p,q}(r) = (1 + 1 + \dots + 1)^p = q^p$$

Nārāyaṇa then gives the following important relation between the coefficients $U_{p,q}(r)$ and the *sāmāsikī-paṅkti* or the generalised Virahāṅka-Fibonacci sequence:

$$\sum_{r=0}^{n-t} U_{n-r,q}(r) = S_n^q$$

where t is such that $(t-1)q < n \leq tq$.

The above relation, gives an important relation between the multinomial coefficients $U_{n-r,q}(r)$ and the elements of the generalised Virahāṅka-Fibonacci sequence $\{S_n^q\}$. It has also an important interpretation in the *prastāra* of the generalised *mātrā-vṛttas* which is the following.

Since $U_{n-r,q}(r)$ is the total number of generalised *mātrā-vṛttas* (sequences of 1, 2, ..., q), which are of length $n-r$ and whose total value or sum is n , the sum of all such numbers for various possible lengths, should clearly be equal to S_n^q , which is the *saṅkhyāṅka* or the total number of all the generalised *mātrā-vṛttas* of value n .

Remark: In mathematical terms, Nārāyaṇa's *matsya-meru* is a part of the triangle of multinomial coefficients. If we tilt this triangle in a certain way and take the sum of the rows, we get the generalised Virahāṅka-Fibonacci numbers. In a similar manner, in case we tilt the Piṅgala-

Pascal triangle and take the sum of the rows, we get the standard Virahāṅka-Fibonacci numbers (Raja Sridharan et al., 2012a)

BIBLIOGRAPHY

1. Source-works

Brhat-saṃhitā of Varāhamihira (c. 6th century): Ed. with the *Vivṛti* of Bhāṭṭotpala (c.967), by Sudhakara Dvivedi, 2 Vols., Varanasi 1897. Ed. with the *Vivṛti* of Bhāṭṭotpala, by Dvivedi K. C., 2 Vols, Sampurnananda Sanskrit University, Varanasi 1968.

Chandaḥ-Sūtra of Piṅgalanāga (c.300 BC): Ed. and Tr. into German in Weber A., *Über die Metrik der Inder* in A. Weber, *Indische Studien*, VIII Band, Harrwitz and Gofsmann, Berlin 1863; Rep. Hilberheim New York, 1973. Ed. with commentary *Mṛtasañjīvanī* of Halāyudha (c. 950), by Viswanatha Sastri, Bibliotheca Indica, Calcutta 1871-1874. Ed. with commentary *Mṛtasañjīvanī* of Halāyudha (c. 950), by Kedaranatha Vasudeva Lakshmana Pansikara, Kavyamala Series No. 91, Bombay 1908, Rep. Chowkhamba 2001.

Gaṇita-kaumudī of Nārāyaṇa Paṇḍita (c.1356): Ed. by Padmakara Dvivedi, 2 Vols., Varanasi 1936, 1942. Chapters 13-14 edited and translated in Kusuba T., *Combinatorics and Magic-squares in India: A Study of Nārāyaṇa Paṇḍita's Gaṇita-kaumudī*, Chapters 13-14, Ph.D. Dissertation (Unpublished), Brown University 1993. Translated into English by Paramanand Singh, *Gaṇita Bhāratī* 1998-2002.

Nārāyaṇī of Nārāyaṇabhaṭṭa (c.1555) on *Vṛtta-ratnākara* of Kedārabhaṭṭa (c.950): Ed. by Vaidyanatha Sastri Varkala, Kashi Sanskrit Series, Varanasi 1927. Ed. by Madhusudanna Sastri, Krishnadasa Academy, Varanasi 1994.

Prākṛta-Piṅgala (in Prākṛta, c.1300): Published in Kavyamala, Bombay 1894. Ed with commentary of Viśvanātha Pañcānana, Vaṃśīdhara etc., by Chandra Mohan Ghosh, Bibliotheca Indica, Calcutta 1902. Ed. by Vyasa B. S., 2 Vols., Prākṛta Text Society, Varanasi, 1959, 1962.

Sanḡita-ratnākara of Niḥśaṅka Śārṅgadeva (c.1225): Ed. with commentary *Kalānidhi* of Kallinātha (c.1420), 2 Vols., Anandasram Sanskrit Series, Pune 1896-7. Ed. with *Kalānidhi* of Kallinātha (c. 1420) and *Sanḡita-sudhākara* of Siṃhabhūpāla (c.1350), by Subramanya Sastri S., Krishnamacharya V. and Sarada S., 4 Vols., Adyar Library, 1943-1986. Chapters 1-4, Ed. with English translation and Notes by Premalata Sharma and Shringy R. K., 2 Vols., Delhi, 1978, 1989.

Vaṇī-bhūṣaṇa of Dāmodara (c.1550): Published in Kavyamala Series, No 53, Bombay 1895.

Vṛtta-jāti-samuccaya or *Kaṣiṭṭha-chanda* (in Prākṛta) of Virahāṅka (c.650): Ed. with commentary of Gopāla (c.1000) by Velankar H. D., JBBRAS 1929, 1932. Ed., with commentary by Gopāla, by Velankar H. D., Rajasthan Prachya Vidya Pratisthana, Jodhpur 1962.

2. Secondary works

Raja Sridharan, Sridharan R. and Srinivas M. D., Combinatorial Methods in Indian Music: *Pratyayas* in *Sanḡita-ratnākara* of Śārṅgadeva, in Seshadri C.S. Ed., *Studies in the History of Indian Mathematics*, Hindustan Publishing Agency, Delhi, 2010, pp. 55-112.

Raja Sridharan, Sridharan R. and Srinivas M. D., Nārāyaṇa Paṇḍita's Enumeration of Combinations and Associated Representations of Numbers as Sums of Binomial Coefficients, *IJHS* 47, 2012, 607-631.

Raja Sridharan, Sridharan R. and Srinivas M. D., Differences in Style, but not in Substance: Ancient Indian Study of Combinatorial Methods, in Sujatha R. et al Ed., *Math Unlimited*, Science Publishers, Enfield NH USA, 2012, pp.451-462.

Sridharan R., Sanskrit prosody, *Piṅgala Sūtras* and Binary Arithmetic, in Emch G. G., Srinivas M. D. and Sridharan R. Eds., *Contributions to the History of Mathematics in India*, Hindustan Publishing Agency, Delhi, 2005, pp. 33-62.

Sridharan R., *Pratyayas* for *Mātrāvṛttas* and Fibonacci Numbers, *Mathematics Teacher*, 42, 2006, 120-137.