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A NEW ALGORITHM FOR GENERATING A RANDOM PACKING OF ELLIPSOIDAL INCLUSIONS TO CONSTRUCT COMPOSITE MICROSTRUCTURE

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ABSTRACT

The subject of the work is a microstructure of a composite which consists of a continuum matrix and a set of isolated particles homogeneously distributed inside the matrix. It is assumed that the reinforcing particles have ellipsoidal shapes, while distribution and orientation are random. The main point of the work is a new computationally-efficient algorithm to generate microstructure of such a composite. In the algorithm the existing "concurrent" method based on an overlap elimination is extended to ellipsoidal shapes of the particles. It begins with randomly distributed and randomly oriented ellipsoidal particles which can overlap each other. During the performance of the algorithm intersections between particles are allowed and at each step the volumes of intersections are minimized by moving the particles. The movement is defined for each pair of particles based on the volume of the intersection: if two particles are overlapped, then the reference point inside the intersection is chosen and then two particles are moved in such a way that the reference point becomes the tangent point for both particles. To define the relative configuration of two particles (separate, tangent or overlapping) and to choose reference point inside the intersection volume the technique based on formulating the problem in four dimensions and then analyzing the roots of the characteristic equation are applied. The algorithm is able to generate close packed microstructures containing arbitrary ellipsoids including prolate and oblate ellipsoids with high aspect ratios (more than 10). The generated packings have a uniform distribution of orientations.

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1. Introduction

Synthesis of a representative volume element (RVE) of a composite material plays a key role for application of numerical simulation methods for determination problems of material effective properties, physical fields at the micro level, load-bearing capacity of the adhesive layer, conditions of microcrack origin, and many other things. Apparently, for such tasks, it is crucially important to model the microstructure with maximum approximation to the real internal structure of the material.

This paper is devoted to the algorithm for construction of spatial microstructures of the dispersion-strengthened composites. Such materials include two following components: the binding homogeneous matrix and evenly distributed filler particles (inclusions) in it. In this case, distribution of the inclusions in the matrix volume and their shape and size are considered as stochastic. Formation of the internal structure of such composites is associated with random arrangement of non-intersecting inclusions of various geometric shapes in a certain volume. To simulate actual geometric shapes of the inclusions, we use ellipsoids taking the form of balls (three semiaxes of the ellipsoid are equal to each other), disks (one of the semiaxes is much smaller than the other two), and stretched fiber (one of the semiaxes is much longer than the other two) in extreme cases.

In recent decades, methods of computer-aided microstructure formation have been actively developed. The methods can be conventionally divided into “successive” and “parallel”. The successive methods are based on piecewise addition of inclusions to the RVE structure, while the parallel algorithms use a certain procedure for distribution of a given number of inclusions to a random configuration. The maximum attainable volume fraction of inclusions and, in the case of non-spherical inclusions, the uniform distribution of inclusion orientation may be used as the evaluation criteria for the algorithms.

The maximum attainable volume fraction depends on the shape and size of the inclusions. There are several experimental works on determination of this value. Most of them are devoted to spherical inclusions. The maximum dense packing of balls with equal radius is $\pi/\sqrt{18} \approx 0,7405$ and it is achieved by formation of regular hexagonal packing (the Kepler's hypothesis formulated in 1611 [1] and proved in 1998 [2]). The maximum ratio for irregular packaging of the balls is not precisely defined, so is the concept itself (see discussion of this question in [3]). Usually, this value equals to 0.64 according to the series of experiments described in [4, 5]. If the inclusions have ellipsoidal shape, the maximum volume fraction with irregular packaging decreases rapidly with the increase of semiaxes ratio [6–13]. There is an experimental evaluation of the maximum attainable volume fraction for stretched fibers [9, 12]:

$$p\alpha = 5,3,$$

where p is the volume fraction of the inclusions; α is the ratio between length and diameter of the fibers. Similar dependence is presented in the paper [11] in the logarithmical form. The maximum attainable volume fraction also decreases with the increase of the semiaxes ratio for disk-shaped ellipsoids [13–17]. It should be noted that the limit volume fraction has not been determined yet for the general case of randomly-shaped ellipsoids of generalized size distribution. The methods of computer-aided microstructure formation can be used to estimate this value.

In the literature, the successive method that was presented in [18] for the first time, dominates other methods of computer-aided microstructure formation. Its essence is as follows. Position of the first inclusion is set randomly. Positions of the inclusions, introduced into the structure after the first one, are also set in a random way, however, the positions are checked for intersection with the previously placed inclusions, and if there are no intersections, the inclusion is added to the RVE structure. Otherwise, a new position is selected until there is one with no intersections. Addition of inclusions to the RVE structure continues until the required volume fraction is reached. This algorithm for ball inclusions of equal radius allows to achieve the volume fraction of 0.385 [19]. For the stretched ellipsoidal fibers, the qualitative dependence of the volume fraction on the semiaxes ratio repeats the experimental evaluations, but the maximum attainable volume fraction is significantly lower [19].

A fundamentally different idea of RVE formation was proposed in the paper [20] for round-shaped inclusions (the 2D-problem). The algorithm is based on the molecular dynamics method and begins with stochastic plotting of the vectors that determine positions and speeds of a given number of zero-volume particles (points). New positions of the particles are determined by integration of motion equations, while dimensions of the particles increase over time and two types of events are tracked: collision of the particles with each other and collision of the particles with the RVE boundaries. The integration process is iterative in the terms of time, and the steps of integration are determined by the time to the nearest event. Movement of the particles in the RVE continues until the required volume fraction is reached. For ball inclusions, collision time of two moving inclusions can be determined using simple analytic formulas, but the situation changes fundamentally when the inclusions are ellipsoidal. In this case, to determine the collision time, it is necessary to apply numerical methods that significantly slow down the algorithm. Thus, in the paper [21, 22], the collision time is determined based on the solution of the optimization problem of extremum seeking for a nonlinear function, and in the paper [23], the time is determined based on the solution of a nonlinear equation with variable coefficients. Therefore, implementation of this parallel algorithm requires great computational efforts, however, it allows to form an RVE with the volume fraction close to the limit values. The paper [24] demonstrate

formation of the RVE for spherical inclusions of equal radius with the volume fraction of 0.637, and the paper [23] contains examples of RVEs for ellipsoidal inclusions with volume fractions close to the experimental evaluations.

The parallel type also includes the algorithm for spherical inclusions presented in the paper [25]. As in the previous algorithm, the position vectors of the inclusion centers are set randomly, the inclusions have a given size determining the volume fraction. Random arrangement leads to intersection of the inclusions at the beginning (if there are no intersections, the RVE is completed and the algorithm performance is finished). After that the iterative process is started where the intersecting inclusions receive displacement vectors to minimize the intersection areas. The process continues until there are no intersections left. The volume fraction of 0.64 can be achieved for spherical inclusions [25]. In the paper [26], this algorithm was extended for cylindrical inclusions and the volume fractions are close to the experimental evaluations. The idea of the algorithm has not been implemented for ellipsoidal inclusions.

The most common algorithms in the literature are listed above. Without going into details, we give references to two more algorithms that are original, but complex in realization and require significant computational resources. They are presented in the papers [27] and [28]. It should be noted that today only one algorithm based on the molecular dynamics method allows to form RVEs for ellipsoidal inclusions with limiting values of volume fractions, but its realization is computationally expensive.

The main goal of this work is to expand the algorithm for spherical and cylindrical inclusions that minimizes the intersection areas [25, 26], to the case of ellipsoidal inclusions. The idea behind the proposed method is as follows. First, positions and orientations of the inclusions of a given size are set randomly. Then, we perform search for pairs of intersecting inclusions. Each of such pairs has a point inside the intersection area, and each of the inclusions is moved in such a way that this point becomes the contact point. The algorithm takes into account the possibility to use the formed microstructure for the problems of numerical homogenization where it is preferable to use periodic RVEs [29]. In this case, distribution of the inclusions inside the RVE is stochastic, however, each inclusion, intersecting the RVE border, has a periodic image on the opposite face of the RVE. This is an auxiliary option, and it can be easily excluded from the algorithm if necessary.

The structure of the article allows to implement the proposed algorithm, therefore all realization steps are described in sufficient detail. The following section introduces the mathematical tools necessary for geometric modeling of RVE inclusions. The third section is directly concerned with realization of the algorithm. The basic steps of the algorithm are explained, and comments are made on their numerical implementation. The fourth section presents the results of the algorithm. In that section, we give examples of RVEs with various forms of inclusions,

evaluate uniform orientation distribution of the inclusions, and compare the algorithm with other algorithms.

2. Inclusions in the Representative volume Element

This is auxiliary section that serves to preserve integrity of the presentation. The information given here is generally known, however, the geometric modeling introduces a certain specificity in the description of location and mutual arrangement of RVE inclusions, therefore, to understand the main part of the work, we dwell on this issue in sufficient detail.

Throughout the article, the capital Latin letters indicate scalars, lowercase bold letters indicate vectors, capital bold letters indicate matrices.

2.1. Location of Inclusion in the Representative volume Element

The article discusses an RVE in the form of an L-side cube with randomly located ellipsoidal inclusions inside. Let us introduce the orthogonal coordinate system $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ so that the point O coincides with one of the vertices of the cube, and we direct the unitary vectors along the sides forming this vertex. Let us consider an ellipsoid randomly located with respect to $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. We denote by O' the center of the ellipsoid and introduce the local coordinate system $O'\mathbf{e}'_1\mathbf{e}'_2\mathbf{e}'_3$ where the axes are directed along the main axes of the ellipsoid. Then, the ellipsoid equation in the local coordinate system takes the canonical form

$$\mathbf{x}'^T \mathbf{A}' \mathbf{x}' = 0,$$

where

$$\mathbf{A}' = \begin{pmatrix} 1/a^2 & 0 & 0 & 0 \\ 0 & 1/b^2 & 0 & 0 \\ 0 & 0 & 1/c^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

a, b, c are the semiaxes of the ellipsoid. The homogeneous coordinates are used to describe location of the points in space, i.e. the vector $\mathbf{x}$ is defined by four numbers $\mathbf{x} = (x_1, x_2, x_3, w)^T$. It is to be recalled that $\mathbf{x} = (x_1/w, x_2/w, x_3/w, w)^T$ is the vector of the corresponding point in the Cartesian coordinate system.

Fig. 1. An ellipsoidal inclusion in the representative volume element

The reason for introduction of homogeneous coordinates is their efficiency for the point transformation expression in numerical simulation, as in this case, rotation and displacement can be described using matrix multiplication. Transition from the local to global coordinates is carried out using the matrix $\mathbf{M}$:

$$\mathbf{x} = \mathbf{M}\mathbf{x}',$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

$\mathbf{R}$ is the rotation matrix; $\mathbf{r}$ is the position vector of the point O' . So, in the global coordinate system, the ellipsoid equation takes the following form

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 0,$$

where

$$\mathbf{A} = \mathbf{M}^{-T} \mathbf{A}' \mathbf{M}^{-1},$$

where $\mathbf{M}^{-T}$ means the inverse matrix of $\mathbf{M}^T$. Due to the fact that $\mathbf{R}$ is an orthogonal matrix, the inverse matrix $\mathbf{M}^{-1}$ can be calculated using the following formula

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

The matrix $\mathbf{R}$ determines rotation at the angle θ around the unitary vector $\mathbf{m}$. We write this matrix using the quaternion which is determined by specification of the scalar α and vector $\mathbf{v}$ according to the following rule

$$\mathbf{q} = (\alpha, \mathbf{v}) = \left(\cos \frac{\theta}{2}, \mathbf{m} \sin \frac{\theta}{2} \right).$$

It should be noted that as $|\mathbf{m}| = 1$, then $|\mathbf{q}| = \sqrt{\alpha^2 + |\mathbf{v}|^2} = 1$, i.e. the quaternion is normalized. Then, the rotation matrix can be represented in the following form

$$\mathbf{R} = (2\alpha^2 - 1)\mathbf{E} + 2\mathbf{v}\mathbf{v}^T + 2\alpha\mathbf{S},$$

where $\mathbf{E}$ is the identity matrix, and the matrix $\mathbf{S}$ is expressed by the components of the vector $\mathbf{v}$

$$\mathbf{S} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix},$$

where v_i is the i^{th} component of the vector $\mathbf{v}$. Let us also recall the multiplication rule for two quaternions, i.e. the turns addition rule. Let the ellipsoid orientation be set by the quaternion $\mathbf{q}_1 = (\alpha_1, \mathbf{v}_1)$ and let it be necessary to rotate this ellipsoid with the quaternion $\mathbf{q}_2 = (\alpha_2, \mathbf{v}_2)$, then the quaternion, defining the new orientation, is calculated using the following formula

$$\mathbf{q}_{12} = (\alpha_{12}, \mathbf{v}_{12}) = (\alpha_1\alpha_2 - \mathbf{v}_1^T \mathbf{v}_2, \alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_1 - \mathbf{v}_1 \times \mathbf{v}_2).$$

2.2. Mutual Arrangement of Two Inclusions

Let us assume that there are two ellipsoids $\mathbf{x}^T \mathbf{A}^i \mathbf{x} = 0$ and $\mathbf{x}^T \mathbf{A}^j \mathbf{x} = 0$. We formulate the following characteristic equation

$$\det(\lambda \mathbf{A}^i - \mathbf{A}^j) = p_1 \lambda^4 + p_2 \lambda^3 + p_3 \lambda^2 + p_4 \lambda + p_5 = 0. \quad (1)$$

This is a fourth power polynomial and, therefore, there are always four roots. The paper [30] shows that the mutual arrangement of two ellipsoids can be determined based on the analysis of the roots of the characteristic equation (1). First, it should be noted that, in the case of ellipsoids, this equation always has two positive roots, so the analysis is based on the remaining two roots. The following options are possible:

a) two aliquant negative roots: $\mathbf{A}^i$ and $\mathbf{A}^j$ with no common points;

b) two multiple negative roots: $\mathbf{A}^i$ and $\mathbf{A}^j$ touching each other at the outside faces of both ellipsoids (i.e. they have only one common point);

c) two complex conjugate roots: $\mathbf{A}^i$ and $\mathbf{A}^j$ with common area;

d) two multiple positive roots: $\mathbf{A}^i$ and $\mathbf{A}^j$ with common area, their borders have one or two contact points;

e) two aliquant positive roots: $\mathbf{A}^i$ and $\mathbf{A}^j$ intersecting each other, or one of the ellipsoids is inside the other.

There are no other possible root combinations.

We can calculate the coefficients of the characteristic polynomial using the formulas given in the paper [31]:

$$p_1 = -\delta_1 \delta_2 \delta_3; \quad p_2 = \delta_2 \delta_3 C_{11} + \delta_1 \delta_3 C_{22} + \delta_1 \delta_2 C_{33} - \delta_1 \delta_2 \delta_3 C_{44};$$

$$p_3 = \delta_1 \delta_2 (C_{33} C_{44} - C_{34} C_{43}) + \delta_2 \delta_3 (C_{11} C_{44} - C_{14} C_{41}) +$$

$$+ \delta_1 \delta_3 (C_{22} C_{44} - C_{24} C_{42}) + \delta_1 (C_{23} C_{32} - C_{22} C_{33}) +$$

$$+ \delta_2 (C_{13} C_{31} - C_{11} C_{33}) + \delta_3 (C_{12} C_{21} - C_{11} C_{22});$$

$$p_4 = \delta_1 (-C_{22} C_{33} C_{44} + C_{22} C_{34} C_{43} + C_{33} C_{42} C_{24} +$$

$$+ C_{44} C_{32} C_{23} - C_{32} C_{24} C_{43} - C_{42} C_{23} C_{34}) +$$

$$+ \delta_2 (-C_{11} C_{33} C_{44} + C_{11} C_{34} C_{43} + C_{33} C_{14} C_{41} +$$

$$+ C_{44} C_{13} C_{31} - C_{31} C_{14} C_{43} - C_{41} C_{13} C_{34}) -$$

$$+ \delta_3 (-C_{11} C_{22} C_{44} + C_{11} C_{24} C_{42} + C_{22} C_{14} C_{41} +$$

$$+ C_{44} C_{12} C_{21} - C_{21} C_{14} C_{42} - C_{41} C_{12} C_{24}) +$$

$$+ C_{11} C_{22} C_{33} - C_{11} C_{23} C_{32} - C_{22} C_{13} C_{31} -$$

$$- C_{33} C_{12} C_{21} + C_{21} C_{13} C_{32} + C_{31} C_{12} C_{23};$$

$$p_5 = \det(\mathbf{A}^j),$$

where

$$\delta_1 = 1/a^{i2}, \quad \delta_2 = 1/b^{i2}, \quad \delta_3 = 1/c^{i2}, \quad \mathbf{C} = \mathbf{M}^{iT} \mathbf{A}^j \mathbf{M}^i.$$

Consequently, the mutual arrangement of two ellipsoids comes to definition of roots of the fourth power polynomial which is easy to implement numerically and achieve performing countable algebraic actions.

We should also mention one more important property of the problem (1) for the generalized eigenvalues. The paper [19] shows that: in the cases b) and d), the corresponding eigenvector (s) indicates the contact point (s); in the case c), the real corresponding eigenvectors (complex conjugate eigenvectors) indicate a point inside the intersection area; in the case e), one of the corresponding eigenvectors indicates a point inside the intersection area. This property will be used below.

2.3. Mutual Arrangement of an Inclusion and the Boundaries of the Representative volume Element

Let us assume that, in the global coordinate system, the ellipsoid equation takes the following form

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 0.$$

The matrix $\mathbf{A}$ is symmetric and can be represented in the following form

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \frac{1}{2} \mathbf{d} \\ \frac{1}{2} \mathbf{d}^T & F \end{pmatrix},$$

where $\mathbf{B}$ is the matrix 3×3 ; $\mathbf{d}$ is the vector 3×1 ; F is the scalar. Then, in the Cartesian coordinate system, the ellipsoid equation takes the following form

$$\mathbf{x}^T \mathbf{B} \mathbf{x} + \mathbf{d}^T \mathbf{x} + F = 0, \quad (2)$$

where $\mathbf{x} = (x, y, z)$ is the Cartesian coordinates. It should be noted that $\mathbf{B}$ is a symmetric positive definite matrix, as the last equation is an ellipsoid equation.

The borders of the RVE are defined using the following equations

$$x_k = \beta, \quad k = 1, 2, 3, \quad \beta = 0, L.$$

I.e., for example, we have two planes $x_1 = 0$ and $x_1 = L$ for the first variable. In order to establish whether the ellipsoid intersects the plane $x_k = \beta$, we introduce a vector on this plane

$$\tilde{\mathbf{x}} = (x_l, x_m)^T, \quad l < m, \quad l \neq k \neq m.$$

Then, a point on the plane can be represented in the following way

$$\mathbf{x} = \mathbf{P} \tilde{\mathbf{x}} + \mathbf{p}, \quad (3)$$

where $\mathbf{P}$ is the matrix 3×2 composed of two unitary vectors of the axes l and m

$$\mathbf{P} = (\mathbf{e}_l, \mathbf{e}_m);$$

and $\mathbf{p}$ is the vector spanned by the unitary vector $\mathbf{e}_k$:

$$\mathbf{p} = \beta \mathbf{e}_k.$$

For example, for the plane $x_1 = L$, the equation (3) takes the following form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} L \\ 0 \\ 0 \end{pmatrix}.$$

Using (3) in (2), we get

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{B}} \tilde{\mathbf{x}} + \tilde{\mathbf{d}}^T \tilde{\mathbf{x}} + \tilde{F} = 0, \quad (4)$$

where

$$\tilde{\mathbf{B}} = \mathbf{P}^T \mathbf{B} \mathbf{P},$$

$$\tilde{\mathbf{d}} = 2\mathbf{P}^T \mathbf{B} \mathbf{p} + \mathbf{P}^T \mathbf{d},$$

$$\tilde{F} = \mathbf{p}^T \mathbf{B} \mathbf{p} + \mathbf{d}^T \mathbf{p} + F.$$

It should be noted that the matrix $\tilde{\mathbf{B}}$ is obtained from $\mathbf{B}$ by crossing out the row k and column k . And since the matrix $\mathbf{B}$ is positive definite, $\tilde{\mathbf{B}}$ is also positive definite.

Let us change variables $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}^* - \tilde{\mathbf{x}}_0$ in (3). We have

$$\tilde{\mathbf{x}}^{*T} \tilde{\mathbf{B}} \tilde{\mathbf{x}}^* + (2\tilde{\mathbf{x}}_0^T \tilde{\mathbf{B}} - \tilde{\mathbf{d}}^T) \tilde{\mathbf{x}}^* + \tilde{F} + \tilde{\mathbf{x}}_0^T \tilde{\mathbf{B}} \tilde{\mathbf{x}}_0 - \tilde{\mathbf{d}}^T \tilde{\mathbf{x}}_0 = 0. \quad (5)$$

We choose $\tilde{\mathbf{x}}_0$, so that the expression in brackets equals to zero

$$\tilde{\mathbf{x}}_0 = \frac{1}{2} \tilde{\mathbf{B}}^{-1} \tilde{\mathbf{d}}.$$

Then, the equation (5) takes the following form

$$\tilde{\mathbf{x}}^{*T} \tilde{\mathbf{B}} \tilde{\mathbf{x}}^* + \tilde{g} = 0, \quad (6)$$

where the scalar $\tilde{g}$ is defined by the expression

$$\tilde{g} = \tilde{F} - \frac{1}{4} \tilde{\mathbf{d}}^T \tilde{\mathbf{B}}^{-1} \tilde{\mathbf{d}}.$$

So, the type of the ellipsoid intersection with the plane $x_k = \beta$ is defined by the equation (6), and since $\tilde{\mathbf{B}}$ is a positive definite matrix, the three following cases are possible:

- 1) If $\tilde{g} < 0$, the ellipsoid intersects the plane and the intersection has the form of an ellipse.
- 2) If $\tilde{g} = 0$, the ellipsoid intersects the plane at one point.
- 3) If $\tilde{g} > 0$, the ellipsoid does not intersect the plane.

3. Formation Algorithm for a Representative volume Element

Let us describe the main steps of the algorithm. First, we randomly set location of the ellipsoid centers in the L-side cube. Orientation of the ellipsoids is also defined

randomly. Dimensions of the ellipsoids are chosen based on the volume fraction that the RVE should have. At each step of the algorithm, we check two types of intersection: intersection of the ellipsoids with the cube boundaries and mutual intersections of the ellipsoids. For the first type of intersection, we create periodic images of the ellipsoids on the opposite side in order to maintain periodicity of the RVE. In the second case, we move the ellipsoids apart to minimize the intersection area. The algorithm performance is finished when there is no mutual intersection of the ellipsoids. Let us consider each type of intersection in more detail.

3.1. Intersection Including the Boundaries of the Representative volume Element

Let us assume that the ellipsoid intersects the boundary of the RVE. In this case, it is necessary to create its periodic image on the opposite side. The distance between the centers of the original ellipsoid and its periodic image is defined by the vector

$$\xi(\beta)\mathbf{e}_k, \quad \xi(\beta) = \begin{cases} L, & \beta = 0, \\ -L, & \beta = L. \end{cases} \quad (7)$$

It should be noted that the periodic image has the same spatial orientation as the original ellipsoid. It is clear that the number of periodic images of an ellipsoid depends on the number of borders it intersects in the area. If the ellipsoid intersects two planes $x_k = \beta_k$ and $x_l = \beta_l$, then besides two periodic images defined by the expression (7), it is necessary to create the third one with the center determined by the vector

$$\xi(\beta_k)\mathbf{e}_k + \xi(\beta_l)\mathbf{e}_l. \quad (8)$$

Finally, if the ellipsoid intersects three planes $x_k = \beta_k$, $x_l = \beta_l$ and $x_m = \beta_m$, then there are seven periodic images. Location of the centers of six periodic images are defined by the vectors (7) and (8), and the seventh location is determined by the vector

$$\xi(\beta_k)\mathbf{e}_k + \xi(\beta_l)\mathbf{e}_l + \xi(\beta_m)\mathbf{e}_m.$$

3.2. Mutual Intersection of Inclusions

Let us assume that two inclusions $\mathbf{A}^i$ and $\mathbf{A}^j$ intersect. The idea behind the proposed method is to search for a new position of the inclusions where they would have only one common point. It is proposed to search for such a position using the iterative way and gradually reducing the intersection area. To do this, we select a new reference point within the intersection area at each step. Next, we plot the displacement vector for the points of the ellipsoids that coincide with the reference point, so the reference point is on the edges of the ellipsoids and, at the same time, is close to their tangency point (the degree of approximation will be

determined below). New position of the ellipsoids is determined by their rotation and displacement, so that the points of the ellipsoids that coincide with the reference point, move to the given vectors.

3.2.1. Selection of the Reference Point in the Intersection Area

It should be noted that the way of the reference point selection is not uniquely determined. It is clear that a good choice can speed up subsequent performance of the algorithm. On the other hand, one should not forget about the time that the search for the "optimal" point may require. For example, we can select the mass center of the intersection area as such a point. However, the search for the mass center can take quite a lot of computing time and slow down operation of the algorithm. Therefore, it is proposed to choose a reference point determined by the eigenvectors for the generalized eigenvalue problem given in § 2.3, as at this step of the algorithm, the eigenvalues are already known and calculation of corresponding eigenvectors does not require great computational efforts.

As mentioned above, there are three possible cases of characteristic polynomial roots for intersecting ellipsoids. If the roots are complex conjugate, then the eigenvectors will also be complex conjugate. In this case, the real eigenvectors indicate a point inside the intersection area. If there are two aliquot positive roots, then one of the eigenvectors indicates a point inside the intersection area. Finally, in the case of two multiple positive roots, the eigenvector (s) indicates the contact point (s). In this case, we propose to choose the point inside the intersection area in the following way. Let us introduce into consideration a straight line passing through the contact point $\mathbf{x}^c$ (in the case of two points, one of them is chosen randomly) and parallel to the normal vector $\mathbf{n}$ to the surfaces of the ellipsoids at this point:

$$\mathbf{x} = \mathbf{x}^c + \mathbf{n}t, \quad t \in \mathbb{R}.$$

To determine the normal vector $\mathbf{n}$, we differentiate the equation of one ellipsoid

$$\mathbf{n} = -\nabla \left(\mathbf{x}^T \mathbf{B} \mathbf{x} + \mathbf{d}^T \mathbf{x} + F \right) \Big|_{\mathbf{x}=\mathbf{x}^c} = -2\mathbf{B}\mathbf{x}^c - \mathbf{d},$$

We choose the negative sign, because we are interested in the normal going inside the area. Let us find the intersection points of the line and the ellipsoids. To do this, we use the equation of the line in the equations of the ellipsoids (2)

$$\left(\mathbf{x}^c + \mathbf{n}t \right)^T \mathbf{B}^k \left(\mathbf{x}^c + \mathbf{n}t \right) + \mathbf{d}^{kT} \left(\mathbf{x}^c + \mathbf{n}t \right) + F^k = 0, \quad k = i, j.$$

Opening the brackets and combining similar terms, we get two quadratic equations

$$\alpha^k t^2 + \beta^k t + \gamma^k = 0,$$

where

$$\alpha^k = \mathbf{e}^T \mathbf{B}^k \mathbf{e}, \quad \beta^k = 2\mathbf{e}^T \mathbf{B}^k \mathbf{e} + \mathbf{d}^{kT} \mathbf{e},$$

$$\gamma^k = \mathbf{x}^{cT} \mathbf{B}^k \mathbf{x}^c + \mathbf{d}^{kT} \mathbf{x}^c + F^k$$

Each of the equations has a zero and positive root. We denote the positive roots by t^i and t^j for the ellipsoids i and j respectively. Location of the reference point is determined by the following expression

$$\mathbf{x}^m = \mathbf{x}^c + \mathbf{n} \min(t^i, t^j) / 2.$$

3.2.2. Definition of the Displacement Vector

Plotting of the displacement vector is identical for both ellipsoids, therefore we consider only one of them. According to the method, the ellipsoid point that coincides with the reference point, must move in such a way so that the reference point is on the surface of the ellipsoid, and the normal vectors of both ellipsoids in the reference point must be collinear. Therefore, it is necessary to determine the normal vector in the new unknown position of the ellipsoid. To do this, we introduce the following function into consideration

$$f(\mathbf{x}) = -\mathbf{x}^T \mathbf{B} \mathbf{x} - \mathbf{d}^T \mathbf{x} - F.$$

This function has its maximum in the middle of the ellipsoid and equals 1 at the maximum point. It equals zero on the border of the ellipsoid and is less than zero outside it. Let us find the gradient of this function

$$\nabla f(\mathbf{x}) = -2\mathbf{B} \mathbf{x} - \mathbf{d}. \quad (9)$$

We approximately determine the normal vector for the new position of the ellipsoid at the reference point $\mathbf{x}^m$ along the gradient of the function $f(\mathbf{x})$:

$$\mathbf{e} = \frac{\nabla f(\mathbf{x}^m)}{|\nabla f(\mathbf{x}^m)|}.$$

Fig. 2 shows the level surfaces of the function $f(\mathbf{x})$. It is clear that the smaller the intersection area, the closer the level surface, where the point $\mathbf{x}^m$ lies, to the surface of the ellipsoid, and the smaller the angle between the vector $\mathbf{e}$ and the normal to the ellipsoid surface at the point closest to $\mathbf{x}^m$.

Fig. 2. Isolines of function $f(\mathbf{x})$

The desired displacement vector is co-directional with the vector $\mathbf{e}$. The displacement distance is defined as the nearest distance from the reference point to the surface of the ellipsoid along the straight line defined by the vector $\mathbf{e}$. To do this, we write the equation of the line in the parametric form

$$\mathbf{x} = \mathbf{x}^m + \mathbf{e}t, \quad t \in R.$$

Let us find intersection points of the line and the ellipsoid. To do this, we use the equation of the line in the equation of the ellipsoid (2)

$$(\mathbf{x}^m + \mathbf{e}t)^T \mathbf{B} (\mathbf{x}^m + \mathbf{e}t) + \mathbf{d}^T (\mathbf{x}^m + \mathbf{e}t) + F = 0.$$

Opening the brackets and combining similar terms, we get the following quadratic equation

$$\alpha t^2 + \beta t + \gamma = 0,$$

where

$$\alpha = \mathbf{e}^T \mathbf{B} \mathbf{e}, \quad \beta = 2\mathbf{e}^T \mathbf{B} \mathbf{x}^m + \mathbf{d}^T \mathbf{e}, \quad \gamma = \mathbf{x}^{mT} \mathbf{B} \mathbf{x}^m + \mathbf{d}^T \mathbf{x}^m + F.$$

This equation gives us two real roots t_1 and t_2 (we will assume that $t_1 < t_2$). Two values of the parameter t correspond to two points

$$\mathbf{x}^1 = \mathbf{x}^m + \mathbf{e}t_1, \quad \mathbf{x}^2 = \mathbf{x}^m + \mathbf{e}t_2.$$

It is apparent that the roots t_1 and t_2 have different signs, because $\mathbf{x}^m$ is inside the ellipsoid. The displacement module will be determined by the negative root, as the movement must be carried out along the vector $\mathbf{e}$. We define the displacement distance of the point $\mathbf{x}^m$ in the following way

$$|\mathbf{u}| = \sqrt{(\mathbf{x}^m - \mathbf{x}^1)^T (\mathbf{x}^m - \mathbf{x}^1)} = \sqrt{t_1 \mathbf{e}^T t_1 \mathbf{e}} = |t_1|.$$

So, the displacement vector of the point is as follows $\mathbf{x}^m$

$$\mathbf{u} = |\mathbf{u}| \mathbf{e}. \quad (10)$$

Such a definition of the displacement vector corresponds to the simple mechanical interpretation where the function $f(\mathbf{x})$ represents the potential of a certain force pushing the ellipsoid out from the intersection area.

3.2.3. Definition of Displacement and Rotation of the Ellipsoid

Once we found the displacement vector of the ellipsoid point coincident with the reference point, it is necessary to determine the new position of the ellipsoid. In the limiting case when the ellipsoid has the form of a ball, the new position is determined by simple transfer of the ball center to the vector $\mathbf{u}$. In the general case, there are several ways to define the new position. For example, we can move the

center of the ellipsoid or rotate the ellipsoid around a certain axis. This article proposes combination of these two possibilities. To do this, we represent the vector $\mathbf{u}$ as the sum of two vectors:

$$\mathbf{u} = u_c \mathbf{e}_c + (\mathbf{u} - u_c \mathbf{e}_c), \quad (11)$$

where the unitary vector $\mathbf{e}_c$ indicates direction from the point $\mathbf{x}^m$ to the center of the ellipsoid

$$\mathbf{e}_c = \frac{\mathbf{r} - \mathbf{x}^m}{\sqrt{(\mathbf{r} - \mathbf{x}^m)^T (\mathbf{r} - \mathbf{x}^m)}}, \quad (12)$$

where $\mathbf{r}$ is the position vector pointing to the center of the ellipsoid. u_c is determined using the scalar product of the vectors $\mathbf{u}$ and $\mathbf{e}_c$

$$u_c = \mathbf{e}_c^T \mathbf{u}.$$

The new position of the ellipsoid center is defined by the following expression

$$\mathbf{r}_N = \mathbf{r} + u_c \mathbf{e}_c. \quad (13)$$

We set rotation of the ellipsoid in such a way so that displacement of the point, coincident with $\mathbf{x}^m$, is determined by the vector $(\mathbf{u} - u_c \mathbf{e}_c)$. To do this, we define the vector that is the axis of rotation by the vector product

$$\mathbf{m} = \frac{(\mathbf{u} - u_c \mathbf{e}_c) \times \mathbf{e}_c}{\sqrt{(\mathbf{u} - u_c \mathbf{e}_c)^T (\mathbf{u} - u_c \mathbf{e}_c)}}.$$

It should be noted that $|\mathbf{m}| = 1$. We define the approximate rotation angle

$$\theta = \frac{\sqrt{(\mathbf{u} - u_c \mathbf{e}_c)^T (\mathbf{u} - u_c \mathbf{e}_c)}}{\sqrt{(\mathbf{r} - \mathbf{x}^m)^T (\mathbf{r} - \mathbf{x}^m)}}.$$

Therefore, the rotation is determined by the quaternion

$$\mathbf{q} = \left(\cos \frac{\theta}{2}, \mathbf{m} \sin \frac{\theta}{2} \right). \quad (14)$$

Fig. 3 shows the iterative process that converts intersecting ellipsoids into contacting ones. At the first step we: choose the reference point inside the intersection area; calculate the gradient of the function $f(\mathbf{x})$ using the formula (9); determine the displacement vectors of the ellipsoid points coincident with the reference point using the expression (10); set the new position and spatial orientation of the ellipsoids defined by the formulas (13) and (14). At the second step, the reference point is located on the ellipsoid faces, but it is not a contact point, as the normal vectors to the two ellipsoids are approximately defined at this point, moreover, they are not collinear, and as the result, the ellipsoids still have a common area, however, it is less than at the first step. After that the iterative process continues with gradual decrease of the intersection area, and the normal vectors, determined at the reference point, gradually become collinear.

Fig. 3. Stages in the elimination of overlaps

4. Results of the Study

4.1. Examples of Packaging

Fig. 4 and Fig. 5 show examples of RVEs for various forms of inclusions with the maximum volume fraction obtained using the proposed algorithm, and Table 1 compares the results with the results of the successive [19] and parallel [23] algorithms. The Table shows that, as it has already been mentioned in the introduction, the maximum volume fraction of the successive algorithm is significantly lower than the values obtained using parallel algorithms. Comparison with the results of the paper [23] shows that the maximum volume fraction of the proposed algorithm is not less than the maximum volume fraction of the parallel algorithms, and it is even bigger for some ratios of semiaxes.

Fig. 4. Representative volume elements contain inclusions of prolate ellipsoidal shape

Fig. 5. Representative volume elements contain inclusions of oblate ellipsoidal shape

Table 1

Comparison of maximum volume fraction obtained with various algorithms

Ratio of the ellipsoid semiaxes a/c	The results given in [19]	The results given in [23]	The results of this work
1/2	0.40	0.60	0.66
1/0	0.25	0.30	0.33
1/30	–	0.10	0.10
2	0.41	0.60	0.65
10	0.26	0.30	0.33
30	–	0.10	0.10

Fig. 6. Representative volume element of polydisperse ellipsoids with the particular size and aspect ratio distributions. volume fraction 36 %, number of ellipsoids 300

In practice, one can often encounter composite materials where the dimensions of the inclusions and the ratio of the semiaxes are stochastic quantities. We use the analysis based on a two-dimensional image of the inclusions to assess forms of the inclusions in such materials. The output data of the analysis are the distribution of the inclusions by their mean diameter (diameter of a circle with an area equal to the area of the inclusion projection) and the ratio of semiaxes. Fig. 6, *b* and *c* show an example of such distributions, and Fig. 6, *a* shows the corresponding RVE. Therefore, the proposed algorithm allows to form a microstructure with ellipsoids of random sizes and shapes.

4.2. Distribution of Ellipsoids by Orientation

One of the important RVE characteristics is distribution of the ellipsoids by their orientation. It is clear that the distribution should be close to uniform (isotropic) distribution for random arrangement of the inclusions. To estimate proximity of the probability function of the inclusion orientation distribution to a constant value, we introduce into consideration the orientation matrix proposed in the paper [32]

$$T_{m,n} = \langle w_m w_n \rangle, \quad m, n = 1, 2, 3,$$

where w_1 , w_2 and w_3 are three components of the unitary vector $\mathbf{w}$ directed along one of the ellipsoid axes; $\langle \bullet \rangle$ means averaging of all ellipsoids in the RVE. It should be noted that the orientation matrix is symmetric and the trace of the matrix equals to 1. In the case of isotropic distribution, the orientation tensor equals to $(1/3)\mathbf{E}$ where $\mathbf{E}$ is the identity matrix. To estimate proximity of the orientation matrix to $(1/3)\mathbf{E}$, we introduce into consideration the relation $\rho = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalues of the orientation matrix. We can estimate proximity of the orientation distribution to the isotropic distribution according to proximity of the number ρ to 1. Fig. 7 shows dependence of the number ρ on the number of inclusions for two types of ellipsoids. We performed 10 realizations of the algorithm for each number of inclusions. The graphs show mean values of ρ with the confidence interval of 95 %. In both cases, the volume fraction of the ellipsoids equals to 0.1. The graphs show that there is $\rho \rightarrow 1$ when $N \rightarrow \infty$, at the same time, the confidence interval narrows too. This suggests that the proposed algorithm forms a truly stochastic RVE.

a) $a = b = 0.1c$

b) $a = b = 10c$

Fig. 7. Ratio ρ as a function of the ellipsoids number. Each point represents the mean value obtained for 10 random realizations. There is also 95 % confidence interval shown

Let us give one more estimation of the inclusion orientation proximity to the isotropic distribution in the created RVEs. It is to be recalled that the algorithm performance begins with a random assignment of location and orientation of the inclusions. In order to make sure that the algorithm does not line up inclusions in a certain direction, one can compare the number ρ at the beginning of the algorithm and its value at the end of the algorithm performance. In this case, as the initial orientation is random, the corresponding value ρ may be considered as a reference for the given number of particles. Table 2 shows the ratio ρ at the beginning and end of the algorithm performance for the RVEs shown in Fig. 4 and Fig. 5. The Table shows that there is no alignment of the inclusions along a certain direction and the numbers ρ are included in the corridor shown in Fig. 7

Table 2

Ratio ρ at the beginning and end of the algorithm performance for the RVEs shown Fig. 4 and Fig. 5

Ratio of the semiaxes a/c	The number of ellipsoids	Ratio ρ at the beginning of the algorithm	Ratio ρ at the end of the algorithm
1/2	30	2.17	1.53
1/10	90	1.44	1.47
1/30	175	1.25	1.19
2	30	1.70	2.03
10	60	1.36	1.99
30	90	1.96	1.53

5. Conclusion

The article presents a new algorithm for formation of spatial microstructures of dispersion-strengthened composite materials with ellipsoidal reinforcing particles. The algorithm allows to create an RVE with the following characteristics:

- uniform stochastic distribution of ellipsoids by orientation and internal volume;
- given distribution of ellipsoids by size and ratios of the semiaxes, i.e. the ellipsoids inside the RVE may have different sizes;
- the volume fraction that is not less, and sometimes more, than the values achieved by currently known algorithms.

The algorithm is simple for implementation and does not require great computational efforts. Determination of eigenvalues and eigenvectors of 4×4 matrices is the most resource-intensive operation of the algorithm. However, in the computational sense, the solution of this problem is a countable combination of simple algebraic actions. These actions are much less resource-intensive compared to solution of a nonlinear equation with variable coefficients necessary for realization of an algorithm based on the

molecular dynamics method [23] (it is to be recalled that, currently, this is the only known algorithm that allows to achieve the volume fractions close to those obtained using the algorithm proposed in this paper). It should be noted that the proposed algorithm allows contact of the particles. An additional artificial increase of dimensions transforms the contact areas into transition layers of a given thickness.

In conclusion of the article, it should be noted that the microstructures of dispersion-strengthened composites can be used in a wide range of problems in physics and mechanics of composite materials. For example, one can study the impact of shape (ratios of the semiaxes) and size distribution of the inclusions on the effective elastic, electrical, and other properties of the composite as well as on the origin of microdefects, and much more.

References

1. Kepler J. Strena seu de nive sexangula. Francofurti ad Moenum, apud Tampach, 1611, 21 p.
2. Hales T.C. A proof of the Kepler conjecture. *Annals of mathematics*, 2005, vol. 162, pp. 1065-1185.
3. Torquato S., Truskett T.M., Debenedetti P.G. Is random close packing of spheres well defined? *Physical Review Letters*, 2000, vol. 84, pp. 2064-2067.
4. Scott G.D. Packing of spheres. *Nature*, 1960, vol. 188, pp. 908-909.
5. Scott G.D., Kilgour D.M. The density of random close packing of spheres. *J. Phys. D: Appl. Phys.*, 1969, vol. 2, pp. 863-866.
6. Milewski J.V. A study of the packing of milled fibreglass and glass beads. *Composites*, 1973, vol. 4, pp. 258-265.
7. Milewski J.V. The combined packing of rods and spheres in reinforcing plastics. *Ind. Engng. Chem. Prod. Res. Dev.*, 1978, vol. 17, pp. 363-366.
8. Nardin M., Papirer E. Contribution a l'etude des empilements au hasard de fibres et/ou de particules spheriques. *Powder Technology*, 1985, vol. 44, pp. 131-140.
9. Evans K.E., Gibson A.G. Prediction of the maximum packing fraction achievable in randomly oriented short-fibre composites. *Composites Science and Technology*, 1986, vol. 25, pp. 149-162.
10. Evans K.E., Ferrar M.D. The packing of thick fibers. *J. Phys. D: Appl. Phys.*, 1989, vol. 22, pp. 354-360.
11. Parkhouse J.G., Kelly A. The random packing of fibres in three dimensions. *Proc. R. Soc. Lond. A*, 1995, vol. 451, pp. 737-746.
12. Philipse A.P. The random contact equation and its implications for (colloidal) rods in packings, suspensions, and anisotropic powders. *Langmuir*, 1996, vol. 12, pp. 1127-1133.
13. Wegner S., Stannarius R., Boese A., Rose G., Szabó B., Somfai E., Börzönyi T. Effects of grain shape on packing and dilatancy of sheared granular materials. *Soft Matter*, 2014, vol. 10, pp. 5157-5167.
14. Donev A., Cisse I., Sachs D., Variano E.A., Stillinger F.H., Connelly R., Torquato S., Chaikin P.M. Improving the density of jammed disordered packings using ellipsoids. *Science*, 2004, vol. 303, pp. 990-993.
15. Man W., Donev A., Stillinger F.H., Sullivan M.T., Russel W.B., Heeger D., Inati S., Torquato S., Chaikin P.M. Experiments on random packings of ellipsoids. *Physical Review Letters*, 2005, vol. 94(19), pp. 198001-1 - 198001-4.
16. Xia C., Zhu K., Cao Y., Sun H., Kou B., Wang Y. X-ray tomography study of the random packing structure of ellipsoids. *Soft Matter*, 2014, vol. 10, pp. 990-996.
17. Schaller F.M., Neudecker M., Saadatfar M., Delaney G.W., Schröder-Turk G.E., Schröder M. Local origin of global contact numbers in frictional ellipsoid packings. *Physical Review Letters*, 2015, vol. 114(15), pp. 158001-1 - 158001-5.
18. Bennett C.H. Serially deposited amorphous aggregates of hard spheres. *J. Appl. Phys.*, 1972, vol. 32(6), pp. 2727-2734.
19. Sherwood J.D. Packing of spheroids in three-dimensional space by random sequential addition. *J. Phys. A: Math. Gen.*, 1997, vol. 30, pp. 839-843.
20. Lubachevsky B.D., Stillinger F.H. Geometric properties of random disk packings. *Journal of Statistical Physics*, 1990, vol. 60, pp. 561-583.
21. Donev A., Torquato S., Stillinger F.H. Neighbor list collision-driven molecular dynamics simulation for nonspherical hard particles. I. Algorithmic details. *Journal of computational physics*, 2005, vol. 202(2), pp. 737-764.
22. Donev A., Torquato S., Stillinger F.H. Neighbor list collision-driven molecular dynamics simulation for nonspherical hard particles. II Application to ellipses and ellipsoids. *Journal of computational physics*, 2005, vol. 202(2), pp. 765-793.
23. Ghossein E., L'évesque M. Random generation of periodic hard ellipsoids based on molecular dynamics: a computationally-efficient algorithm. *Journal of Computational Physics*, 2013, vol. 253, pp. 471-490.
24. Lubachevsky B.D., Stillinger F.H., Pinson E.N. Disks vs. spheres: contrasting properties of random packings. *Journal of Statistical Physics*, 1991, vol. 64, pp. 501-524.
25. Jodrey W.S. Computer simulation of close random packing of equal spheres. *Physical Review A*, 1985, vol. 32, no. 4, pp. 2347-2351.
26. Williams S.R., Philipse A.P. Random packings of spheres and spherocylinders simulated by mechanical contraction. *Physical Review E*, 2003, vol. 67, pp. 051301-1 - 051301-9.
27. Stillinger F.H., Weber T.A. Inherent structure of liquids in the hard-sphere limit. *J. Chem. Phys.*, 1985, vol. 83(9), pp. 4767-4775.
28. Zinchenko A. Algorithm for random close packing of spheres with periodic boundary conditions. *J. Comp. Phys.*, 1994, vol. 114, pp. 298-307.
29. Kanit T., Forest S., Galliet I., Mounoury V., Jeulin D. Determination of the size of the representative volume element for random composites: statistical and numerical approach. *International Journal of Solids and Structures*, 2003, vol. 40, pp. 3647-3679.
30. Chan K. A simple mathematical approach for determining intersection of quadratic surfaces. Multiscale optimization methods and applications. New York, 2006, pp. 271-298.
31. Choi Y., Chang J., Wang W., Kim M., Elber G. Continuous collision detection for ellipsoids. *Visualization and computer Graphics, IEEE Transactions*, 2009, vol. 15, pp. 311-325.
32. Advani S.G., Tucker C.L. The use of tensors to describe and predict fiber orientation in short fiber composites. *Journal of rheology*, 1987, vol. 31, pp. 751-784.

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