TECHNICAL SCIENCE

THE SOLUTION OF THE PROBLEM OF ELASTICITY THEORY FOR A HALF-SPACE WITH A LONGITUDINAL CYLINDRICAL PIPE AND AN ELASTIC LAYER LINKED TO THE HALF-SPACE, WITH DISPLACEMENTS GIVEN ON THE BOUNDARY SURFACES

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Abstract

A solution to the spatial problem of the theory of elasticity is proposed for a composite in the form of a half-space with a longitudinal thick-walled circular cylindrical tube and a layer rigidly attached to the surface of the half-space. Layer, half-space and pipe - elastic homogeneous isotropic materials different from each other. On the free surface of the layer and the inner surface of the pipe, displacements are specified. At the boundary of the layer and half-space, as well as at the boundary of half-space and the outer surface of the pipe, the matching conditions are coupling. It is necessary to evaluate the stress state of the multilayer medium. The solution of the spatial problem of the theory of elasticity is obtained on the basis of the generalized Fourier method in cylindrical coordinates associated with the pipe and Cartesian coordinates associated with the layer and half-space. Satisfying the boundary and conjugation coupling, we obtain infinite systems of linear algebraic equations that are solved by the reduction method. As a result, displacements and stresses were obtained at various points of the layer, half-space, and.

Key words: composite, thick-walled pipe in half-space, coupling conditions, generalized Fourier method.

Introduction

When designing complex geotechnical structures, as well as structures and parts made of composites, one has to deal with design schemes in the form of rigidly interlinked layers, pipes and half-space. In view of the combination of different boundary surfaces and the spatial formulation of the problem, the solution is much more complicated.

For such problems, the generalized Fourier method [1] is used in the numerical analytical approach.

So, using this method, the problems are solved for a half-space, with a surface parallel to it, a cylindrical cavity or inclusion [2–8], for a cylinder

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with cylindrical inclusions [9], for a layer with one cylindrical cavity, inclusion or pipe [10–13].

The presented work is also based on the generalized Fourier method.

Formulation of the problem

In an elastic homogeneous half-space parallel to its surface, there is a circular cylindrical thick-walled pipe with an outer radius of R_1 and an inner one of R_2 . A layer of height h_1 is rigidly adhered to the surface of the half-space.

We will consider the pipe in a cylindrical coordinate system (ρ, φ, z) , the half-space in the Cartesian coordinate system (x_2, y_2, z_2) , which is identically oriented and combined with the coordinate system of the pipe. The half-space boundary is located at $y_2=h_2$. The layer will be considered in the Cartesian coordinate system (x_1, y_1, z_1) located on the lower surface of the layer (the boundary of the coupling with half-space and, accordingly, shifted relative to the coordinate system by $y_2=h_2$).

It is necessary to find a solution to the Lamé equations $\Delta \vec{U}_j + (1 - 2\sigma_j)^{-1} \nabla \text{div} \vec{U}_j = 0$, where σ_j – Poisson's ratio of the layer ($j=1$), half-space ($j=2$) or pipes ($j=3$).

At the upper boundary of the layer, displacements are specified $\vec{U}_1(x_1, z_1)|_{y_1=h_1} = \vec{U}_h^0(x_1, z_1)$, displacements are set on the inner surface of the pipe $\vec{U}_2(\phi, z)|_{\rho=R_2} = \vec{U}_R^0(\phi, z)$, where $\vec{U}_1$ — displacements in layer; $\vec{U}_2$ — displacements in half-space;

$$\begin{aligned} \vec{U}_h^0(x_1, z_1) &= U_x^{(1)} \vec{e}_1^{(1)} + U_y^{(1)} \vec{e}_2^{(1)} + U_z^{(1)} \vec{e}_3^{(1)}, \\ \vec{U}_R^0(\phi, z) &= U_\rho^{(R)} \vec{e}_1^{(2)} + U_\phi^{(R)} \vec{e}_2^{(2)} + U_z^{(R)} \vec{e}_3^{(2)} \end{aligned} \quad (1)$$

known functions; $\vec{e}_j^{(k)}$, $j = 1, 2, 3$ – the unit vectors of the Cartesian ($k=1$) and cylindrical ($k=2$) coordinate systems.

On the boundary of the layer and half-space, the coupling conditions are given

$$\vec{U}_1|_{y_1=0} = \vec{U}_2|_{y_2=h_2}, \quad (2)$$

$$F_1 \vec{U}_1|_{y_1=0} = F_2 \vec{U}_2|_{y_2=h_2}, \quad (3)$$

at the boundary of the half-space and the pipe, the coupling conditions

$$\vec{U}_2(\phi, z)|_{\rho=R_1} = \vec{U}_3(\phi, z)|_{\rho=R_1}, \quad (4)$$

$$F \vec{U}_2(\phi, z)|_{\rho=R_1} = F \vec{U}_3(\phi, z)|_{\rho=R_1}, \quad (5)$$

where
$$F_j \vec{U}_j = 2G_j \left[\frac{\sigma_j}{1-2\sigma_j} \vec{n} \operatorname{div} U_j + \frac{\partial}{\partial n} \vec{U}_j + \frac{1}{2} (\vec{n} \times \operatorname{rot} \vec{U}_j) \right];$$

$$G_j = \frac{E_j}{2(1+\sigma_j)}; \sigma_j, E_j - \text{Poisson's ratio and modulus of elasticity of the}$$

layer ($j=1$), half-space ($j=2$) or pipes ($j=3$); $\vec{U}_j$ — displacements in pipes.

The specified functions will be considered as rapidly decreasing from the origin of coordinates along the z axis for the cylinder and along the z and x axes for the layer boundaries.

Solution Method

We take the basic solutions of the Lamé equation in the form [1]

$$\vec{u}_k^{(\pm)}(x, y, z; \lambda, \mu) = N_k^{(d)} e^{i(\lambda z + \mu x) \pm \gamma y};$$

$$\vec{R}_{k,m}(\rho, \phi, z; \lambda) = N_k^{(p)} I_m(\lambda \rho) e^{i(\lambda z + m\phi)}; \quad (6)$$

$$\vec{S}_{k,m}(\rho, \phi, z; \lambda) = N_k^{(p)} \left[(\operatorname{sign} \lambda)^m K_m(|\lambda| \rho) \cdot e^{i(\lambda z + m\phi)} \right]; k = 1, 2, 3;$$

$$N_1^{(d)} = \frac{1}{\lambda} \nabla; \quad N_2^{(d)} = \frac{4}{\lambda} (\sigma - 1) \vec{e}_2^{(i)} + \frac{1}{\lambda} \nabla(y \cdot);$$

$$N_3^{(d)} = \frac{i}{\lambda} \operatorname{rot}(\vec{e}_3^{(i)} \cdot); \quad N_1^{(p)} = \frac{1}{\lambda} \nabla;$$

$$N_2^{(p)} = \frac{1}{\lambda} \left[\nabla \left(\rho \frac{\partial}{\partial \rho} \right) + 4(\sigma - 1) \left(\nabla - \vec{e}_3^{(z)} \frac{\partial}{\partial z} \right) \right]; \quad N_3^{(p)} = \frac{i}{\lambda} \operatorname{rot}(\vec{e}_3^{(z)} \cdot);$$

$$\gamma = \sqrt{\lambda^2 + \mu^2}, \quad -\infty < \lambda, \mu < \infty,$$

where $I_m(x)$, $K_m(x)$ – modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, $k=1, 2, 3$ – respectively, internal and external solutions of the Lamé equation for the cylinder; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ – solutions of the Lamé equation for a layer and half-space.

The solution to the problem is presented in the form

$$\vec{U}_1 = \sum_{k=1}^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(H_k^{(i)}(\lambda, \mu) \cdot \vec{u}_k^{(+)}(x_1, y_1, z_1; \lambda, \mu; \sigma_1) + \right. \\ \left. + \tilde{H}_k^{(i)}(\lambda, \mu) \cdot \vec{u}_k^{(-)}(x_1, y_1, z; \lambda, \mu; \sigma_1) \right) d\mu d\lambda, \quad (7)$$

$$\begin{aligned} \vec{U}_2 = & \sum_{k=1}^3 \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{k,m}(\lambda) \cdot \vec{S}_{k,m}(\rho, \phi, z; \lambda; \sigma_1) d\lambda + \\ & + \sum_{k=1}^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H_k^{(2)}(\lambda, \mu) \cdot \vec{u}_k^{(+)}(x_2, y_2, z_2; \lambda, \mu; \sigma_2)) d\mu d\lambda, \end{aligned} \quad (8)$$

$$\vec{U}_3 = \sum_{k=1}^3 \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{k,m}(\lambda) \cdot \vec{R}_{k,m}(\rho, \phi, z; \lambda) + \tilde{A}_{k,m}(\lambda) \cdot \vec{S}_{k,m}(\rho, \phi, z; \lambda) d\lambda, \quad (9)$$

where $\vec{S}_{k,m}(\rho, \phi, z; \lambda)$, $\vec{R}_{k,m}(\rho, \phi, z; \lambda)$, $\vec{u}_k^{(+)}(x, y, z; \lambda, \mu)$ и $\vec{u}_k^{(-)}(x, y, z; \lambda, \mu)$ – basic solutions given by formulas (6), unknown functions $H_k^{(1)}(\lambda, \mu)$, $\tilde{H}_k^{(1)}(\lambda, \mu)$, $B_{k,m}(\lambda)$, $H_k^{(2)}(\lambda, \mu)$, $A_{k,m}(\lambda)$ и $\tilde{A}_{k,m}(\lambda)$ must be found from the boundary conditions (1) and coupling conditions (2 – 5).

For the transition of basic solutions between coordinate systems, we use the formulas [12].

To satisfy the boundary conditions at the upper boundary of the layer, we equate vector (7), for $y_1=h_1$, to the given one $\vec{U}_h^0(x, z)$, represented through the double Fourier integral. So we get three equations (one for each projection) with six unknowns $H_k^{(1)}(\lambda, \mu)$, $\tilde{H}_k^{(1)}(\lambda, \mu)$.

To satisfy the coupling conditions at the boundary of the layer and half-space in displacements, we substitute the right-hand sides (7) and (8) in (2). In this case, writing expression $\vec{U}_2(x_2, z_2)|_{y_2=0}$, it is necessary to use the formulas for the transition from solutions $\vec{S}_{k,m}$ of the cylinder to solutions $\vec{u}_k^{(-)}$ [12, formula (7)]. In a similar way, we can write three additional equations for stresses (3).

So we get nine infinite systems of equations with unknown functions $H_k^{(1)}(\lambda, \mu)$, $\tilde{H}_k^{(1)}(\lambda, \mu)$, $H_k^{(2)}(\lambda, \mu)$ и $B_{k,m}(\lambda)$.

This system of equations has a unique solution, since the determinant Δ of this system has the form $\Delta = -16 \cdot \gamma^6 \cdot e^{-3\gamma(h_1-h_2)} \cdot \Phi(\gamma) / \lambda^6$, where $\Phi(\gamma)$ – the function, for $\gamma > 0$, has only positive values and does not vanish.

From the resulting system of equations, we express $H_k^{(1)}(\lambda, \mu)$,

$\tilde{H}_k^{(1)}(\lambda, \mu)$ and $H_k^{(2)}(\lambda, \mu)$ through $B_{k,m}(\lambda)$.

To satisfy the coupling conditions at the boundary of the half-space and the pipe, we set $\rho=R_1$ in (8) and (9). We expand in (8) the basic solutions of $\vec{u}_k^{(+)}$ by the solutions of $\vec{R}_{k,m}$ using the transition formulas [12, formula (8)]. The resulting vector, as well as vector (9), for $\rho=R_1$, we substitute in (4). So we get three infinite systems of equations for the coupling of half-space and pipe in displacements. Similarly, we write three equations for stresses (5).

To fulfill the boundary conditions on the inner surface of the pipe, we equate (9), for $\rho=R_2$, given by $\vec{U}_R^0(\phi, z)$, represented by the integral and the Fourier series.

Having received 9 infinite equations, instead of $H_k^{(2)}(\lambda, \mu)$, we substitute the previously expressed functions through $B_{k,m}(\lambda)$, free ourselves from the series in m and the integrals in λ . As a result, we obtain a set of nine infinite systems of linear algebraic equations for determining unknown $B_{k,m}(\lambda)$, $A_{k,m}(\lambda)$ and $\tilde{A}_{k,m}(\lambda)$. These infinite systems have the properties of equations of the second kind and, as a consequence, the reduction method can be applied to them.

Having solved this system of equations, we find the unknowns $A_{k,m}(\lambda)$, $\tilde{A}_{k,m}(\lambda)$ and $B_{k,m}(\lambda)$.

Found from an infinite system of equations $B_{k,m}(\lambda)$, we substitute in the expressions for $H_k^{(1)}(\lambda, \mu)$, $\tilde{H}_k^{(1)}(\lambda, \mu)$ and $H_k^{(2)}(\lambda, \mu)$. This will identify all unknown tasks.

Summary

Based on the generalized Fourier method, a method is proposed for solving the spatial problem of the theory of elasticity for a multilayer medium in the form of a layer and a half-space in which a longitudinal cylindrical pipe is located. The problem is reduced to an infinite system of linear algebraic equations, allowing the application of the reduction method to it.

The solution method can be used in the design of composites, machine parts, building structures and underground structures, the design scheme of which coincides with the formulation of this problem.

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