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THE FOURIER PROBLEM FOR WEAKLY NONLINEAR INTEGRO-DIFFERENTIAL ELLIPTIC-PARABOLIC SYSTEMS

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The Fourier problem or, in other words, the problem without initial conditions for weakly nonlinear elliptic-parabolic systems is considered in this paper. The existence and uniqueness of solutions of the problem are proved. The estimates of these solutions are obtained.

Introduction. Let n, N be natural numbers, and $\mathbb{R}^n$ (respectively, $\mathbb{R}^N$) be the finite-dimensional real linear space consisting of vector-rows $x = (x_1, \dots, x_n)$ (respectively, vector-columns $\rho = \text{col}(\rho_1, \dots, \rho_N)$) with the norm $|x| := (|x_1|^2 + \dots + |x_n|^2)^{1/2}$ (respectively, $|\rho| := (|\rho_1|^2 + \dots + |\rho_N|^2)^{1/2}$); $\mathbb{M}_{N \times n}$ be the linear space consisting of $N \times n$ real matrices $\zeta = (\zeta_{kl}) = (\zeta_{kl}; k = \overline{1, N}, l = \overline{1, n})$ with the norm

$$|\zeta| = \left(\sum_{i=1}^N \sum_{j=1}^n |\zeta_{ij}|^2 \right)^{1/2}.$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with the piecewise smooth boundary $\partial\Omega$. Denote $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma := \partial\Omega \times S$, $Q_{t_1, t_2} := \Omega \times (t_1, t_2)$ for arbitrary real $t_1, t_2 \in \mathbb{R}$ ($t_1 < t_2$).

We suppose that the following assumption is valid the paper:

(B) $b_1, \dots, b_N$ are measurable functions which are defined on Ω and such that for each $i \in \{1, \dots, N\}$ there exists an open set $\Omega_i^0 \subset \Omega$ providing $0 < b_i(x) \leq 1$ for almost every (a.e.) $x \in \Omega_i^0$ and $b_i(x) = 0$ for a.e. $x \in \Omega \setminus \Omega_i^0$.

Let us consider the *Fourier* (or, in other words, the *problem without initial conditions*) *problem for elliptic-parabolic system*: to find a vector function $u = \text{col}(u_1 \dots u_N): \overline{Q} \rightarrow \mathbb{R}^N$, satisfying (in some sense) the system

$$\begin{aligned} \frac{\partial}{\partial t}(b_i(x)u_i) - \sum_{j=1}^n \frac{d}{dx_j} a_{ij}(x, t, u, \nabla u) + a_{i0}(x, t, u, \nabla u) + \int_{\Omega} c_i(x, y, t, u(y, t)) dy = \\ = f_{i0}(x, t) - \sum_{j=1}^n \frac{\partial}{\partial x_j} f_{ij}(x, t), \quad (x, t) \in Q, \quad i = \overline{1, N}, \end{aligned} \quad (1)$$

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and the boundary conditions

$$u_i \Big|_{\Sigma} = 0, \quad i = \overline{1, N}, \quad (2)$$

where

$$a_{ij}: Q \times \mathbb{R}^N \times \mathbb{M}_{N \times n} \rightarrow \mathbb{R}, \quad c_i: \Omega \times \Omega \times S \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad f_{ij}: Q \rightarrow \mathbb{R}, \quad i = \overline{1, N}, j = \overline{0, n},$$

are given functions, $\nabla u := (u_{x_1}, \dots, u_{x_n}) \equiv (u_{i,x_j}; i = \overline{1, N}, j = \overline{1, n})$.

An example of system (1) is

$$\begin{aligned} & (b_i(x)u_i)_t - \sum_{j=1}^n \left(\sum_{k=1}^N \sum_{l=1}^n \tilde{a}_{ij}^{kl}(x, t) u_{k,x_l} \right)_{x_j} + \sum_{k=1}^N d_{ik}(x, t) u_k + \\ & + \int_{\Omega} \sum_{k=1}^N \tilde{c}_{ik}(x, y, t) u_k(y, t) dy = f_i(x, t), \quad (x, t) \in Q, \quad i = \overline{1, N}, \end{aligned} \quad (3)$$

where $\tilde{a}_{ij}^{kl}$, d_{ik} , $\tilde{c}_{ik}$ ($i, k = \overline{1, N}$, $j, l = \overline{1, n}$), are measurable and bounded functions, f_i ($i = \overline{1, N}$) are integrable functions.

The Fourier problem or, in other words, the problem without initial conditions for systems of evolution equations appears in the modeling of various dynamic processes in nature and economy, when the process is so far removed from the current moment that the initial data practically does not affect the situation at that moment (see, for example, [9]). Such a problem was considered in papers of many mathematicians, in particular, in [4]–[14]. A fairly complete overview of these results can be found in [13].

Note that the unknown function can enter both in the differential part and in the integral part of the equation in system (1). The integro-differential elliptic-parabolic equations and systems arise in the simulation of complex phenomena in modern science, economics and technology, for example, in the description of exchange fluctuations in the value of options, in the theory of nuclear reactions, in the study of the process of neutron deceleration, in the diffusion of charged particles in plasma and so on (see [15]–[17]).

The Fourier problem for equations or systems in the case of strong nonlinearity can be uniquely solved without any conditions at infinity (see [4]–[8]). The problem without initial conditions for systems (1) is investigated in [7]. But there was considered the case of weak nonlinearity when the Fourier problem for the system (1) is well-posedness only for some restrictions on the behavior of solutions at $t \rightarrow -\infty$ (will be listed later). The Fourier problem for such type of equations is considered in [8].

We examine the question of well-posedness of the Fourier problem for weakly nonlinear systems (1). Earlier, the Fourier problem of such type was not considered.

The paper is organized as follows. Section 1 contains the main notation and auxiliary facts. The statement of the problem and the formulation of the main result are given in Section 2. Section 3 encloses the auxiliary statements. The main result is proved in Section 4.

1. Preliminaries. Let $L_{\text{loc}}^{\infty}(\overline{Q})$ be a linear space of measurable functions on Q such that their restrictions on Q_{t_1, t_2} belongs to $L^{\infty}(Q_{t_1, t_2})$ for each $t_1, t_2 \in S$ ($t_1 < t_2$).

Let X be an arbitrary Hilbert space with the scalar product $(\cdot, \cdot)_X$ and a norm $\|\cdot\|_X$. Denote by $L_{\text{loc}}^2(S; X)$ the linear space of functions $g: S \rightarrow X$ such that the restriction of g on $[t_1, t_2] \subset S$ belongs to $L^2(t_1, t_2; X)$ for each $t_1, t_2 \in S$ ($t_1 < t_2$).

We define by $C_c^1(I)$ (I is an interval of the real axis) the linear space of continuously differentiable on I functions with compact support (if $I = (t_1, t_2)$ then we write $C_c^1(t_1, t_2)$ instead of $C_c^1((t_1, t_2))$).

Denote by $H^1(\Omega) := \{v \in L_2(\Omega) \mid v_{x_j} \in L_2(\Omega), j = \overline{1, n}\}$ the Sobolev space with the Hilbert scalar product

$$(v, w)_{H^1(\Omega)} := \int_{\Omega} \left[vw + \sum_{j=1}^n v_{x_j} w_{x_j} \right] dx$$

and the norm

$$\|v\|_{H^1(\Omega)} := \left(\|v\|_{L_2(\Omega)}^2 + \sum_{j=1}^n \|v_{x_j}\|_{L_2(\Omega)}^2 \right)^{1/2}.$$

Also we define by $\mathring{H}^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Hereafter $C_c^\infty(\Omega)$ is the linear space of infinitely differentiable bounded real-valued functions defined on Ω .

For each $i \in \{1, \dots, N\}$ we denote $\tilde{b}_i(x) := b_i(x)$ if $x \in \Omega_i^0$, and $\tilde{b}_i(x) := 1$ if $x \in \Omega \setminus \Omega_i^0$, and we define $H_{b_i}(\Omega)$ as the linear space of the functions $w = \tilde{b}_i^{-1/2} v$ with $v \in L_2(\Omega)$. We introduce a seminorm on $H_{b_i}(\Omega)$ as

$$\|w\|_{H_{b_i}(\Omega)} := \left(\int_{\Omega} b_i(x) |w(x)|^2 dx \right)^{1/2}.$$

It is easy to check that for each $i \in \{1, \dots, N\}$ the space $H_{b_i}(\Omega)$ is the completion of $\mathring{H}^1(\Omega)$ with respect to the seminorm $\|\cdot\|_{H_{b_i}}$ (see [18, I.3.3]).

For each $i \in \{1, \dots, N\}$ we define $C(S; H_{b_i}(\Omega))$ as a linear space of functions $h: S \rightarrow H_{b_i}(\Omega)$ such that $b_i^{1/2} h \in C(S; L_2(\Omega))$.

We denote

$$K := \inf_{v \in \mathring{H}^1(\Omega), v \neq 0} \left\{ \int_{\Omega} |\nabla v|^2 dx / \int_{\Omega} |v|^2 dx \right\}. \quad (4)$$

It is known that K is finite and coincides with the first eigenvalue of the eigenvalue problem: $-\Delta v = \lambda v$, $v|_{\partial\Omega} = 0$. From (4) it follows the Friedrichs inequality:

$$\int_{\Omega} |\nabla v|^2 dx \geq K \int_{\Omega} |v|^2 dx \quad \forall v \in \mathring{H}^1(\Omega). \quad (5)$$

2. Formulation of the problem and main results. To define a weak solution of problem (1)-(2) we introduce the corresponding classes of data-in.

Denote by $\mathcal{A}$ a set of matrix functions $(a_{ij}; i = \overline{1, N}, j = \overline{0, n}) \equiv (a_{ij})$ such that the next three conditions hold:

(A₁) for each $i \in \{1, \dots, N\}$ and $j \in \{0, \dots, n\}$, $a_{ij}: Q \times \mathbb{R}^N \times \mathbb{M}_{N \times n} \rightarrow \mathbb{R}$ is the Caratheodory function, i. e., for a.e. $(x, t) \in Q$ the function $a_{ij}(x, t, \cdot, \cdot): \mathbb{R}^N \times \mathbb{M}_{N \times n} \rightarrow \mathbb{R}$ is a continuous and for all $(\rho, \xi) \in \mathbb{R}^N \times \mathbb{M}_{N \times n}$ the function $a_{ij}(\cdot, \cdot, \rho, \xi): Q \rightarrow \mathbb{R}$ is Lebesgue measurable; in addition, for any $i \in \{1, \dots, N\}$ and $j \in \{0, \dots, n\}$ the equality $a_{ij}(x, t, 0, 0) = 0$ holds for a.e. $(x, t) \in Q$;

(A₂) for each $i \in \{1, \dots, N\}$ and $j \in \{0, \dots, n\}$, for a.e. $(x, t) \in Q$ and for all $\rho \in \mathbb{R}^N$, $\xi \in \mathbb{M}_{N \times n}$ we have

$$|a_{ij}(x, t, \rho, \xi)| \leq h_{ij}(x, t)(|\rho| + |\xi|) + \tilde{h}_{ij}(x, t),$$

where $h_{ij} \in L^\infty_{\text{loc}}(\overline{Q})$, $\tilde{h}_{ij} \in L^2_{\text{loc}}(\overline{Q})$;

(A₃) for a.e. $(x, t) \in Q$ and for arbitrary $(\rho^1, \xi^1), (\rho^2, \xi^2) \in \mathbb{R}^N \times \mathbb{M}_{N \times n}$ the following inequality

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=0}^n (a_{ij}(x, t, \rho^1, \xi^1) - a_{ij}(x, t, \rho^2, \xi^2))(\xi_{ij}^1 - \xi_{ij}^2) + \\ & + \sum_{i=1}^N (a_{i0}(x, t, \rho^1, \xi^1) - a_{i0}(x, t, \rho^2, \xi^2))(\rho_i^1 - \rho_i^2) \geq \\ & \geq \gamma_1(t) \sum_{i=1}^N \sum_{j=1}^n |\xi_{ij}^1 - \xi_{ij}^2|^2 + \gamma_2(t) \sum_{i=1}^N |\rho_i^1 - \rho_i^2|^2, \end{aligned}$$

holds where $\gamma_1, \gamma_2 \in C(S)$, $\gamma_1(t) > 0$, $\gamma_2(t) \geq 0 \quad \forall t \in S$.

Let $\mathcal{C}$ be the set of vector functions $\text{col}(c_1 \dots c_N)$, such that the next two conditions are valid:

(C₁) for each $i \in \{1, \dots, N\}$ $c_i: \Omega \times \Omega \times S \times \mathbb{R}^N \rightarrow \mathbb{R}$ is the Caratheodory function, i. e. for a.e. $(x, y, t) \in \Omega \times \Omega \times S$ the function $c_i(x, y, t, \cdot): \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous and for all $\rho \in \mathbb{R}^N$ the function $c_i(\cdot, \cdot, \cdot, \rho): \Omega \times \Omega \times S \rightarrow \mathbb{R}$ is Lebesgue measurable; in addition, for any $i \in \{1, \dots, N\}$ the equality $c_i(x, y, t, 0) = 0$ holds for a.e. $(x, y, t) \in \Omega \times \Omega \times S$;

(C₂) for each $i \in \{1, \dots, N\}$, for a.e. $(x, y, t) \in \Omega \times \Omega \times S$ and arbitrary $\rho^1, \rho^2 \in \mathbb{R}^N$ the inequality

$$|c_i(x, y, t, \rho^1) - c_i(x, y, t, \rho^2)| \leq L_i \gamma(t) |\rho^1 - \rho^2|$$

holds, where $L_i \geq 0$ is a constant,

$$\gamma(t) := NK\gamma_1(t) + \gamma_2(t) \quad \forall t \in S. \quad (6)$$

Let $\mathbb{F}$ be the set of matrix functions $(f_{ij}; i = \overline{1, N}, j = \overline{0, n}) \equiv (f_{ij})$ such that the next condition is true:

(F) $f_{ij} \in L^2_{\text{loc}}(\overline{Q})$ for each $i \in \{1, \dots, N\}$, $j \in \{0, \dots, n\}$.

Definition 1. Let $b_1, \dots, b_N$ be such that condition (B) is valid, $(a_{ij}) \in \mathcal{A}$, $(f_{ij}) \in \mathbb{F}$, $\text{col}(c_1, \dots, c_N) \in \mathcal{C}$. A vector function $u = \text{col}(u_1 \dots u_N)$ is called a *weak solution* of (1), (2), provided $u_i \in L^2_{\text{loc}}(S; \mathring{H}^1(\Omega)) \cap C(S; H_{b_i}(\Omega))$, $i = \overline{1, N}$, and the following integral identity holds

$$\begin{aligned} & \iint_Q \sum_{i=1}^N \left[-b_i(x) u_i v_i \varphi_i' + \sum_{j=1}^n a_{ij}(x, t, u, \nabla u) v_{i,x_j} \varphi_i + a_{i0}(x, t, u, \nabla u) v_i \varphi_i + \right. \\ & \quad \left. + v_i \varphi_i \int_\Omega c_i(x, y, t, u(y, t)) dy \right] dx dt = \\ & = \iint_Q \sum_{i=1}^N \left[f_{i0} v_i \varphi_i + \sum_{j=1}^n f_{ij} v_{i,x_j} \varphi_i \right] dx dt, \quad v_i \in \mathring{H}^1(\Omega), \varphi_i \in C_c^1(-\infty, 0), i = \overline{1, N}. \quad (7) \end{aligned}$$

Note that problem (1)–(2) can have many weak solutions. For example, let $\Omega := (0, 2\pi)$, $Q := (0, 2\pi) \times (-\infty, 0]$,

$$b(x) := \begin{cases} 0, & \text{if } x \in (0, \pi], \\ 1, & \text{if } x \in (\pi, 2\pi), \end{cases} \quad \widehat{c}(x, y, t) := \begin{cases} \frac{1}{\pi} \sin x \sin y, & \text{if } x \in (0, \pi], y \in (0, 2\pi), t \in S, \\ \frac{c_0}{\pi} \sin x \sin y, & \text{if } x \in (\pi, 2\pi), y \in (0, 2\pi), t \in S, \end{cases} \quad (8)$$

where $c_0 \in \mathbb{R}$ is an arbitrary number.

It is easy to check that the problem

$$(b(x)u)_t - u_{xx} + \int_0^{2\pi} \widehat{c}(x, y, t) u(y, t) dy = 0, \quad (x, t) \in Q, \quad u|_{x=0 \vee x=2\pi} = 0,$$

has the weak solutions $u = Ae^{-\lambda t} \sin x$, $(x, t) \in \overline{Q}$, where $\lambda := c_0 + 1$, $A \in \mathbb{R}$ is arbitrary.

Hence, it follows that for the uniqueness of the weak solution of problem (1)–(2) it is necessary to restrict his behavior at $t \rightarrow -\infty$.

We will find a weak solution of problem (1)–(2), satisfying the analogue of the initial condition

$$\lim_{t \rightarrow -\infty} e^{\omega \int_0^t \gamma(s) ds} \|u_i(\cdot, t)\|_{H_{b_i}(\Omega)} = 0, \quad i = \overline{1, N}. \quad (9)$$

Here ω is some real number, γ is a function from (6).

To formulate the main results of the paper we need a function space. Let us denote

$$L_{\omega, \beta}^2(S; L^2(\Omega)) := \left\{ f \in L_{\text{loc}}^2(S; L^2(\Omega)) \mid \int_S \beta(t) e^{2\omega \int_0^t \gamma(s) ds} \|f(\cdot, t)\|_{L^2(\Omega)}^2 dt < \infty \right\}$$

with the scalar product

$$(f, g)_{L_{\omega, \beta}^2(S; L^2(\Omega))} = \int_S \beta(t) e^{2\omega \int_0^t \gamma(s) ds} (f(\cdot, t), g(\cdot, t))_{L^2(\Omega)} dt,$$

and the norm

$$\|f\|_{L_{\omega, \beta}^2(S; L^2(\Omega))} := \left(\int_S \beta(t) e^{2\omega \int_0^t \gamma(s) ds} \|f(\cdot, t)\|_{L^2(\Omega)}^2 dt \right)^{1/2},$$

where $\omega \in \mathbb{R}$, $\beta \in C(S)$, $\beta(t) > 0$ for all $t \in S$, γ is defined in (6). It is easy to check that it is a Hilbert space.

The main result of the paper is following.

Theorem 1. *Let $b_1, \dots, b_N$ be such that **(B)** is true, $(a_{ij}) \in \mathcal{A}$, $(f_{ij}) \in \mathbb{F}$, $\text{col}(c_1, \dots, c_N) \in \mathcal{C}$, and, in addition, if $\min_{1 \leq i \leq N} b_i^0 = 0$, where $b_i^0 := \text{ess inf}_{x \in \Omega} b_i(x) = 0$, we have*

$$L \text{mes}_n \Omega < 1, \quad (10)$$

where $L := \left(\sum_{i=1}^n L_i^2 \right)^{1/2}$, $\text{mes}_n \Omega$ is the Lebesgue measure of Ω .

Suppose that $\omega < 1 - L \text{mes}_n \Omega$ if condition (10) holds, and $\omega < (1 - L \text{mes}_n \Omega) / \min_{1 \leq i \leq N} b_i^0$ otherwise, and also

$$f_{i0} \in L_{\omega, 1/\gamma}^2(S; L^2(\Omega)), \quad f_{ij} \in L_{\omega, 1/\gamma_1}^2(S; L^2(\Omega)), \quad i = \overline{1, N}, j = \overline{1, n}. \quad (11)$$

Then there exists a unique weak solution of (1),(2),(9), and its estimate

$$\begin{aligned} e^{\omega \int_0^\tau \gamma(s) ds} \sum_{i=1}^N \|u_i(\cdot, \tau)\|_{H_{b_i}(\Omega)} + \sum_{i=1}^N \left[\|u_i\|_{L_{\omega, \gamma}^2(S_\tau; L^2(\Omega))} + \sum_{j=1}^n \|u_{i, x_j}\|_{L_{\omega, \gamma_1}^2(S_\tau; L^2(\Omega))} \right] \leq \\ \leq C_1 \sum_{i=1}^N \left[\|f_{i0}\|_{L_{\omega, 1/\gamma}^2(S_\tau; L^2(\Omega))} + \sum_{j=1}^n \|f_{ij}\|_{L_{\omega, 1/\gamma_1}^2(S_\tau; L^2(\Omega))} \right], \quad \tau \in S, \end{aligned} \quad (12)$$

holds, where $S_\tau := (-\infty, \tau]$ $\forall \tau \in (-\infty, 0]$ ($S_0 = S$), and $C_1 > 0$ is a constant which depends only on $\text{mes}_n \Omega$, L , ω and b_i^0 , $i = \overline{1, N}$.

3. Auxiliary statements. We denote

$$\partial_0 w := w, \quad \partial_j w := w_{x_j}, \quad j = \overline{1, n}.$$

Lemma 1. Let $b_1, \dots, b_N$ be such that condition (B) is true, $w_i \in L^2(0, T; \mathring{H}^1(\Omega))$ and $g_{ij} \in L^2(\Omega \times (0, T))$, $i = \overline{1, N}$, $j = \overline{0, n}$, where $T > 0$ is an arbitrary number. Assume that the function $w = \text{col}(w_1 \dots w_N)$ satisfies the integral identity

$$\int_0^T \int_\Omega \sum_{i=1}^N [-b_i w_i v_i \varphi'_i + \sum_{j=0}^n g_{ij} \partial_j v_i \varphi_i] dx dt = 0, \quad v_i \in \mathring{H}^1(\Omega), \quad \varphi_i \in C_c^1(0, T), \quad i = \overline{1, N}. \quad (13)$$

Then for each $i \in \{1, \dots, N\}$ the function w_i belongs to $C([0, T]; H_{b_i}(\Omega))$ and for all $\theta_i \in C^1([0, T])$, $i = \overline{1, N}$, and $t_1, t_2 \in [0, T]$ ($t_1 < t_2$) the following equality

$$\begin{aligned} \frac{1}{2} \theta_i(t_2) \sum_{i=1}^N \|w_i(\cdot, t_2)\|_{H_{b_i}(\Omega)}^2 - \frac{1}{2} \theta_i(t_1) \sum_{i=1}^N \|w_i(\cdot, t_1)\|_{H_{b_i}(\Omega)}^2 - \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^N \|w_i(\cdot, t)\|_{H_{b_i}(\Omega)}^2 \theta'_i(t) dt + \\ + \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^N \left[\sum_{j=0}^n g_{ij} \partial_j w_i \right] \theta_i dx dt = 0 \end{aligned} \quad (14)$$

holds.

Proof of Lemma 1. This statement can be proved similarly to Lemma 2 in [3]. $\square$

Lemma 2. Let $b_1, \dots, b_N$ be such that condition (B) is true, $(a_{ij}) \in \mathcal{A}$, $\text{col}(c_1, \dots, c_N) \in \mathcal{C}$ and, moreover, if $\min_{1 \leq i \leq N} b_i^0 = 0$, then condition (10) holds. Suppose that $\omega < 1 - L \text{mes}_n \Omega$, if condition (10) holds, and $\omega < (1 - L \text{mes}_n \Omega) / \min_{1 \leq i \leq N} b_i^0$ otherwise. Let for each $k \in \{1, 2\}$ the function $u^k = \text{col}(u_1^k \dots u_N^k)$ be a weak solution of problem (1),(2),(9) with $f_{ij} = f_{ij}^k$, where

$$f_{i0}^k \in L_{\omega, 1/\gamma}^2(S; L^2(\Omega)), \quad f_{ij}^k \in L_{\omega, 1/\gamma_1}^2(S; L^2(\Omega)), \quad k = \overline{1, 2}, \quad i = \overline{1, N}, \quad j = \overline{0, n}. \quad (15)$$

Then the estimate

$$\begin{aligned}
& e^{\omega \int_0^\tau \gamma(s) ds} \sum_{i=1}^N \|u_i^1(\cdot, \tau) - u_i^2(\cdot, \tau)\|_{H_{b_i}(\Omega)} + \sum_{i=1}^N \left[\|u_i^1 - u_i^2\|_{L_{\omega, \gamma}^2(S_\tau; L^2(\Omega))} + \right. \\
& \quad \left. + \sum_{j=1}^n \|u_{i, x_j}^1 - u_{i, x_j}^2\|_{L_{\omega, \gamma_1}^2(S_\tau; L^2(\Omega))} \right] \leq \\
& \leq C_1 \sum_{i=1}^N \left[\|f_{i0}^1 - f_{i0}^2\|_{L_{\omega, 1/\gamma}^2(S_\tau; L^2(\Omega))} + \sum_{j=1}^n \|f_{ij}^1 - f_{ij}^2\|_{L_{\omega, 1/\gamma_1}^2(S_\tau; L^2(\Omega))} \right], \quad \tau \in S, \quad (16)
\end{aligned}$$

holds, where S_τ and $C_1 > 0$ are same as in (12).

Proof of Lemma 2. Put

$$\begin{aligned}
\widehat{u}_i(x, t) &:= u_i^1(x, t) - u_i^2(x, t), \quad i = \overline{1, N}, \\
\widehat{a}_{ij}(x, t) &:= a_{ij}(x, t, u^1(x, t), \nabla u^1(x, t)) - a_{ij}(x, t, u^2(x, t), \nabla u^2(x, t)), \quad i = \overline{1, N}, \quad j = \overline{0, n}, \\
\widehat{c}_i(x, y, t) &:= c_i(x, y, t, u^1(y, t)) - c_i(x, y, t, u^2(y, t)), \quad i = \overline{1, N}, \\
\widehat{f}_{ij}(x, t) &:= f_{ij}^1(x, t) - f_{ij}^2(x, t), \quad i = \overline{1, N}, \quad j = \overline{0, n},
\end{aligned}$$

for almost all $(x, t) \in Q$.

From (7) we obtain

$$\begin{aligned}
& \iint_Q \sum_{i=1}^N \left[-b_i \widehat{u}_i v_i \varphi_i' + \sum_{j=0}^n \widehat{a}_{ij} \partial_j v_i \varphi_i + v_i \varphi_i \int_\Omega \widehat{c}_i(x, y, t) dy \right] dx dt = \\
& = \iint_Q \sum_{i=1}^N \left[\sum_{j=0}^n \widehat{f}_{ij} \partial_j v_i \varphi_i \right] dx dt. \quad (17)
\end{aligned}$$

Using Lemma 1 for (17) with $w_i = \widehat{u}_i$, $\theta_i = \theta$, $g_{ij} = \widehat{a}_{ij}$, $g_{i0} = \widehat{a}_{i0} + \int_\Omega \widehat{c}_i dy$, $i = \overline{1, N}$, $j = \overline{1, n}$, from this equality we get

$$\begin{aligned}
& \frac{1}{2} \int_\Omega \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, t)|^2 \theta(t) dx \Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_\Omega \sum_{i=1}^N b_i |\widehat{u}_i|^2 \theta' dx dt + \\
& + \int_{\tau_1}^{\tau_2} \int_\Omega \sum_{i=1}^N \left[\sum_{j=0}^n \widehat{a}_{ij} \partial_j \widehat{u}_i + \widehat{u}_i \int_\Omega \widehat{c}_i(x, y, t) dy \right] \theta dx dt = \int_{\tau_1}^{\tau_2} \int_\Omega \sum_{i=1}^N \left[\sum_{j=0}^n \widehat{f}_{ij} \partial_j \widehat{u}_i \right] \theta dx dt, \quad (18)
\end{aligned}$$

where $\tau_1, \tau_2 \in S$, $\tau_1 < \tau_2$, $\theta \in C^1(S)$ are arbitrary.

By virtue of the Cauchy inequality

$$ab \leq \frac{\varepsilon}{2} |a|^2 + \frac{1}{2\varepsilon} |b|^2, \quad a, b \in \mathbb{R}, \quad \varepsilon > 0, \quad (19)$$

we estimate the right-hand side of equality (18) as follows

$$\int_{\tau_1}^{\tau_2} \int_\Omega \sum_{i=1}^N \left[\sum_{j=1}^n \widehat{f}_{ij} \partial_j \widehat{u}_i \right] \theta dx dt \leq \frac{\varepsilon_1}{2} \int_{\tau_1}^{\tau_2} \int_\Omega \gamma_1 \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] \theta dx dt + \quad (20)$$

$$\begin{aligned}
& + \frac{1}{2\varepsilon_1} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma_1} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] \theta \, dx dt, \\
\int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N \widehat{f}_{i0} \widehat{u}_i \theta \, dx dt & \leq \frac{\varepsilon_2}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma \left[\sum_{i=1}^N |\widehat{u}_i|^2 \right] \theta \, dx dt + \frac{1}{2\varepsilon_2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] \theta \, dx dt, \quad (21)
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ are arbitrary positive numbers.

From condition $(\mathbf{A}_3)$ we obtain

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N \left[\sum_{j=0}^n \widehat{a}_{ij} \partial_j \widehat{u}_i \right] \theta \, dx dt \geq \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1(t) \left[\sum_{i=1}^N \sum_{j=1}^n |\partial_j \widehat{u}_i|^2 \right] \theta \, dx dt + \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_2(t) \left[\sum_{i=1}^N |\widehat{u}_i|^2 \right] \theta \, dx dt = \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1(t) \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] \theta \, dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_2(t) |\widehat{u}|^2 \theta \, dx dt. \quad (22)
\end{aligned}$$

Using conditions $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ and the Cauchy-Bunyakovsky-Schwarz inequality, we receive

$$\begin{aligned}
& \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N \widehat{u}_i(x, t) \left(\int_{\Omega} \widehat{c}_i(x, y, t) \, dy \right) \theta(t) \, dx dt \right| \leq \\
& \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N |\widehat{u}_i(x, t)| \left(\int_{\Omega} |\widehat{c}_i(x, y, t)| \, dy \right) \theta(t) \, dx dt \leq \\
& \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma(t) \left(\sum_{i=1}^N L_i |\widehat{u}_i(x, t)| \right) \left(\int_{\Omega} |\widehat{u}(y, t)| \, dy \right) \theta(t) \, dx dt. \quad (23)
\end{aligned}$$

Since for almost all $(x, t) \in Q$ from the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\sum_{i=1}^N L_i |\widehat{u}_i(x, t)| \leq \left(\sum_{i=1}^N L_i^2 \right)^{1/2} \left(\sum_{i=1}^N |\widehat{u}_i(x, t)|^2 \right)^{1/2} = L |\widehat{u}(x, t)|, \quad (24)$$

inequality (23) provides validity of the inequality

$$\begin{aligned}
& \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N \widehat{u}_i(x, t) \left(\int_{\Omega} \widehat{c}_i(x, y, t) \, dy \right) \theta(t) \, dx dt \right| \leq \\
& \leq L \int_{\tau_1}^{\tau_2} \gamma(t) \left(\int_{\Omega} |\widehat{u}(x, t)|^2 \, dx \right) \theta(t) \, dt \leq L \operatorname{mes}_n \Omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma(t) |\widehat{u}(x, t)|^2 \theta(t) \, dx dt. \quad (25)
\end{aligned}$$

Using (20)–(22), (25), from (18) we obtain

$$\frac{1}{2} \theta(\tau_2) \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_2)|^2 \, dx - \frac{1}{2} \theta(\tau_1) \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_1)|^2 \, dx -$$

$$\begin{aligned}
& -\frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \sum_{i=1}^N b_i |\widehat{u}_i|^2 \theta' dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] \theta dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_2 |\widehat{u}|^2 \theta dx dt - \\
& -Lmes_n \Omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma |\widehat{u}|^2 \theta dx dt \leq \frac{\varepsilon_1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] \theta dx dt + \\
& + \frac{1}{2\varepsilon_1} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma_1} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] \theta dx dt + \frac{\varepsilon_2}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma |\widehat{u}|^2 \theta dx dt + \frac{1}{2\varepsilon_2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] \theta dx dt,
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ are arbitrary positive numbers.

Setting $\theta(t) := 2e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds}$, $t \in S$, we get

$$\begin{aligned}
& e^{\frac{2\omega}{\gamma} \int_0^{\tau_2} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_2)|^2 dx - e^{\frac{2\omega}{\gamma} \int_0^{\tau_1} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_1)|^2 dx - \\
& -2\omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N b_i |\widehat{u}_i|^2 \right] dx dt + 2(\delta + (1 - \delta)) \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] dx dt + \\
& + 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_2 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt - 2Lmes_n \Omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt \leq \\
& \leq \varepsilon_1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] dx dt + \frac{1}{\varepsilon_1} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma_1} e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] dx dt + \\
& + \varepsilon_2 \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt + \frac{1}{\varepsilon_2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma} e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] dx dt,
\end{aligned}$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\delta \in (0, 1)$ are arbitrary numbers.

Hence and from (5) we obtain the inequality

$$\begin{aligned}
& e^{\frac{2\omega}{\gamma} \int_0^{\tau_2} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_2)|^2 dx - e^{\frac{2\omega}{\gamma} \int_0^{\tau_1} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_1)|^2 dx - \\
& -2\omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N b_i |\widehat{u}_i|^2 \right] dx dt + 2(1 - \delta) \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] dx dt + \\
& + 2\delta NK \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt + 2\delta \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_2 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt - \\
& -2Lmes_n \Omega \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt \leq \\
& \leq \varepsilon_1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\nabla \widehat{u}_i|^2 \right] dx dt + \frac{1}{\varepsilon_1} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma_1} e^{\frac{2\omega}{\gamma} \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] dx dt +
\end{aligned}$$

$$+\varepsilon_2 \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{2\omega \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt + \frac{1}{\varepsilon_2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] dx dt. \quad (26)$$

Using notation (6), from (26) we get

$$\begin{aligned} & e^{2\omega \int_0^{\tau_2} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_2)|^2 dx - e^{2\omega \int_0^{\tau_1} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_1)|^2 dx + \\ & + [2(1 - \delta) - \varepsilon_1] \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma_1 e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N |\nabla \widehat{u}_i|^2 dx dt + \\ & + [2(\delta - L \text{mes}_n \Omega - \max_{1 \leq i \leq N} \text{ess sup}_{x \in \Omega} \{\omega b_i(x)\}) - \varepsilon_2] \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma e^{2\omega \int_0^t \gamma(s) ds} |\widehat{u}|^2 dx dt \leq \\ & \leq \frac{1}{\varepsilon_1} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma_1} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] dx dt + \frac{1}{\varepsilon_2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] dx dt. \end{aligned} \quad (27)$$

Note that

$$1 - L \text{mes}_n \Omega - \max_{1 \leq i \leq N} \text{ess sup}_{x \in \Omega} \{\omega b_i(x)\} > 0. \quad (28)$$

Let $1 - L \text{mes}_n \Omega > 0$. Then $\omega < 1 - L \text{mes}_n \Omega$. Let us consider two cases: 1) $\omega \leq 0$; 2) $0 < \omega < 1 - L \text{mes}_n \Omega$. In the first case we have $\text{ess sup}_{x \in \Omega} \{\omega b_i(x)\} \leq 0$, and therefore inequality (28) holds. In the second case we have

$$\text{ess sup}_{x \in \Omega} \{\omega b_i(x)\} = \omega \text{ess sup}_{x \in \Omega} b_i(x) \leq \omega < 1 - L \text{mes}_n \Omega,$$

whence inequality (28) holds. Now let $1 - L \text{mes}_n \Omega \leq 0$. Then

$$\min_{1 \leq i \leq N} b_i^0 > 0, \quad \omega \min_{1 \leq i \leq N} b_i^0 < 1 - L \text{mes}_n \Omega,$$

in particular, $\omega < 0$. Since

$$\max_{1 \leq i \leq N} \text{ess sup}_{x \in \Omega} \{\omega b_i(x)\} = \omega \min_{1 \leq i \leq N} \text{ess inf}_{x \in \Omega} b_i(x) = \omega \min_{1 \leq i \leq N} b_i^0,$$

in this situation inequality (28) holds too.

We choose $\delta \in (0, 1)$ in (27) such that

$$\begin{aligned} & \delta - L \text{mes}_n \Omega - \max_{1 \leq i \leq N} \text{ess sup}_{x \in \Omega} \{\omega b_i(x)\} > 0, \\ & \varepsilon_1 = 1 - \delta, \quad \varepsilon_2 = \delta - L \text{mes}_n \Omega - \max_{1 \leq i \leq N} \text{ess sup}_{x \in \Omega} \{\omega b_i(x)\}. \end{aligned}$$

As a result we obtain

$$e^{2\omega \int_0^{\tau_2} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_2)|^2 dx - e^{2\omega \int_0^{\tau_1} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau_1)|^2 dx +$$

$$\begin{aligned}
& + C_3 \int_{\tau_1}^{\tau_2} \int_{\Omega} [\gamma_1 e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N |\nabla \widehat{u}_i|^2 + \gamma e^{2\omega \int_0^t \gamma(s) ds} |\widehat{u}|^2] dx dt \leq \\
& \leq C_4 \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(\frac{1}{\gamma_1} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] + \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N |\widehat{f}_{i0}|^2 \right] \right) dx dt, \quad (29)
\end{aligned}$$

where C_3, C_4 are positive constants depending only on $\text{mes}_n \Omega$, L , ω and b_i^0 , $i = \overline{1, N}$.

From (9) we have the condition

$$\int_{\Omega} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N b_i(x) |\widehat{u}_i(x, t)|^2 \right] dx \rightarrow 0 \quad \text{as } t \rightarrow -\infty. \quad (30)$$

We fix arbitrary number $\tau_2 = \tau \in S$, and take the limit as $\tau_1 \rightarrow -\infty$ in (29), using (15) and (30). As a result we get

$$\begin{aligned}
& e^{2\omega \int_0^{\tau} \gamma(s) ds} \int_{\Omega} \sum_{i=1}^N b_i(x) |\widehat{u}_i(x, \tau)|^2 dx + \\
& + C_3 \int_{-\infty}^{\tau} \int_{\Omega} [\gamma_1 e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N |\nabla \widehat{u}_i|^2 + \gamma e^{2\omega \int_0^t \gamma(s) ds} |\widehat{u}|^2] dx dt \leq \\
& \leq C_4 \int_{-\infty}^{\tau} \int_{\Omega} \left(\frac{1}{\gamma_1} e^{2\omega \int_0^t \gamma(s) ds} \left[\sum_{i=1}^N \sum_{j=1}^n |\widehat{f}_{ij}|^2 \right] + \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N |\widehat{f}_{i0}|^2 \right) dx dt. \quad (31)
\end{aligned}$$

Hence, we obtain estimate (16). $\square$

4. Proof of the main results.

Proof of Theorem 1. First we prove that there exists at most one weak solution of problem (1), (2), (9). Assume the contrary. Let $u^1 = \text{col}(u_1^1, \dots, u_N^1)$, $u^2 = \text{col}(u_1^2, \dots, u_N^2)$ be (distinct) weak solutions of this problem. Using Lemma 2 (see (16)) we have

$$\iint_Q \left[\sum_{i=1}^N \gamma e^{2\omega \int_0^t \gamma(s) ds} |u_i^1(x, t) - u_i^2(x, t)|^2 \right] dx dt \leq 0. \quad (32)$$

From this inequality we get $u^1(x, t) = u^2(x, t)$ for almost all $(x, t) \in Q$. The obtained contradiction proves our statement.

Now we prove of the existence of the weak solution of problem (1), (2), (9) and its estimate. We start from a priori estimate of the weak solution. Suppose that u is the weak solutions (1), (2), (9). Using Lemma ($\mathbf{A}_1$), it easy to see that $u = 0$ is the weak solution of problem (1), (2), (9) with $f_{ij} = 0$, $i = \overline{1, N}$, $j = \overline{0, n}$. Then from estimate (16) (see Lemma 2) with $u_i^1 = u$, $f_{ij}^1 = f_{ij}$ and $u_i^2 = 0$, $f_{ij}^2 = 0$, $i = \overline{1, N}$, $j = \overline{0, n}$, we obtain estimate (12).

Now for each $m \in \mathbb{N}$, $i \in \{1, \dots, N\}$, $j \in \{0, \dots, n\}$ we define $f_{ij}^m(\cdot, t) := f_{ij}(\cdot, t)$, if $-m < t \leq 0$, and $f_{ij}^m(\cdot, t) := 0$, if $t \leq -m$, and consider a problem of finding of a vector

function $u_i^m \in L^2(-m, 0; \overset{\circ}{H}^1(\Omega)) \cap C([-m, 0]; H_{b_i}(\Omega))$, $i = \overline{1, N}$, which satisfies the initial condition

$$u_i^m|_{t=-m} = 0, \quad i = \overline{1, N}, \quad (33)$$

(as elements of spaces, respectively, $C([-m, 0]; H_{b_i}(\Omega))$) and system (1) in $Q_m := \Omega \times (-m, 0]$ in the sense of integral identity

$$\begin{aligned} \iint_{Q_m} \sum_{i=1}^N \left[-u_i^m v_i \varphi'_i + \sum_{j=0}^n a_{ij}(x, t, u^m, \nabla u^m) \partial_j v_i \varphi_i + v_i \varphi_i \int_{\Omega} c_i(x, y, t, u^m(y, t)) dy \right] dx dt = \\ = \iint_{Q_m} \sum_{i=1}^N \sum_{j=0}^n f_{ij}^m \partial_j v_i \varphi_i dx dt, \quad v_i \in \overset{\circ}{H}^1(\Omega), \quad \varphi_i \in C_c^1(-m, 0), \quad i = \overline{1, N}. \end{aligned} \quad (34)$$

The existence and uniqueness of the function u_m follow from a well-known fact (see, for example, [16]). We extend u_m on Q by zero and this extension is denoted by u_m again. Note that for each $m \in \mathbb{N}$ the function $u^m = \text{col}(u_1^m, \dots, u_N^m)$ such that for each $i \in \{1, \dots, N\}$ function u_i belongs to $L^2(S; \overset{\circ}{H}^1(\Omega)) \cap C(S; H_{b_i}(\Omega))$ and the integral identity (7) holds with f_{ij}^m instead of f_{ij} , $i = \overline{1, N}$, $j = \overline{0, n}$, that is

$$\begin{aligned} \iint_Q \sum_{i=1}^N \left[-u_i^m v_i \varphi'_i + \sum_{j=0}^n a_{ij}(x, t, u^m, \nabla u^m) \partial_j v_i \varphi_i + v_i \varphi_i \int_{\Omega} c_i(x, y, t, u^m(y, t)) dy \right] dx dt = \\ = \iint_Q \sum_{i=1}^N \left[\sum_{j=0}^n f_{ij}^m \partial_j v_i \right] \varphi_i dx dt, \quad v_i \in \overset{\circ}{H}^1(\Omega), \quad \varphi_i \in C_c^1(-\infty, 0), \quad i = \overline{1, N}. \end{aligned} \quad (35)$$

So, we arrive at the conclusion that the function $u^m = \text{col}(u_1^m, \dots, u_N^m)$ is the weak solution of problem (1), (2), (9) with (f_{ij}^m) instead of (f_{ij}) . In particular, we obtain (see (12)) the estimates

$$\begin{aligned} e^{2\omega \int_0^\tau \gamma(s) ds} \sum_{i=1}^N \|u_i^m(\cdot, \tau)\|_{H_{b_i}(\Omega)}^2 \leq \\ \leq C_5 \int_{-\infty}^\tau \left[\frac{1}{\gamma_1} e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N \sum_{j=1}^n \|f_{ij}(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N \|f_{i0}(\cdot, t)\|_{L^2(\Omega)}^2 \right] dt, \quad \tau \in S, \end{aligned} \quad (36)$$

$$\begin{aligned} \sum_{i=1}^N \left[\|\nabla u_i^m\|_{L_{\omega, \gamma_1}^2(S; L^2(\Omega))} + \|u_i^m\|_{L_{\omega, \gamma}^2(S; L^2(\Omega))} \right] \leq \\ \leq C_1 \sum_{i=1}^N \left[\sum_{j=0}^n \|f_{ij}\|_{L_{\omega, 1/\gamma_1}^2(S; L^2(\Omega))} + \|f_{i0}\|_{L_{\omega, 1/\gamma}^2(S; L^2(\Omega))} \right], \end{aligned} \quad (37)$$

where $C_5 > 0$ is a constant, which depends only on $\text{mes}_n \Omega$, L , ω and b_i^0 , $i = \overline{1, N}$.

Let k, l be arbitrary natural numbers, $l > k$. Using Lemma 2 for the functions u^k and u^l , we receive the similar estimate to (16), namely

$$\sup_{\tau \in S} \left[e^{2\omega \int_0^\tau \gamma(s) ds} \sum_{i=1}^N \|u_i^k(\cdot, \tau) - u_i^l(\cdot, \tau)\|_{H_{b_i}(\Omega)}^2 \right] +$$

$$\begin{aligned}
& + \sum_{i=1}^N \left[\|\nabla u_i^k - \nabla u_i^l\|_{L_{\omega,\gamma_1}^2(S;L^2(\Omega))}^2 + \|u_i^k - u_i^l\|_{L_{\omega,\gamma}^2(S;L^2(\Omega))}^2 \right] \leq \\
& \leq C_6 \int_{-l}^{-k} \left[\frac{1}{\gamma_1} e^{2\omega \int_0^t \gamma(s) ds} \sum_{i=1}^N \sum_{j=1}^n \|f_{ij}^k(\cdot, t) - f_{ij}^l(\cdot, t)\|_{L^2(\Omega)}^2 + \right. \\
& \quad \left. + \frac{1}{\gamma} e^{2\omega \int_0^t \gamma(s) ds} \|f_{i0}^k(\cdot, t) - f_{i0}^l(\cdot, t)\|_{L^2(\Omega)}^2 \right] dt, \tag{38}
\end{aligned}$$

where $C_6 > 0$ is a constant, depending only on $\text{mes}_n \Omega, L, \omega$ and $b_i^0, i = \overline{1, N}$.

From condition (11) we have that the right-hand side of inequality (38) converges to zero when k and l tend to $+\infty$. It means that the sequences $\{u_i^m\}_{m=1}^\infty, i = \overline{1, N}$, are the Cauchy sequences, respectively, in $L_{\omega,\gamma}^2(S; L^2(\Omega))$ and $C(S; H_{b_i}(\Omega)), i = \overline{1, N}$, and the sequences $\{\partial_j u_i^m\}_{m=1}^\infty, i = \overline{1, N}, j = \overline{0, n}$, are the Cauchy sequences in $L_{\omega,\gamma_1}^2(S; L^2(\Omega))$. It follows that there exist functions $u_i \in L_{\text{loc}}^2(S; \dot{H}^1(\Omega)) \cap L_{\omega,\gamma}^2(S; L^2(\Omega)) \cap C(S; H_{b_i}(\Omega))$ such that $\partial_j u_i \in L_{\omega,\gamma_1}^2(S; L^2(\Omega)), i = \overline{1, N}, j = \overline{1, n}$, and

$$u_i^m \xrightarrow{m \rightarrow \infty} u_i \quad \text{strongly in} \quad C(S; H_{b_i}(\Omega)) \text{ and } L_{\omega,\gamma}^2(S; L^2(\Omega)), \quad i = \overline{1, N}, \tag{39}$$

$$\partial_j u_i^m \xrightarrow{m \rightarrow \infty} \partial_j u_i \quad \text{strongly in} \quad L_{\omega,\gamma_1}^2(S; L^2(\Omega)), \quad i = \overline{1, N}, j = \overline{1, n}. \tag{40}$$

Using condition $(\mathbf{A}_2)$ and estimate (37), for all $t_1, t_2 \in S$ ($t_1 < t_2$) we obtain

$$\int_{t_1}^{t_2} \int_{\Omega} |a_{ij}(u^m)|^2 dx dt \leq C_7 \int_{t_1}^{t_2} \int_{\Omega} \left(|h_{ij}|^2 (|u^m|^2 + \sum_{k=1}^N |\nabla u_k^m|^2) + |\tilde{h}_{ij}|^2 \right) dx dt \leq C_8, \tag{41}$$

where C_7, C_8 are positive constants which do not depend on m .

Hence, from (41) we get for each $i \in \{1, \dots, N\}$ and $j \in \{0, \dots, n\}$ the sequence $\{a_{ij}(u^m)\}_{m=1}^\infty$ is bounded in $L_{\text{loc}}^2(S; L^2(\Omega))$. From here and (40) it follows the existence of a subsequence sequence $\{u^m\}_{m=1}^\infty$ (we also denote it by $\{u^m\}_{m=1}^\infty$) and functions $\chi_{ij} \in L_{\text{loc}}^2(S; L_2(\Omega)), i = \overline{1, N}, j = \overline{0, n}$, such that

$$\partial_j u_i^m \xrightarrow{m \rightarrow \infty} \partial_j u_i \quad \text{almost everywhere on } Q, \quad i = \overline{1, N}, j = \overline{0, n}, \tag{42}$$

$$a_{ij}(u^m) \xrightarrow{m \rightarrow \infty} \chi_{ij} \quad \text{weakly in } L_{2,\text{loc}}(S; L_2(\Omega)), \quad i = \overline{1, N}, j = \overline{0, n}. \tag{43}$$

From condition $(\mathbf{A}_1)$ and (42) it follows that

$$a_{ij}(u^m) \xrightarrow{m \rightarrow \infty} a_{ij}(u) \quad \text{almost everywhere on } Q, \quad i = \overline{1, N}, j = \overline{0, n}. \tag{44}$$

In view of [20, Chapter I, Lemma 1.3], from (43) and (44) we obtain $\chi_{ij} = a_{ij}(u)$ for each $i \in \{1, \dots, N\}, j \in \{0, \dots, n\}$, that is

$$a_{ij}(u^m) \xrightarrow{m \rightarrow \infty} a_{ij}(u) \quad \text{weakly in } L_{\text{loc}}^2(S; L_2(\Omega)), \quad i = \overline{1, N}, j = \overline{0, n}. \tag{45}$$

Using condition $(\mathbf{C}_2)$, for each $i \in \{1, \dots, n\}$ and for all $t_1, t_2 \in S$ ($t_1 < t_2$) we obtain

$$\int_{t_1}^{t_2} \int_{\Omega} \left| \int_{\Omega} c_i(x, y, t, u^m(y, t)) dy - \int_{\Omega} c_i(x, y, t, u(y, t)) dy \right|^2 dx dt \leq$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} \int_{\Omega} \left| \int_{\Omega} |c_i(x, y, t, u^m(y, t)) - c_i(x, y, t, u(y, t))| dy \right|^2 dx dt \leq \\
&\leq L_i^2 \text{mes}_n \Omega \int_{t_1}^{t_2} \gamma^2(t) \left(\int_{\Omega} |u^m(y, t) - u(y, t)| dy \right)^2 dt \leq \\
&\leq L_i^2 (\text{mes}_n \Omega)^2 \max_{t \in [t_1, t_2]} \gamma^2(t) \int_{t_1}^{t_2} \int_{\Omega} |u^m - u|^2 dx dt.
\end{aligned} \tag{46}$$

From (39) we have

$$\int_{t_1}^{t_2} \int_{\Omega} |u^m - u|^2 dx dt \xrightarrow{m \rightarrow \infty} 0. \tag{47}$$

Taking into account (46) and (47) we get

$$\int_{\Omega} c_i(\cdot, y, \cdot, u_m(y, \cdot)) dy \xrightarrow{m \rightarrow \infty} \int_{\Omega} c_i(\cdot, y, \cdot, u(y, \cdot)) dy \text{ strongly in } L_{\text{loc}}^2(S; L^2(\Omega)), \quad i = \overline{1, n}. \tag{48}$$

It leaves to prove that the function $u = \text{col}(u_1 \dots u_N)$ is the weak solution of problem (1),(2),(9). We pass to limit as m tends to $+\infty$ in identity (35). Using (39), (45), (48) and definition f_{ij}^m , $i = \overline{1, N}$, $j = \overline{0, n}$, $m \in \mathbb{N}$, we obtain identity (7). Taking into account (39), we evaluate the limit in (36) as m converges to $+\infty$. From the obtained inequality and conditions (11) we get that condition (9) holds. So, we conclude that the function u is the weak solution of (1), (2), (9). $\square$

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