УДК 517.53

P. SAHOO, G. BISWAS

MEROMORPHIC FUNCTION THAT SHARE A SET OF SMALL FUNCTIONS WITH ITS DERIVATIVE

P. Sahoo, G. Biswas. *Meromorphic function that share a set of small functions with its derivative*, Mat. Stud. **46** (2016), 130–136.

In this paper, we investigate the uniqueness problem of meromorphic function that share a set of small functions with its derivative and obtain some results which improve and generalize the recent results due to S. Majumder [Arab J. Math. Sci., 22 (2016), 265–274].

1. Introduction, definitions and results. In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We shall use the standard notations of the Nevanlinna theory of meromorphic functions as explained in [7], [14], and [18]. For a nonconstant meromorphic function f , we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for $r \rightarrow +\infty$ outside a possible exceptional set of finite logarithmic measure. We denote by $S(f)$ the family of all small functions α of meromorphic function f satisfying $T(r, \alpha) = S(r, f)$. For convenience, we assume that $S(f)$ includes all constant functions and $\tilde{S}(f) = S(f) \cup \{\infty\}$. For $\alpha \in \tilde{S}(f)$ and $S \subset \tilde{S}(f)$, we define

$$E(S, f) = \bigcup_{\alpha \in S} \{z: f(z) - \alpha = 0, \text{ counting multiplicity}\},$$

$$\overline{E}(S, f) = \bigcup_{\alpha \in S} \{z: f(z) - \alpha = 0, \text{ ignoring multiplicity}\}.$$

Let f and g be two nonconstant meromorphic functions. If $E(S, f) = E(S, g)$, we say that f and g share the set S CM and if $\overline{E}(S, f) = \overline{E}(S, g)$, we say that f and g share the set S IM. Especially, if $S = \{\alpha\}$, we say that f and g share the value α CM if $E(S, f) = E(S, g)$ and α IM if $\overline{E}(S, f) = \overline{E}(S, g)$.

Many research works on entire and meromorphic function f and its derivative $f^{(k)}$ have been done by many mathematicians in the world (see [2], [5], [11], [15], [17], [19], [21]). Recently, there have been an increasing interest in studying entire and meromorphic functions sharing a set of small functions with their derivative. In this direction we need the following definitions.

Definition 1 ([8, 9]). Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k , we denote by $N_k(r, a; f)$ the *counting function of those a -points of f*

2010 *Mathematics Subject Classification*: 30D35.

Keywords: meromorphic function; small function; set sharing; derivative.

doi:10.15330/ms.46.2.130-136

whose multiplicities are not greater than k and by $\overline{N}_k(r, a; f)$ the corresponding reduced counting function. We denote $N_{(k+1)}(r, a; f)$ by the counting function of those a -points of f whose multiplicities are greater than k and $\overline{N}_{(k+1)}(r, a; f)$ by the corresponding reduced counting function.

Definition 2 ([9]). Let f be a nonconstant meromorphic function and k be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_k(r, a; f)$ the *counting function of a -points of f* , where an a -point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_{(2)}(r, a; f) + \dots + \overline{N}_{(k)}(r, a; f).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

In 1996, the following conjecture was proposed by R. Brück ([3]).

Conjecture 1. Let f be a nonconstant entire function and $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f' - a}{f - a} = c,$$

for some nonzero constant c , where $\rho_1(f)$ is the first iterated order of f defined by

$$\rho_1(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

In 1996, R. Brück ([3]) proved that the conjecture is true if $a = 0$ or $N(r, 0; f') = S(r, f)$. In 1998, G. G. Gundersen and L. Z. Yang ([6]) proved that the conjecture is true if f is of finite order and fails, in general, for meromorphic functions. In 2004, Z. X. Chen and K. H. Shon ([4]) proved that the conjecture is true for entire function of order $\rho_1(f) < \frac{1}{2}$. In 2005, A. Al-Khaladi ([1]) proved that the conjecture is true for meromorphic function f when $N(r, 0; f') = S(r, f)$.

In 2008, L. Z. Yang and J. L. Zhang ([16]) considered the case f^n , where f is a nonconstant meromorphic function and obtained the following result.

Theorem A. Let f be a nonconstant meromorphic function and $n \geq 12$ be an integer. If f^n and $(f^n)'$ share 1 CM, then $f^n = (f^n)'$, and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where c is a nonzero constant.

In 2009, J. L. Zhang and L. Z. Yang ([20]) improved Theorem A as follows.

Theorem B. Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k are positive integers such that $n > k + 1 + \sqrt{k+1}$. If f^n and $(f^n)^{(k)}$ share $\alpha(z)$ CM, then $f^n = (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c is a nonzero constant and $\lambda^k = 1$.

In the same paper [20] the authors also proved the following result.

Theorem C. Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k are positive integers such that $n > 2k + 3 + \sqrt{(2k+3)(k+3)}$. If f^n and $(f^n)^{(k)}$ share $\alpha(z)$ IM, then $f^n = (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c is a nonzero constant and $\lambda^k = 1$.

In 2012, L. Sheng and G. Zongsheng ([12]) proved the following result.

Theorem D. *Let f be a nonconstant meromorphic function such that $\overline{N}(r, \infty; f) = S(r, f)$. Suppose f^n and $(f^n)'$ share 1 CM. If (1) $n \geq 3$, or (2) $n = 2$ and $N(r, 0; f) = O(N_3(r, 0; f))$, then $f^n = (f^n)'$ and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where c is a nonzero constant.*

Regarding Theorems B-D, one may ask the following question.

Question 1. *What happen if one replace the polynomial f^n by $f^n P(f)$ where $P(z) = a_s z^s + a_{s-1} z^{s-1} + \dots + a_1 z + a_0$, where $a_s (\neq 0)$, $a_{s-1}, \dots, a_1, a_0$ are complex constants.*

Recently S. Majumder ([10]) answered the above question and proved the following results which improve Theorems B-D.

Theorem E. *Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, s are positive integers such that $n > k + 2$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share $\alpha(z)$ CM, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, \dots, s\}$ and $f^{n+i} = (f^{n+i})^{(k)}$, where f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where c is a nonzero constant and $\lambda^k = 1$.*

Theorem F. *Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, s are positive integers such that $n > 2k + s + 2$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share $\alpha(z)$ IM, then the conclusion of Theorem E holds.*

Theorem G. *Let f be a nonconstant meromorphic function such that $\overline{N}(r, \infty; f) = S(r, f)$, $N(r, 0; f) = O(N_2(r, 0; f))$ and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, s are positive integers such that $n > k + s$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share $\alpha(z)$ IM, then the conclusion of Theorem E holds.*

Regarding Theorems E-G, it is natural to ask the following question which is the motive of the present paper.

Question 2. *What happen if sharing a small function is replaced by sharing a set S_m of small functions in Theorems E-G where $S_m = \{\alpha(z), \alpha(z)\omega, \dots, \alpha(z)\omega^{m-1}\}$, where $\omega = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$, m is a positive integer?*

In the paper our aim is to find the possible answer of the above question. We prove the following results which extend and improve Theorems E-G respectively.

Theorem 1. *Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, m, s are positive integers such that $n > k + \frac{2}{m}$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share the set S_m CM, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, \dots, s\}$ and $f^{n+i} = t(f^{n+i})^{(k)}$ for a constant t satisfying $t^m = 1$, and f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where c is a nonzero constant and $\lambda^{km} = 1$.*

Theorem 2. *Let f be a nonconstant meromorphic function and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, m, s are positive integers such that $n > 2k + \frac{s+2}{m}$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share the set S_m IM, then the conclusion of Theorem 1 holds.*

Theorem 3. *Let f be a nonconstant meromorphic function such that $\overline{N}(r, \infty; f) = S(r, f)$, $N(r, 0; f) = O(N_2(r, 0; f))$ and $\alpha(z) (\neq 0, \infty)$ be a small function of f . Suppose n, k, m, s are positive integers such that $n > k + \frac{s}{m}$. If $f^n P(f)$ and $(f^n P(f))^{(k)}$ share the set S_m IM, then the conclusion of Theorem 1 holds.*

2. Lemmas.

Lemma 1 ([13]). *Let f be a nonconstant meromorphic function and $a_n(z) (\neq 0), a_{n-1}(z), \dots, a_1(z), a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \dots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([10]). *Let f be a nonconstant meromorphic function and n, k, s be positive integers such that $n \geq k + 1$. If $f^n P(f) = \{f^n P(f)\}^{(k)}$, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, \dots, s\}$; and $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where c is a nonzero constant and $\lambda^k = 1$.*

Arguing similarly as in Lemma 2, we can get the following lemma.

Lemma 3. *Let f be a nonconstant meromorphic function and n, k, m be positive integers such that $n \geq k + \frac{1}{m}$. If $\{f^n P(f)\}^m = [\{f^n P(f)\}^{(k)}]^m$, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, \dots, s\}$; and $f^{n+i} \equiv t(f^{n+i})^{(k)}$ for a constant t satisfying $t^m = 1$, where f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where c is a nonzero constant and $\lambda^{km} = 1$.*

3. Proof of the theorems.

Proof of Theorem 1. Let $F = (F_1)^m$ and $G = (G_1)^m$, where $F_1 = \frac{f^n P(f)}{\alpha(z)}$ and $G_1 = \frac{\{f^n P(f)\}^{(k)}}{\alpha(z)}$. Since $f^n P(f)$ and $\{f^n P(f)\}^{(k)}$ share S_m CM, F and G share 1 CM except the zeros and poles of $\alpha(z)$, and so $\overline{N}(r, 1; F) = \overline{N}(r, 1; G) + S(r, f)$. Let

$$V = \frac{1}{F} \left(\frac{G'}{G-1} - \frac{F'}{F-1} \right) = \frac{G}{F} \left(\frac{G'}{G-1} - \frac{G'}{G} \right) - \left(\frac{F'}{F-1} - \frac{F'}{F} \right). \quad (1)$$

We now discuss following two cases:

Case 1. Let $V = 0$. Then simplifying (1), we get $\frac{G'}{G-1} = \frac{F'}{F-1}$. Integrating we obtain

$$F - 1 = c(G - 1), \quad (2)$$

where c is a nonzero constant. It is obvious from (2) that $\overline{N}(r, \infty; f) = S(r, f)$. If $c \neq 1$, then using Lemma 1 and (2) we obtain

$$\begin{aligned} m(n+s)T(r, f) &= T(r, F) + O(1) \leq \\ &\leq T\left(r, \frac{G}{F}\right) + S(r, f) \leq mT\left(r, \frac{G_1}{F_1}\right) + S(r, f) \leq mN\left(r, \infty; \frac{G_1}{F_1}\right) + S(r, f) \leq \\ &\leq mN_k(r, 0; F_1) + mk\overline{N}(r, \infty; f) + S(r, f) \leq mk\overline{N}(r, 0; f) + msT(r, f) + S(r, f). \end{aligned}$$

This gives $nT(r, f) \leq k\overline{N}(r, 0; f) + S(r, f)$, a contradiction as $n > k + \frac{1}{m}$. Therefore, we have $c = 1$ and hence from (2) we have $F = G$. Now the conclusion of the theorem follows from Lemma 3.

Case 2. Let $V \neq 0$. Then $F \neq G$. From (1), we see that $m(r, \infty; V) = S(r, f)$ and

$$m(r, \infty; F) \leq m(r, 0; V) + S(r, f). \quad (3)$$

Now from (1), we obtain

$$\begin{aligned} N(r, \infty; F) - \overline{N}(r, \infty; F) &\leq N(r, 0; V) + S(r, f) \leq T\left(r, \frac{1}{V}\right) - m(r, 0; V) + S(r, f) \\ &\leq T(r, V) - m(r, 0; V) + S(r, f) = N(r, \infty; V) + m(r, \infty; V) - m(r, 0; V) + S(r, f) \leq \\ &\leq N_{mk+1}(r, 0; F) - m(r, 0; V) + S(r, f) \leq (mk+1)\overline{N}(r, 0; f) + msT(r, f) - \\ &\quad - m(r, 0; V) + S(r, f). \end{aligned} \quad (4)$$

Now (3), (4) and Lemma 1 together yields

$$\begin{aligned} m(n+s)T(r, f) &= T(r, F) + O(1) \leq N(r, \infty; F) + m(r, \infty; F) + S(r, f) \leq \\ &\leq (mk+1)\overline{N}(r, 0; f) + msT(r, f) + \overline{N}(r, \infty; f) + S(r, f). \end{aligned} \quad (5)$$

We now consider the function

$$U = F \left(\frac{F'}{F-1} - \frac{G'}{G-1} \right)^k. \quad (6)$$

Clearly $U \neq 0$ and

$$m(r, \infty; U) \leq m(r, \infty; F) + S(r, f), \quad (7)$$

$$m(r, 0; F) \leq m(r, 0; U) + S(r, f). \quad (8)$$

Since $n > k + \frac{2}{m}$, we obtain from (6)–(8)

$$\begin{aligned} N(r, 0; F) + k\overline{N}(r, 0; f) &\leq N(r, 0; U) + S(r, f) \leq T\left(r, \frac{1}{U}\right) - m(r, 0; U) + S(r, f) \leq \\ &\leq T(r, U) - m(r, 0; U) + S(r, f) = N(r, \infty; U) + m(r, \infty; U) - m(r, 0; U) + S(r, f) \leq \\ &\leq N(r, \infty; F) + k\overline{N}(r, \infty; f) + m(r, \infty; F) - m(r, 0; U) + S(r, f) \leq \\ &\leq T(r, F) + k\overline{N}(r, \infty; f) - m(r, 0; U) + S(r, f) \leq \\ &\leq N(r, 0; F) + m(r, 0; F) + k\overline{N}(r, \infty; f) - m(r, 0; U) + S(r, f). \end{aligned}$$

This gives $\overline{N}(r, 0; f) \leq \overline{N}(r, \infty; f) + S(r, f)$. Using this in (5), we get $nT(r, f) \leq (k + \frac{2}{m})\overline{N}(r, \infty; f) + S(r, f)$, a contradiction with the assumption that $n > k + \frac{2}{m}$. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Let F, G, F_1 and G_1 be similarly defined as in Theorem 1. Since $f^n P(f)$ and $\{f^n P(f)\}^{(k)}$ share the set S_m IM, F and G share 1 IM except the zeros and poles of $\alpha(z)$, and hence $\overline{N}(r, 1; F) = \overline{N}(r, 1; G) + S(r, f)$. We assume that $F \neq G$. Now

$$\begin{aligned} \overline{N}(r, 1; F) &\leq \overline{N}\left(r, 1; \frac{G}{F}\right) + S(r, f) \leq T\left(r, \frac{G}{F}\right) + S(r, f) \leq \\ &\leq N\left(r, \infty; \frac{G}{F}\right) + m\left(r, \infty; \frac{G}{F}\right) + S(r, f) \leq \end{aligned}$$

$$\begin{aligned}
&\leq mN\left(r, \infty; \frac{G_1}{F_1}\right) + S(r, f) \leq mN_k(r, 0; f^n P(f)) + mk\bar{N}(r, \infty; f) + S(r, f) \leq \\
&\leq mk\bar{N}(r, 0; f) + msT(r, f) + mk\bar{N}(r, \infty; f) + S(r, f).
\end{aligned} \tag{9}$$

Using (9), Lemma 1 and the second fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned}
m(n+s)T(r, f) &\leq T(r, F) + S(r, f) \leq \bar{N}(r, 0; F) + \bar{N}(r, 1; F) + \bar{N}(r, \infty; F) + S(r, f) \leq \\
&\leq (mk+1)(\bar{N}(r, 0; f) + \bar{N}(r, \infty; f)) + (m+1)sT(r, f) + S(r, f).
\end{aligned}$$

This gives $(mn-2mk-s-2)T(r, f) \leq S(r, f)$, a contradiction with the fact that $n > 2k + \frac{s+2}{m}$. Therefore $F = G$. Now the conclusion follows from Lemma 3. $\square$

Proof of Theorem 3. Let F, G, F_1 and G_1 be defined similarly as in Theorem 1. Since $f^n P(f)$ and $\{f^n P(f)\}^{(k)}$ share the set S_m IM, F and G share 1 IM except the zeros and poles of $\alpha(z)$, and hence $\bar{N}(r, 1; F) = \bar{N}(r, 1; G) + S(r, f)$. Let us consider the function

$$W = \frac{F'(F-G)}{F(F-1)}. \tag{10}$$

We now discuss following two cases:

Case (i). Suppose that $W \not\equiv 0$. Clearly $m(r, \infty; W) = S(r, f)$. Let z_0 ($\alpha(z_0) \neq 0$) be a zero of f of multiplicity p . Then z_0 will be a zero of F with multiplicity mnp and a zero of G with multiplicity $m(np-k)$. Then by a simple calculation we see that

$$W = O\left((z-z_0)^{m(np-k)-1}\right).$$

Since $n > k + \frac{s}{m}$, it follows that W is analytic at z_0 . From this and by the assumption of the theorem, it is clear that $N(r, \infty; W) = S(r, f)$ and consequently $T(r, W) = S(r, f)$. By (10), we have $m(r, 0; F) = S(r, f)$ and so $m(r, 0; f) = S(r, f)$. Thus from (10), we obtain

$$N_2(r, 0; f) \leq N(r, 0; W) + S(r, f) \leq T(r, W) + S(r, f) = S(r, f).$$

Therefore by the assumption of the theorem we have $N(r, 0; f) = O(N_2(r, 0; f)) = S(r, f)$. Thus we obtain $T(r, f) = S(r, f)$, which is not possible.

Case (ii). Suppose $W = 0$. This gives $F = G$ and the result follows from Lemma 3. $\square$

REFERENCES

1. A. Al-Khaladi, *On meromorphic functions that share one value with their derivatives*, Analysis, **25** (2005), 131–140.
2. A. Banerjee, *Weighted sharing of a small function by a meromorphic function and its derivative*, Comput. Math. Appl., **53** (2007), 1750–1761.
3. R. Brück, *On entire functions which share one value CM with their first derivative*, Results Math., **30** (1996), 21–24.

4. Z.X. Chen, K.H. Shon, *On conjecture of R. Brück concerning entire function sharing one value CM with its derivative*, Taiwanese J. Math., **8** (2004), 235–244.
5. G.G. Gundersen, *Meromorphic functions that share finite values with their derivatives*, J. Math. Anal. Appl., **75** (1980), 441–446.
6. G.G. Gundersen, L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., **223** (1998), 88–95.
7. W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
8. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161** (2001), 193–206.
9. I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241–253.
10. S. Majumder, *Values shared by meromorphic functions and their derivatives*, Arab J. Math. Sci., **22** (2016), 265–274.
11. E. Mues, N. Steinmetz, *Meromorphe Funktionen, die mit ihrer und zweiten Ableitung einen endlichen Wertteilen*, Complex Var. Theory Appl., **6** (1986), 51–71.
12. L. Sheng, G. Zongsheng, *Results on a question of Zhang and Yang*, Acta Math. Sci., **32** (2012), 717–723.
13. C.C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107–112.
14. L. Yang, *Value distribution theory*, Springer-Verlag Berlin, 1993.
15. L.Z. Yang, *Entire functions that share finite values with their derivatives*, Bull. Aust. Math. Soc., **41** (1990), 337–342.
16. L.Z. Yang, J.L. Zhang, *Nonexistence of meromorphic solutions of Fermat type function equation*, Aequationes Math., **76** (2008), 140–150.
17. H.X. Yi, *On characteristic function of a meromorphic function and its derivative*, Indian J. Math., **33** (1991), 119–133.
18. H.X. Yi, C.C. Yang, *Uniqueness theory of meromorphic functions*, Science Press, Beijing, New York, 1995.
19. J.L. Zhang, L.Z. Yang, *Some results related to a conjecture of R. Brück*, J. Inequal. Pure Appl. Math., **8** (2007), Art. 18.
20. J.L. Zhang, L.Z. Yang, *A power of a meromorphic function sharing a small function with its derivative*, Ann. Acad. Sci. Fenn. Math., **34** (2009), 249–260.
21. Q.C. Zhang, *Meromorphic function that shares one small function with its derivative*, J. Inequal. Pure Appl. Math., **6** (2005), Art. 116.

Department of Mathematics, University of Kalyani
 West Bengal, India
 sahoopulak1@gmail.com
 gdb.math@gmail.com

Received 16.09.2016