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ON THE CONVERGENCE CLASSES FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Results on the belonging of entire functions to generalized convergence classes earlier obtained by the authors are carried over on entire functions of several complex variables.

1. Introduction. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (1)$$

be an entire transcendental function, $M(r, f) = \max\{|f(z)| : |z| = r\}$ and $\mu(r, f) = \max\{|a_k| : k \geq 0\}$ be the maximal term of the series (1). Let L be class of continuous functions α such that $\alpha(x) \geq 0$ for $x \geq x_0$, $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and on $[x_0, +\infty)$ the function α increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha(x(1+o(1))) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$ and remark that if $\alpha \in L^0$ then $[1] \lim_{x \rightarrow +\infty} \frac{\alpha(\lambda x)}{\alpha(x)} = c(\lambda) < +\infty$ for each $\lambda \in [1, +\infty)$.

For $\alpha \in L$ and $\beta \in L$ we say that f belongs to the *generalized convergence class* if

$$\int_{r_0}^{\infty} \frac{\alpha(\ln M(r, f))}{r\beta(\ln r)} dr < +\infty. \quad (2)$$

We remark that $\mu(r, f) \leq M(r, f) \leq 2\mu(2r, f)$. Therefore, if $\alpha \in L^0$ and $\beta \in L^0$, then (2) holds iff

$$\int_{r_0}^{\infty} \frac{\alpha(\ln \mu(r, f))}{r\beta(\ln r)} dr < +\infty. \quad (3)$$

Theorem 1 from [2] for entire Dirichlet series implies the following statement.

Theorem A. *Let $\alpha \in L$ be a concave function on $[x_0, +\infty)$ and $\alpha(e^x) \in L^0$. Let a function $\beta \in L^0$ satisfy the condition $x\beta'(x)/\beta(x) \geq h > 0$ on $[x_0, +\infty)$ and*

$$\int_{x_0}^{\infty} \frac{\alpha(x)}{\beta(x)} dx < +\infty. \quad (4)$$

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In order that f belong to generalized convergence class it is necessary and in the case when $|a_k|/|a_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that

$$\sum_{k=0}^{\infty} (\alpha(k) - \alpha(k-1)) \beta_1 \left(\frac{1}{k} \ln \frac{1}{|a_k|} \right) < +\infty, \quad \beta_1(x) = \int_x^{\infty} \frac{dt}{\beta(t)}. \quad (5)$$

By Ω we denote the class of positive functions Φ unbounded on $(-\infty, +\infty)$ such that the derivative Φ' is continuously differentiable, positive and increasing to $+\infty$ on $(-\infty, +\infty)$ and we say that an entire function (1) belongs to the *convergence Φ -class* if

$$\int_{r_0}^{\infty} \frac{\Phi'(\ln r) \ln M(r, f)}{\Phi^2(\ln r)} d \ln r < +\infty. \quad (6)$$

In [3] it is proved that if $\Phi \in \Omega$ and

$$\int_{r_0}^{\infty} \frac{\Phi'(\sigma) \ln \Phi'(\sigma)}{\Phi^2(\sigma)} d\sigma < +\infty, \quad (7)$$

then (6) holds iff

$$\int_{r_0}^{\infty} \frac{\Phi'(\ln r) \ln \mu(r, f)}{\Phi^2(\ln r)} d \ln r < +\infty. \quad (8)$$

On the other hand, Theorem 1 from [4] implies the following statement.

Theorem B. Let $\Phi \in \Omega$ and

$$0 < h \leq \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \leq H < +\infty, \quad (9)$$

In order that (8) hold it is necessary and in the case when $|a_k|/|a_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that

$$\sum_{k=0}^{\infty} \frac{1}{\Phi' \left(\frac{1}{k} \ln \frac{1}{|a_k|} \right)} < +\infty. \quad (10)$$

Clearly, by condition (9) from Theorem B we get the necessary and sufficient condition for belonging of f to the convergence Φ -class.

Here we carry over these results on entire functions of several complex variables.

We remark that in [7] it is investigated the belonging to convergence classes of the functions represented by series, similar to the Taylor-Dirichlet series.

2. Generalized convergence class. Let $Z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We consider the exhaustion of the space $\mathbb{C}^n$ by the system $(G_R)_{R \geq 0}$ of a complete multiple-circular domains with the center at the point $(0, \dots, 0) \in \mathbb{C}^n$. We assume that this system satisfies the conditions:

- i) $\bigcup_{R \geq 0} G_R = \mathbb{C}^n$;
- ii) $(\forall R_1 < R_2) : G_{R_1} \subset G_{R_2}$;
- iii) $(z_1, \dots, z_n) \in G_1 \iff (\forall R > 0) : (Rz_1, \dots, Rz_n) \in G_R$;
- iv) $(z_1, \dots, z_n) \in G_R \implies (\forall (\theta_1, \dots, \theta_n) \in \mathbb{R}^n) : (z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) \in G_R$.

We denote $d_{\overline{G}}(k_1, \dots, k_n) = \max\{|z_1|^{k_1} \dots |z_n|^{k_n} : Z \in \overline{G} = \overline{G_1}\}$ and suppose that if $Z = (z_1, z_1, \dots, z_n) \in \overline{G}$ then $|z_j| \geq l > 0$ for all $j = 1, \dots, n$.

Let

$$f(Z) = f(z_1, z_1, \dots, z_n) = \sum_{k_1 \dots k_n=0}^{\infty} a_{k_1, \dots, k_n} z_1^{k_1} \dots z_n^{k_n} \quad (11)$$

be an entire transcendental function. We put $M_{\overline{G}}(R) = M_{\overline{G}}(R, f) = \max\{|f(Z)| : Z \in \overline{G}_R\}$ and say that the function (11) belongs to the *generalized convergence class* if

$$\int_{R_0}^{\infty} \frac{\alpha(\ln M_{\overline{G}}(R, f))}{\beta(\ln R)} d \ln R < +\infty, \quad (12)$$

where $\alpha \in L$ and $\beta \in L$. The following theorem is true.

Theorem 1. *Suppose that the functions α and β satisfy the assumptions of Theorem A and $A_k = \max\{|a_{k_1, \dots, k_n}| : k_1 + \dots + k_n = k\}$. In order that the function (11) belong to the generalized convergence class it is necessary and in the case when $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that*

$$\sum_{k=0}^{\infty} (\alpha(k) - \alpha(k-1)) \beta_1 \left(\frac{1}{k} \ln \frac{1}{|A_k|} \right) < +\infty, \quad \beta_1(x) = \int_x^{\infty} \frac{dt}{\beta(t)}. \quad (13)$$

Proof. We use a method of A.A. Gol'dberg [5], applied for finding of formulas for the determination of the order and the type of the function $\ln M_{\overline{G}}(R, f)$.

We denote

$$P_k(\xi_1, \dots, \xi_n) = \sum_{k_1 + \dots + k_n = k} a_{k_1, \dots, k_n} \xi_1^{k_1} \dots \xi_n^{k_n}.$$

Then [5]

$$\max\{M_{\overline{G}}(1, P_k)R^k : k > 0\} \leq M_{\overline{G}}(R, f) \leq F_1(R) =: \sum_{k=1}^{\infty} M_{\overline{G}}(1, P_k)R^k. \quad (14)$$

Since $\max\{M_{\overline{G}}(1, P_k)R^k : k > 0\} = \mu(R, F_1)$ and the functions $\ln \mu(R, F_1)$ and $\ln F_1(R)$ belong to the same generalized convergence class, the relation (12) holds iff

$$\int_{R_0}^{\infty} \frac{\alpha(\ln \mu(R, F_1))}{\beta(\ln R)} d \ln R < +\infty. \quad (15)$$

Let

$$F_2(R) = \sum_{k=1}^{\infty} B_k R^k, \quad B_k =: \max_{k_1 + \dots + k_n = k} (|a_{k_1, \dots, k_n}| d_{\overline{G}}(k_1, \dots, k_n)). \quad (16)$$

By Lemma 3 from [5] $B_k \leq M_{\overline{G}}(1, P_k) \leq B_k(k+1)^n$ and, therefore,

$$\begin{aligned} \ln \mu(R, F_2) &\leq \ln \mu(R, F_1) \leq \max\{\ln B_k + nk + k \ln R : k > 0\} = \\ &= \max\{\ln B_k + k(\ln R + n) : k > 0\} = \ln \mu(Re^n, F_2). \end{aligned}$$

Hence it follows that

$$\int_{R_0}^{\infty} \frac{\alpha(\ln \mu(R, F_2))}{\beta(\ln R)} d \ln R \leq \int_{R_0}^{\infty} \frac{\alpha(\ln \mu(R, F_1))}{\beta(\ln R)} d \ln R \leq \int_{R_0}^{\infty} \frac{\alpha(\ln \mu(Re^n, F_2))}{\beta(\ln R)} d \ln R$$

and in view of the condition $\beta \in L^0$ the relation (15) is equivalent to the relation

$$\int_{R_0}^{\infty} \frac{\alpha(\ln \mu(R, F_2))}{\beta(\ln R)} d \ln R < +\infty. \quad (17)$$

Since the closed domain $\overline{G}$ is bounded, contained in some polycylinder $Z_L = \{Z : |z_j| \leq L\}$. Therefore, $d_{\overline{G}}(k_1, \dots, k_n) = d_k^k$, where $l \leq d_k \leq L$ and, thus,

$$B_k = d_k^k \max_{k_1 + \dots + k_n = k} |a_{k_1, \dots, k_n}| = A_k d_k^k$$

and $\ln \mu(R, F_2) = \max\{\ln A_k + k \ln d_k + k \ln R\}$, that is

$$\ln \mu(lR, F_3) \leq \ln \mu(R, F_2) \leq \ln \mu(LR, F_3), \quad (18)$$

where $F_3(R) = \sum_{n=1}^{\infty} A_k R^k$. From (18) in view of the condition $\beta \in L^0$ it follows that the relation (17) and

$$\int_{R_0}^{\infty} \frac{\alpha(\ln \mu(R, F_3))}{\beta(\ln R)} d \ln R < +\infty. \quad (19)$$

are equivalent. By Theorem A in order that inequality (19) hold, it is necessary and in the case $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that (14) holds. Theorem 1 is proved. $\square$

3. Convergence Φ -class. We say that the function (11) belongs to the *convergence Φ -class* ($\Phi \in \Omega$) if

$$\int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln M_{\overline{G}}(R, f)}{\Phi^2(\ln R)} d \ln R < +\infty. \quad (20)$$

The results of the section are announced in [6] without the proof.

Theorem 2. *Let the function $\Phi \in \Omega$ satisfy the condition (7), (9), and $\Phi'(\sigma + O(1)) \asymp \Phi'(\sigma)$ as $\sigma \rightarrow +\infty$. In order that (20) hold it is necessary and in the case when $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that*

$$\sum_{k=k_0}^{\infty} \frac{1}{\Phi'(\frac{1}{k} \ln \frac{1}{|A_k|})} < +\infty \quad (21)$$

Proof. In view of a result from [3] and the inequalities (14) the relation (20) holds if and only if

$$\int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(R, F_1)}{\Phi^2(\ln R)} d \ln R < +\infty. \quad (22)$$

For the function (16) as above we have $\ln \mu(R, F_2) \leq \ln \mu(R, F_1) \leq \ln \mu(Re^n, F_2)$, whence

$$\begin{aligned} \int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(R, F_2)}{\Phi^2(\ln R)} d \ln R &\leq \int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(R, F_1)}{\Phi^2(\ln R)} d \ln R \leq \\ &\leq \int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(Re^n, F_2)}{\Phi^2(\ln R)} d \ln R \end{aligned}$$

and according to the condition $\Phi'(\sigma + O(1)) \asymp \Phi'(\sigma)$ as $\sigma \rightarrow +\infty$ the relation (22) is equivalent to the condition

$$\int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(R, F_2)}{\Phi^2(\ln R)} d \ln R < +\infty. \quad (23)$$

For the function $F_3(R) = \sum_{n=1}^{\infty} A_n R^n$ we have (18) and, therefore, in view of the condition $\Phi'(\sigma + O(1)) \asymp \Phi'(\sigma)$ as $\sigma \rightarrow +\infty$ the relation (23) hold if and only if

$$\int_{R_0}^{\infty} \frac{\Phi'(\ln R) \ln \mu(R, F_3)}{\Phi^2(\ln R)} d \ln R < +\infty. \quad (24)$$

By Theorem B in order that inequality (24) holds, it is necessary and in the case when $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that (21) holds. $\square$

Choosing properly the function α and β or Φ from Theorems 1 and 2 we obtain suitable corollaries. If we choose for example $\Phi(\sigma) = e^{\rho\sigma}$, $0 < \rho < +\infty$, then we obtain the following statement for an entire function of finite order.

Corollary 1. *In order that $\int_{R_0}^{\infty} \frac{\ln M_{\overline{G}}(R, f)}{R^{\rho+1}} dR < +\infty$, it is necessary and in the case when $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that $\sum_{k=k_0}^{\infty} |A_k|^{\rho/k} < +\infty$.*

For an entire function of finite logarithmic order $p \in (1, +\infty)$ we choose $\Phi(\sigma) = \sigma^p$ for $\sigma \geq \sigma_0$.

Corollary 2. *In order that $\int_{R_0}^{\infty} \frac{\ln M_{\overline{G}}(R, f)}{R \ln^{p+1} R} dR < +\infty$, $p > 1$, it is necessary and in the case when $|A_k|/|A_{k+1}| \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ it is sufficient that*

$$\sum_{k=k_0}^{\infty} \left(\frac{1}{k} \ln \frac{1}{|A_k|} \right)^{1-p} < +\infty.$$

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