ANALYSIS OF A DYNAMICAL COURNOT DUOPOLY GAME WITH DISTRIBUTED TIME DELAY

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DOI: 10.1515/tjeb-2015-0001

The aim of the paper is to analyze the dynamic model of the Cournot duopoly game with bounded rationality associated to two firms. We consider the cost function of the first firm as nonlinear and for the second firm as linear. The players do not have a complete knowledge of the market and they follow a bounded rationality adjustment process based on the estimation of the marginal profit. Also, the distributed time delay is introduced, because the decisions at the current time depend on the average past decisions. The mathematical model is described by a distributed delay differential system with two nonlinear equations. The study for the local stability of the Nash equilibrium point is carried out in the case of two types of kernels: weak (exponential) and Dirac. A change in local stability of the equilibrium point, from stable to unstable, implies a Hopf bifurcation. The delays are considered as bifurcation parameters. In some conditions of the parameters of the model, we have proved that a family of periodic solutions bifurcates from the equilibrium point when the bifurcation parameter passes through a critical value. Numerical simulations are performed to illustrate the effectiveness of our results. Finally, conclusions and future researches are provided.

Keywords: Cournot duopoly game, Cournot Model, Bounded Rationality, distributed time delay, local stability, Hopf bifurcation.

JEL Classification: C61, C73, D43, D84.

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1. Introduction

Augustin Cournot studied the oligopoly markets operation where each company acts knowing that the volume of production affects the market price (Binmore, 1999). He defined the balance as a situation where each company chooses the output which can maximize its profit while taking into account the output forecast by the other companies. He showed that such balance leads to a price above the marginal productivity (Sirghi, 2008).

Lately, there are many articles that studied Cournot game with bounded rationality, considering different demand and cost functions (linear and nonlinear) (Aziga, 1999; Aziga & Elsadany, 2003, 2004; Aziga *et al.*, 2001, 2002, 2013; Elsadany, 2010; Kopel, 1996; Sheng *et al.*, 2013; Yassen & Aziga, 2003). Also, heterogeneous players into Cournot game were taken into consideration. (Aziga & Elsadany, 2003, 2004; Dubiel-Teleszynski, 2011; Onozaki *et al.* 2003; Zhang *et al.*, 2007).

Due to incomplete information and delayed decisions, the delay is introduced in nonlinear discrete Cournot duopoly games and the dynamics is analyzed (Dubiel-Teleszynski, 2011; Ma & Ji, 2009; Sirghi, 2008).

Based on Binmore (1999), Ma and Ji (2009), and Mircea, Neamțu and Opriș (2004), in this paper a dynamic nonlinear Cournot duopoly is described by an economic game, where two firms are on the market with a homogenous consumption product. The firms do not cooperate, which means there is no collusion. Each firm's output decision affects the good's price. Moreover, firms compete in quantities and choose the quantities simultaneously. Because the decisions of the economic agents at time t take into account the past decisions, we introduce the distributed time delays as weak and Dirac kernels. We determine the Nash equilibrium point and we analyze the stability and the existence of the Hopf bifurcation.

The paper is organized as follows. In Section 2, the dynamic Cournot duopoly with bounded rationality and distributed time delay is presented. In Section 3, for the weak and Dirac kernels, the local stability and the existence of the Hopf bifurcation are analyzed. In Section 4, numerical simulations are performed in order to illustrate the theoretical findings. Finally, some conclusions are drawn.

2. The Mathematical Models

Suppose that there are two representative firms F_1, F_2 . The firms F_1, F_2 make the optimal output decision, and suppose the t -outputs are $q_i(t)$, $i=1,2$, respectively for $t \geq 0$.

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For $t \geq 0$, the price $P(t)$, is determined by the inverse demand function given by (Kopel, 1996):

$$P(t) = P(Q(t)) = m - nQ(t)^2 \quad (1)$$

where $Q(t) = q_1(t) + q_2(t)$ is the total output, $m > n > 0$.

According to (Ma & Ji, 2009), the cost function of F_1 is nonlinear and the cost function of F_2 is linear. The second firm has lower variable cost than the first firm. The cost functions for F_1 , respectively F_2 , are given by:

$$\begin{aligned} C_1(t) &= a_1 + b_1 q_1(t) + c_1 q_1(t)^2 \\ C_2(t) &= a_2 + b_2 q_2(t) \end{aligned} \quad (2)$$

The profits of the firms are:

$$\begin{aligned} \pi_1(t) &= q_1(t)(m - nQ(t)^2) - C_1(t) - \gamma q_1(t) \\ \pi_2(t) &= q_2(t)(m - nQ(t)^2) - C_2(t) - \gamma q_2(t) \end{aligned} \quad (3)$$

where $1 > \gamma > 0$ is the depreciation.

For the game between the firms and long-term repeated dynamic process in market, the assumptions are: the players (the firms) do not have a complete knowledge of the market and they follow a bounded rationality adjustment process based on the estimation of the marginal profit (Agiza & Elsadany, 2003; Onozaki et al., 2003). The firms decide to increase/decrease their output if they have a positive/negative marginal profit. The dynamic adjustment of this repeated Cournot duopoly game with bounded rational players is as follows:

$$\begin{aligned} \dot{q}_1(t) &= \alpha_1 q_1(t) \frac{\partial \pi_1(t)}{\partial q_1} \\ \dot{q}_2(t) &= \alpha_2 q_2(t) \frac{\partial \pi_2(t)}{\partial q_2} \end{aligned} \quad (4)$$

where $\alpha_i \in (0, 1]$ $i = 1, 2$ are output adjustment speed parameters.

Combining (1), (2), (3), (4), a dynamic duopoly game with bounded rationality, associated to two firms, has the following form:

$$\begin{aligned} \dot{q}_1(t) &= \alpha_1 q_1(t) (-3nq_1(t)^2 - nq_2(t)^2 - 4nq_1(t)q_2(t) - 2c_1q_1(t) + m - b_1 - \gamma) \\ \dot{q}_2(t) &= \alpha_2 q_2(t) (-3nq_2(t)^2 - nq_1(t)^2 - 4nq_1(t)q_2(t) + m - b_2 - \gamma) \end{aligned} \quad (5)$$

The parameters of (5) are nonnegative real numbers.

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In what follows, we consider the distributed delay structure, because the decisions made by the economic agents at time t depend on past observed variables. The functional relationships describing the dynamics of the model may not only depend on the current state, but also in a nontrivial manner on past states.

We propose that the distributed delay decisions are described by $k_i : [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2$, where k_i are piecewise continuous and:

$$\int_0^\infty k_i(s)ds = 1, \quad \int_0^\infty s k_i(s)ds < \infty. \quad (6)$$

The functions k_i , are called kernels. For $q_i(t)$, $i = 1, 2$, we define:

$$r_i(t) = \int_{-\infty}^0 k_i(t-s)q_i(s)ds, \quad i = 1, 2. \quad (7)$$

Therefore, the dynamic game model (5) with distributed delays is as follows:

$$\begin{aligned} \dot{q}_1(t) &= \alpha_1 q_1(t) (-3nq_1(t)^2 - n(w_2 q_2(t) + (1-w_2)r_2(t))^2 - \\ &\quad - 4nq_1(t)(w_2 q_2(t) + (1-w_2)r_2(t)) - 2c_1 q_1(t) + m - b_1 - \gamma) \\ \dot{q}_2(t) &= \alpha_2 q_2(t) (-3nq_2(t)^2 - n(w_1 q_1(t) + (1-w_1)r_1(t))^2 - \\ &\quad - 4n(w_1 q_1(t) + (1-w_1)r_1(t))q_2(t) + m - b_2 - \gamma) \end{aligned} \quad (8)$$

where $w_i \in [0, 1]$ $i = 1, 2$ are the weights of choosing the output of mutual firms.

In what follows we analyze system (8) for:

$$k_i(s) = d_i e^{-sd_i}, \quad d_i > 0, \quad k_i(s) = \delta(s - \tau_i), \quad \tau_i \geq 0 \quad i = 1, 2,$$

where δ is the Dirac distribution.

3. Stability analysis of systems (5) and (8)

The systems (5) and (8) have the same non-zero equilibrium point and it is obtained by solving the algebraic system:

$$\begin{aligned} 3nq_1^2 + nq_2^2 + 4nq_1q_2 + 2c_1q_1 - m + b_1 + \gamma &= 0. \\ 3nq_2^2 + nq_1^2 + 4nq_1q_2 - m + b_2 + \gamma &= 0. \end{aligned} \quad (9)$$

Let:

$$a_0 = \frac{m - b_2 - \gamma}{n}, \quad b_0 = \frac{2(b_2 - b_1)n - c_1^2}{4n^2} \quad (10)$$

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System (9) is given by:

$$\begin{aligned}(q_1 + 2q_2)^2 - q_2^2 &= a_0. \\ \left(q_1 + \frac{c_1}{2n}\right)^2 - q_2^2 &= b_0.\end{aligned}\tag{11}$$

Proposition 1

If

$$(c_1^2 + 4n^2(a_0 - b_0))^2 > 16n^2c_1^2a_0, \quad 0 < n < 1,\tag{12}$$

the equilibrium point is $E_0(q_{10}, q_{20})$ where q_{20} is the positive root of equation:

$$((4nx - c_1)^2 + 4n^2(a_0 - b_0))^2 - 16n^2(4x - c_1)^2(x^2 + a_0) = 0\tag{13}$$

and q_{10} is given by:

$$q_{10} = \frac{4n^2(a_0 - b_0) - (16n^2q_{20}^2 - c_1^2)}{4n(4nq_{20} - c_1)}.\tag{14}$$

Proof. From (11) we obtain (12). Eq. (12) is written as:

$$\begin{aligned}16n^2(1 - n^2)x^4 + 16n^2c_1(n - 8)x^3 + 16n^2(16a_0 - 5c_1^2 - 8n^2(a_0 - b_0))x^2 + \\ + 16nc_1(4n^2(a_0 - b_0) + c_1^2 - 8na_0)x + 16n^2c_1^2a_0 - (c_1^2 + 4n^2(a_0 - b_0))^2 = 0\end{aligned}\tag{15}$$

Because (12) holds, then eq. (15) has a positive root.

Consider the transformation $x(t) = q_1(t) - q_{10}$, $y(t) = q_2(t) - q_{20}$.In order to analyze the stability of the equilibrium point E_0 , we study the linearized system of (5) and (8) in E_0 , obtained by the above translation. For system (5) we obtain:

$$\begin{aligned}\dot{x}(t) &= a_{11}x(t) + a_{12}y(t) \\ \dot{y}(t) &= a_{21}x(t) + a_{22}y(t)\end{aligned}\tag{16}$$

where:

$$\begin{aligned}a_{11} &= -2\alpha_1q_{10}(3nq_{10} + 2nq_{20} + c_1) \\ a_{12} &= -2n\alpha_1q_{10}(2q_{10} + q_{20}) \\ a_{21} &= -2n\alpha_2q_{20}(2q_{20} + q_{10}) \\ a_{22} &= -2n\alpha_2q_{20}(2q_{10} + 3q_{20})\end{aligned}\tag{17}$$

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For system (8), we obtain:

$$\begin{aligned}\dot{x}(t) &= a_{11}x(t) + d_{12}y(t) + b_{12}y_2(t) \\ \dot{y}(t) &= d_{21}x(t) + a_{22}y(t) + b_{21}x_1(t)\end{aligned}\quad (18)$$

where

$$x_1(t) = \int_{-\infty}^0 k_1(t-s)x(s)ds, \quad y_2(t) = \int_{-\infty}^0 k_2(t-s)y(s)ds, \quad (19)$$

and

$$\begin{aligned}b_{12} &= (1-w_2)a_{12}, & b_{21} &= (1-w_1)a_{21}, \\ d_{12} &= w_2a_{12}, & d_{21} &= w_1a_{21}.\end{aligned}\quad (20)$$

The characteristic equation of (16) is:

$$\lambda^2 - (a_{11} - a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \quad (21)$$

The characteristic equation of (18) is:

$$\begin{aligned}\lambda^2 - (a_{11} - a_{22})\lambda + a_{11}a_{22} - w_1w_2a_{12}a_{21} - w_2(1-w_1)a_{12}a_{21}\int_{-\infty}^0 k_1(-s)e^{-\lambda s}ds - \\ - w_1(1-w_2)a_{12}a_{21}\int_{-\infty}^0 k_2(-s)e^{-\lambda s}ds - (1-w_1)(1-w_2)a_{12}a_{21}\left(\int_{-\infty}^0 k_1(-s)e^{-\lambda s}ds\right)\left(\int_{-\infty}^0 k_2(-s)e^{-\lambda s}ds\right) = 0.\end{aligned}\quad (22)$$

Proposition 2

The characteristic equation (21) has the roots with negative real part. System (5) is locally asymptotically stable in E_0 .

Proof. From (17), we have:

$$\begin{aligned}a_{11} + a_{22} &= -2(\alpha_1q_{10}(3nq_{10} + 2nq_{20} + c_1) + \alpha_2q_{10}n(3q_{20} + 2q_{10})) < 0 \\ a_{11}a_{22} - a_{12}a_{21} &= 4n\alpha_1\alpha_2q_{10}q_{20}(4n(q_{10} + q_{20})^2 + c_1(2q_{10} + 3q_{20})) > 0.\end{aligned}$$

Thus eq. (21) has the roots with negative real part.

Therefore, system (5) is locally asymptotically stable in E_0 .

Șirghi, N., Neamțu, M., Străin, P. C. (2015).*Analysis of a Dynamical Cournot Duopoly Game with Distributed Time Delay***3.1. Stability analysis of system (8) for $k_1(s) = d_1 e^{-d_1 s}$, $k_2(s) = \delta(s - \tau_2)$**

If $k_1(s) = d_1 e^{-d_1 s}$, $k_2(s) = \delta(s - \tau_2)$, with $d_1 > 0$, $\tau_2 \geq 0$, the characteristic equation (22) is given by:

$$\lambda^3 + \delta_3 \lambda^2 + \delta_1 \lambda + \delta_0 - (\delta_3 \lambda + \delta_4) e^{-\lambda \tau_2} = 0. \quad (23)$$

where

$$\begin{aligned} \delta_4 &= (1 - w_2) d_1 a_{12} a_{21} \\ \delta_3 &= w_1 (1 - w_2) a_{12} a_{21} \\ \delta_2 &= d_1 - a_{11} - a_{22} \\ \delta_1 &= a_{11} a_{22} - w_1 w_2 a_{12} a_{21} - d_1 (a_{11} + a_{22}) \\ \delta_0 &= d_1 (a_{11} a_{22} - w_2 a_{12} a_{21}). \end{aligned} \quad (24)$$

Proposition 3

If $\tau_2 = 0$ eq. (23) has the roots with negative real part for all $d_1 > 0$. If $\tau_2 \neq 0$ and $d_1 = d_{10} > 0$ and $\tau_2 = \tau_{20}$, eq. (23) has a pair of purely imaginary roots $\lambda = \pm i \omega_{20}$ where:

$$\tau_{20} = \frac{1}{\omega_{20}} \left(\arccos \left(\frac{\delta_4 (\delta_0 - \delta_2 \omega_{20}^2) + \delta_3 (\delta_1 \omega_{20} - \omega_{20}^3)}{\delta_3^2 + \delta_4^2} \right) + 2n\pi \right), \quad n = 0, 1, 2, \quad (25)$$

and ω_{20} is a positive real root of the equation:

$$\omega^6 + (\delta_2^2 - 2\delta_1) \omega^4 + (\delta_1^2 - 2\delta_0 \delta_2) \omega^2 + \delta_0^2 - \delta_3^2 - \delta_4^2 = 0 \quad (26)$$

and d_{10} satisfies the inequality:

$$(1 - (1 - w_2)^2 a_{12}^2 a_{21}^2) d_1^2 - 2(a_{11} + a_{22}) d_1 + (a_{11} + a_{22})^2 - w_1^2 (1 - w_2^2) a_{12}^2 a_{21}^2 < 0. \quad (27)$$

Moreover,

$$\text{sign} \left\{ \frac{d\lambda(\tau_2)}{d\tau_2} \bigg|_{\lambda = i\omega_{20}, \tau_2 = \tau_{20}} \right\} = \text{sign} \{ G_{20}(\omega_{20}) \} \quad (28)$$

where

$$\begin{aligned} G_{20}(\omega_{20}) &= \delta_3^2 \omega_{20} + \omega_{20} (3\delta_3 \omega_{20}^2 + 2\delta_2 \delta_4 - \delta_1 \delta_3) \cos(\omega_{20} \tau_{20}) + \\ &+ ((2\delta_2 \delta_3 - \delta_4) \omega_{20}^2 + \delta_1 \delta_4) \sin(\omega_{20} \tau_{20}). \end{aligned} \quad (29)$$

Sîrghi, N., Neamțu, M., Străin, P. C. (2015).*Analysis of a Dynamical Cournot Duopoly Game with Distributed Time Delay***Proof.** If $\tau_2 = 0$, eq. (23) is given by:

$$\lambda^3 + \delta_2 \lambda^2 + (\delta_1 - \delta_3) \lambda + \delta_0 - \delta_4 = 0. \quad (30)$$

Because $\delta_2 > 0$, $\delta_1 - \delta_3 > 0$, $\delta_0 - \delta_4 > 0$, for all $d_1 > 0$, according to the Routh-Hurwitz criterion, eq. (30) has the roots with negative real part for all $d_1 > 0$.

If $\tau_2 \neq 0$, let $\lambda = i\omega$ be the root of (23). We have:

$$\begin{aligned} \delta_4 \cos(\omega\tau_2) + \delta_3 \omega \sin(\omega\tau_2) &= \delta_0 - \delta_2 \omega^2 \\ \delta_4 \sin(\omega\tau_2) - \delta_3 \omega \cos(\omega\tau_2) &= \omega^3 - \delta_1 \omega \end{aligned} \quad (31)$$

From (31), we obtain (26). Consider $d_1 = d_{10}$, so that (27) holds, and let $\omega = \omega_{20}$ be the positive root of (26). From (31) we obtain (25). Let $\lambda(\tau_2)$ be the root of (23). Deriving (23) with respect to τ_2 we have:

$$\left(\frac{d\lambda(\tau_2)}{d\tau_2} \right)^{-1} = \frac{(3\lambda^2 + 2\delta_2 \lambda + \delta_1) e^{\lambda\tau_2}}{\lambda(\delta_3 \lambda + \delta_4)} - \frac{\tau_2}{\lambda}. \quad (32)$$

From (32) we get (28).

Proposition 4

Assuming the hypotheses from Proposition 3 hold. Then, we have:

1. If $\tau_2 \in [0, \tau_{20})$ then all the roots of (23) have negative real part. One can get that the equilibrium point E_0 of system (8) is locally asymptotically stable.
2. If $\tau_2 = \tau_{20}$ and $G_{20}(\omega_{20}) \neq 0$ system (8) undergoes Hopf bifurcation at E_0 .

3.2. Stability analysis of system (8) for $k_1(s) = \delta(s - \tau_1)$, $k_2(s) = d_2 e^{-d_2 s}$

If $k_1(s) = \delta(s - \tau_1)$, $k_2(s) = d_2 e^{-d_2 s}$, with $c_1 \geq 0$, $d_2 > 0$, the characteristic equation (22) is given by:

$$\lambda^3 + \varepsilon_2 \lambda^2 + \varepsilon_1 \lambda + \varepsilon_0 - (\varepsilon_4 \lambda + \varepsilon_3) e^{-\lambda\tau_1} = 0. \quad (33)$$

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where

$$\begin{aligned}
\varepsilon_4 &= (1 - w_1) a_{12} a_{21} d_2 \\
\varepsilon_3 &= w_2 (1 - w_1) a_{12} a_{21} \\
\varepsilon_2 &= d_2 - a_{11} - a_{22} \\
\varepsilon_1 &= a_{11} a_{22} - w_1 w_2 a_{12} a_{21} - d_2 (a_{11} + a_{22}) \\
\delta_0 &= d_2 (a_{11} a_{22} - w_1 a_{12} a_{21}).
\end{aligned} \tag{34}$$

Proposition 5

If $\tau_1 = 0$ then eq. (33) has the roots with negative real part for all $d_2 > 0$. If $\tau_1 \neq 0$ and $d_2 = d_{20} > 0$ and $\tau_1 = \tau_{10}$, eq. (33) has a pair of purely imaginary roots $\lambda = \pm i\omega_{10}$ where:

$$\tau_{10} = \frac{1}{\omega_{10}} \left(\arccos \left(\frac{\varepsilon_4 (\varepsilon_0 - \varepsilon_2 \omega_{10}^2) + \varepsilon_3 (\varepsilon_1 \omega_{10} - \omega_{10}^3)}{\varepsilon_3^2 + \varepsilon_4^2} \right) + 2n\pi \right), \quad n = 0, 1, 2, \tag{35}$$

and ω_{10} is the positive root of the equation:

$$\omega^6 + (\varepsilon_2^2 - 2\varepsilon_1)\omega^4 + (\varepsilon_1^2 - 2\varepsilon_0\varepsilon_2)\omega^2 + \varepsilon_0^2 - \varepsilon_3^2 - \varepsilon_4^2 = 0. \tag{36}$$

and d_{20} satisfies the inequality:

$$(1 - (1 - w_1)^2 a_{12}^2 a_{21}^2) d_2^2 - 2(a_{11} + a_{22}) d_1 + (a_{11} + a_{22})^2 - w_2^2 (1 - w_1^2) a_{12}^2 a_{21}^2 < 0. \tag{37}$$

Moreover,

$$\left. \text{sign} \left\{ \frac{d\lambda(\tau_1)}{d\tau_1} \right\} \right|_{\lambda=i\omega_{10}, \tau_1=\tau_{10}} = \text{sign}\{G_{10}(\omega_{10})\} \tag{38}$$

where

$$\begin{aligned}
G_{10}(\omega_{10}) &= \varepsilon_3^2 \omega_{10} + \omega_{10} (3\varepsilon_3 \omega_{10}^2 + 2\varepsilon_2 \varepsilon_4 - \varepsilon_1 \varepsilon_3) \cos(\omega_{10} \tau_{10}) + \\
&\quad + ((2\varepsilon_2 \varepsilon_3 - \varepsilon_4) \omega_{10}^2 + \varepsilon_1 \varepsilon_4) \sin(\omega_{10} \tau_{10}).
\end{aligned} \tag{39}$$

The proof is similar to the proof of Proposition 3. Therefore, we have:

Sîrghi, N., Neamțu, M., Străin, P. C. (2015).*Analysis of a Dynamical Cournot Duopoly Game with Distributed Time Delay***Proposition 6**

Assuming the hypotheses of *Proposition 5* hold. Then we have:

1. If $\tau_1 \in [0, \tau_{10})$ then all the roots of (33) have negative real part. One can get that the equilibrium point E_0 of system (8) is locally asymptotically stable.
2. If $\tau_1 = \tau_{10}$ and $G_{10}(\omega_{10}) \neq 0$ then system (8) undergoes Hopf bifurcation at E_0 .

4. Numerical Simulation

For the numerical simulation we consider the following parameters: $m=5.20$, $n=0.95$, $b_1=0.42$, $c_1=0.25$, $b_2=0.37$, $\gamma=0.15$, $\alpha_1=0.02$, $\alpha_2=0.03$, $w_1=0.5$, $w_2=0.5$. We use Maple and Matlab.

Using (14) and (15) we obtain the equilibrium point: $q_{10}=0.9697533802$, $q_{20}=0.754967037$.

If there is no distributed delay, the characteristic equation (21) has the roots: $\lambda_1=-0.7371360357$, $\lambda_2=-0.2797361134$. The equilibrium point is locally asymptotically stable. The total output, $Q(t)$, the profits $\pi_1(t)$, $\pi_2(t)$ and the costs $C_1(t)$, $C_2(t)$ has the following orbits :

Figure 1. The total output ($t, Q(t)$)

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Analysis of a Dynamical Cournot Duopoly Game with Distributed Time Delay **Figure 2.** The profit of the first firm ($t, \pi_1(t)$) **Figure 3.** The profit of the second firm ($t, \pi_2(t)$) **Figure 4.** The cost of the first firm ($t, C_1(t)$) **Figure 5.** The cost of the second firm ($t, C_2(t)$)

5. Conclusions

In this paper, the dynamics of a nonlinear duopoly game with bounded rationality and distributed time delay is analyzed. The game is described by a nonlinear differential system with two equations which contain distributed time delays, because the decisions made by the economic agents at time t depend on the past decisions.

For two types of kernels, weak and Dirac, the stability of equilibria and bifurcation are investigated. It is found that if there is no delay the equilibrium point is locally asymptotically stable. If $k_1(s) = d_1 e^{-d_1 s}$, $k_2(s) = \delta(s - \tau_2)$, with $d_1 = d_{10} > 0$, $\tau_2 > 0$, there exists a value τ_{20}

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for which a Hopf bifurcation occurs and the orbits of system (8) has a limit cycle. If $k_1(s) = \delta(s - \tau_1)$, $k_2(s) = d_2 e^{-d_2 s}$, with $\tau_1 > 0$, $d_2 = d_{20} > 0$, there exists a value τ_{10} for which a Hopf bifurcation occurs and the orbits of system (8) has a limit cycle.

A similar analysis will be carried out for $k_1(s) = d_1 e^{-d_1 s}$, $k_2(s) = d_2 e^{-d_2 s}$ and other types of kernels: strong, uniform, or Gauss. Because the parameters of the real models are subject to perturbations, then in our next paper we will consider the stochastic model with distributed time delay.

Acknowledgements:

We would like to offer special thanks to Dumitru Opris, professor at West University of Timisoara for the numerous discussions on the topic of the paper.

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