MOTION OF A POINT MASS IN A ROTATING DISC: A QUANTITATIVE ANALYSIS OF THE CORIOLIS AND CENTRIFUGAL FORCE

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ABSTRACT. In Newtonian mechanics, the non-inertial reference frames is a generalization of Newton's laws to any reference frames. While this approach simplifies some problems, there is often little physical insight into the motion, in particular into the effects of the Coriolis force. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths. In this paper, a mathematical solution based on differential equations in non-inertial reference is used to study different types of motion in rotating system. In addition, the experimental data measured on a turntable device, using a video camera in a mechanics laboratory was conducted to compare with mathematical solution in case of parabolically curved, solving non-linear least-squares problems, based on Levenberg-Marquardt's and Gauss-Newton algorithms.
KEY WORDS: Non-inertial frame, mathematical solution, Maple calculation, experimental data, algorithm.

1. Introduction

Observation of motion from a rotating frame of reference introduces many curious features [1]. This motion, observed in a rotating frame of reference, is generally explained by invoking inertial force. These forces do not exist, they are invented to preserve the Newtonian world view in reference systems, where it does not apply [2]. Centrifugal and Coriolis forces arise in rotating reference systems (i. e., that are accelerated) and are examples of inertial forces. For instance, a point mass is launched directly away from the center of the rotating turntable. The point mass will be rolling without forces acting on it; therefore its trajectory of motion is a straight line (path relative to Earth), if we neglect friction and a path curved to the right on the rotating turntable surface (from the view of the rotating reference systems) [3, 4] (Fig. 1). The

earth is considered an inertial frame of reference with little or no worry about effects due to its rotation. An observer standing next to the rotating turntable sees the point mass rolling straight and the rotating turntable rotate at angular speed underneath it. In the accelerated coordinate system, we explain the apparent curve to the right by using a fictitious force, called the Coriolis force, that causes the point mass to curve to the right [3, 5]. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allowing us to apply Newtons Laws in non-inertial frames of reference [6].

Fig. 1. To the stationary observer the point mass follows a straight path, so the observer in the rotating system follows a curved trajectory

The general approach in model fitting is to select a merit or objective function, that is a measure of the agreement between observed and modelled data, and which is directly or indirectly related to the adjustable parameters to be fitted. The goodness of fit parameters is obtained by minimizing (or maximizing, depending on how the function is defined) this objective function [7]. Many techniques have been developed to solve the non-linear minimization or maximization problem [8, 9]. Most of these methods are iterative algorithms (e. g. Levenberg-Marquardt's, Gauss-Newton, Gradient-Descent...) used to solve non-linear least squares problems.

The Levenberg-Marquardt's method is actually a combination of two minimization methods: the Gradient-Descent method and the Gauss-Newton method. In the gradient descent method, the sum of the squared errors is reduced by updating the parameters in the direction of the greatest reduction

of the least squares objective [10]. In the Gauss-Newton method, the sum of the squared errors is reduced by assuming the least squares function, which is locally quadratic, and finding the minimum of the quadratic. The Levenberg-Marquardt's method acts more like a Gradient-Descent method, when the parameters are far from their optimal value, and act more like the Gauss-Newton method, when the parameters are close to their optimal value [10].

A mathematical solution is used to compute the different types of motion of point mass with respect to a non-inertial reference frame. A good fit is obtained between the computed and experimental data by minimizing the objective function, based on the non-linear model fitting of the motion, observed of a point mass in a non-inertial system parabolically curved, with the Levenberg-Marquardt's and Gauss-Newton algorithms. In order to compare the two methods, we will give an explanation of each method's steps.

Moreover, rotating reference frames play an important role, because

Fig. 2. The wind directions in the Northern Hemisphere are deflected to the right, while those in the Southern Hemisphere are deflected to the left

they simplify reasoning and calculations, when looking at rotating systems. In addition, the inertial force study is the key to the explanation of many phenomena in connection with the winds (e. g. the Coriolis effect is that wind directions in the Northern Hemisphere are deflected to the right, while those in the Southern Hemisphere are deflected to the left) (see Fig. 2) and currents of the ocean. The cause of the Coriolis effect is the earth's rotation. Those phenomena that is crucial to any analysis of weather systems and the large-scale climatology of the earth. The use of a rotating frame can also simplify the study of certain mechanics problems, that involve rotating bodies in the laboratory.

2. Mathematical solution

It is known, that Newton's second law applies specifically in inertial frames, in fact, in exactly those frames, for which the first law holds, therefore, definitely not in rotating axes. This means, that it is usually best to use frames related by Galilean transformations. Sometimes, it is far more convenient to do the calculations in a non-inertial frame. In such cases, the thing to do is to apply the second law in an inertial frame and then transform to the non-inertial frame.

The basic equation in Newtonian mechanics: $F = m.a^{in}$, a^{in} is the acceleration, relative to the inertial frame. Alternatively, the forces applied on the point mass as seen by an observation co-moving with non-inertial system, is cited below [11, 12]:

$$(1) \quad m.a = F - \underbrace{m.\dot{\omega}}_{\text{Euler force}} \wedge r - \underbrace{2.m.\omega \wedge V}_{\text{Coriolis force}} - \underbrace{m.\omega \wedge (\omega \wedge r)}_{\text{Centrifugal force}},$$

with:

$$\vec{r} = x_1 \vec{i}_1 + y_1 \vec{j}_1,$$

where: ω is the angular velocity (rad/s).

The velocity in the rotating reference frame in function of the position of the point mass, is cited below:

$$(2) \quad \vec{V} = \dot{x}_1 \vec{i}_1 + \dot{y}_1 \vec{j}_1.$$

The Coriolis force is proportional to the rotation rate and the centrifugal force is proportional to its square. The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating reference systems [1, 3, 13]. The centrifugal force acts outwards in the

radial direction and is proportional to the distance of the body from the axis of the rotating reference systems.

Equation (1) is a mathematical representation of what is meant by the statement, that Newton's Second Law does not apply in a non-inertial reference frame. It is not, that the physics dealt with Newtonian mechanics cannot be analyzed in a non-inertial frame, but that the form of the equations of motion is different [6].

The final expression for differential equations of motion in the rotating reference frame can be written and decomposed into x_1, y_1 as follows:

$$(3) \quad m\ddot{x}_1 - 2m\omega\dot{y}_1 - \omega^2 mx_1 = 0.$$

$$(4) \quad m\ddot{y}_1 + 2m\omega\dot{x}_1 - \omega^2 my_1 = 0.$$

The initial coordinates are:

$$\begin{aligned} x_1(0) &= x_{10}, & y_1(0) &= y_{10} \\ \dot{x}_1(0) &= \dot{x}_{10}, & \dot{y}_1(0) &= \dot{y}_{10}. \end{aligned}$$

The x_1 and y_1 coordinates specify the position of the point mass, as seen by an observer on rotating frame. Differential equations of motion, see below, can be described by system of coupled equations. The following is an analytic solution of the point mass in the rotating frame system, using a change of variable (complex number).

$$(5) \quad Z = x_1 + I y_1.$$

Multiply the equation (4) by I and add the equation (3).

$$(6) \quad \ddot{x}_1 + I\ddot{y}_1 + 2\omega I(\dot{x}_1 + I\dot{y}_1) - \omega^2(x_1 + I y_1) = 0.$$

The initial coordinates are:

$$\begin{aligned} x(0) + I y(0) &= (x_{10} + I y_{10}) \\ \dot{x}_1(0) + I \dot{y}_1(0) &= (\dot{x}_{10} + I \dot{y}_{10}). \end{aligned}$$

The auxiliary variable Z occurs naturally in the unique differential equation, as follows:

$$(7) \quad \ddot{Z} + 2\omega I \dot{Z} - \omega^2 Z = 0,$$

with:

$$\begin{aligned} Z(0) &= Z_0 \\ \dot{Z}(0) &= \dot{Z}_0. \end{aligned}$$

The solution for Z is well-known, the final expression in the form of:

$$(8) \quad Z(t) = Z_0.(1 + \omega.I.t). \exp(-I.\omega.t) + V_0 t \exp(-I.\omega.t).$$

The parametric equations for the rotating coordinate system (in term of $x_1(t)$ and $y_1(t)$), are as follows:

$$(9) \quad x_1(t) = [x_{10} - y_{10}.\omega.t + V_{x0}t] \cos(\omega.t) + [y_{10} + x_{10}.\omega.t + V_{y0}t] . \sin(\omega.t).$$

$$(10) \quad y_1(t) = [y_{10} + x_{10}.\omega.t + V_{y0}t] \cos(\omega.t) + [-x_{10} + y_{10}.\omega.t - V_{x0}t] . \sin(\omega.t).$$

3. Levenberg–Marquardt and Gauss–Newton algorithms

The non-linear least squares' problem is closely related to the problem of solving a non-linear system of equations, and is a special case of the general optimization problem in R^n [14]. This implies that the function to be minimized is of the following special form [14, 15]:

$$(11) \quad F(x) = \frac{1}{2} \sum_{m=1}^M r_m^2(\theta),$$

where $r : R^N \rightarrow R^M$ is an M -dimensional, non-linear vector function of N parameters, θ , where $M \geq N$.

One important area, in which non-linear least squares problems arise, is in data fitting [16]. The Gauss–Newton method for the problem above: A start with an initial guess $x^{(0)}$ for the minimum, the method proceeds by the iterations [16, 17].

$$(12) \quad \theta^{(i+1)} = \theta^{(i)} - [J_r^T J_r]^{-1} J_r^T r(\theta^{(i)}),$$

where, if r and θ are column vectors, the entries of the Jacobian matrix, defined as:

$$(13) \quad (J_r)_{m,m'} = \frac{\partial r_m(\theta^{(i)})}{\partial \theta_{m'}}.$$

If, $M = N$, the iteration simplifies to:

$$(14) \quad \theta^{(i+1)} = \theta^{(i)} - [J_r]^{-1} r(\theta^{(i)}),$$

which is a direct generalization of Newton's method in one-dimension. In data fitting, where the goal is to find the parameters θ , such that a given model function $y = f(x, \theta)$ good fits some data points (x_m, y_m) , the functions r_m are the residuals:

$$(15) \quad r_m(\theta) = y_m - f(x_m, \theta).$$

Then, the Gauss-Newton method can be expressed in terms of the Jacobian J_f of the function f , as [10]:

$$(16) \quad \theta^{(i+1)} = \theta^{(i)} + [J_f^T J_f]^{-1} J_f^T r(\theta^{(i)}).$$

Also, the assumption $M \geq N$ in the algorithm statement is necessary, as otherwise the matrix $J_r^T J_r$ is not invertible and the normal equations cannot be solved (at least uniquely). The Gauss-Newton algorithm can be derived by linearly approximating vector of functions r_m . The Levenberg-Marquardt algorithm adaptively varies the parameter updates between the gradient descent update and the Gauss-Newton update. The step for the Levenberg algorithm, denoted h_l , is defined as [18, 19]:

$$(17) \quad [H + \lambda I] . h_l = J^T (y - f(\theta)),$$

$H = J^T J$ is the Hessian matrix, λ is the damping factor (adjusted at each iteration) and I is the identity matrix, where small values of the algorithmic parameter λ result in a Gauss-Newton update and large values of λ result in a gradient descent update. The parameter λ is initialized to be large. If iteration happens to result in a worse approximation, λ is increased. As the solution approaches the minimum, λ is decreased, the Levenberg-Marquardt's method approaches the Gauss-Newton method, and the solution typically converges rapidly to the local minimum [10, 18, 20, 21]. Therefore, Marquardt's contribution is to replace the identity matrix in (Eq. 17) with the diagonal of the Hessian resulting in the Levenberg-Marquardt update rule [18, 19]:

$$(18) \quad [H + \lambda \text{diag}(H)] . h_{lm} = J^T (y - f(\theta)).$$

Since the Hessian is proportional to the curvature of f , (Eq. 18) implies a large step in the direction with low curvature (i.e., an almost flat terrain) and a small step in the direction with high curvature (i.e., a steep incline).

4. Experimental set-up

The experimental set-up used in this study: two digital camera (Sony Cybers, 10 megapixels) in a video mode, first is fixed to the floor of the laboratory, shows the view of the inertial reference frame; a further digital camera, fixed on the turntable, shows the view of the rotating reference frame (assuming negligible friction). The rest of the experimental set-up is known: A point mass is rolling down an inclined plane; a motor is rotating the disc with angular velocity (see Fig. 3). In addition, in a computer, the movement of the point mass was followed frame by frame for a time interval 26 ms using Latis-Pro program. In each experiment, the image plane was carefully arranged to be parallel to the plane of motion in order to minimize systematic errors from the projection. The results of the inertial reference frame are not included in this paper.

Fig. 3. Experimental device (i.e. rotating disc used in this study)

5. Results and discussion

Computed scenarios of the different spatial trajectories types of motion in rotating reference systems $Y_1(X_1)$, (as described below) show the inertial forces effect (assuming negligible friction). The computed of spatial trajectories is in function of the effect of initial velocity of launch $V_0(V_{x0}, V_{y0})$, the angular velocity (ω) and initial position vector of launch (X_{10}, Y_{10}), see Figs 4 and 5. All motions will be reduced outward, w the right, the degree of deflection determined by the motion relative to the rotating turntable. The difference between the deflections constitutes the Coriolis effect.

- Initial velocity effect V_0 :

We fix the angular velocity $\omega = 0.5 \text{ rad/s}$, $(X_{10}, Y_{10}) = (0.0)$ and $V_0 = (0, 0.1)$; $(0, 0.4)$; $(0, 0.6) \text{ m/s}$ see Figure 4(left).

- Angular velocity effect ω :

$V_0 = 0.6 \text{ m/s}$ fixed; $\omega = 0.2 \text{ rad/s}$; 0.6 rad/s ; 1.5 rad/s , Figure 4 (right).

Fig. 4. Spatial trajectory simulation, initial velocity and angular velocity effect (V_0), (ω) in a rotating frame

- Initial position (X_{10}, Y_{10}) and direction of launch (V_{x0}, V_{y0}) effect:

We fix the angular velocity $\omega = 0.9 \text{ rad/s}$, $(X_{10}, Y_{10}) = (0, -1)$ and $(V_{x0}, V_{y0}) = (-0.9, 0.57)$; $(-0.57, 0.9)$; $(-0.9, 0.9) \text{ m/s}$ Figure 5 (left). Also, $(V_{x0}, V_{y0}) = (0, 0.9)$; $(0, -0.6)$; $(0, -1.15) \text{ m/s}$ Figure 5 (right).

Fig. 5. Spatial trajectory simulation, initial position and direction of launch effect (X_{10}, Y_{10}) , (V_{x0}, V_{y0}) in a rotating frame

5.1. Experimental observation

We launch the point mass from the origin of our coordinate system $X_{10} = 0; Y_{10} = 0$ with initial conditions of launch $V_{x0} = 0, V_{y0} = V$ and an angular velocity (ω). The analytical solution for differential equations of motion in rotating experimental systems is:

$$(19) \quad \begin{aligned} x_1(t) &= (v.t) \cdot \sin(\omega.t) \\ y_1(t) &= (v.t) \cdot \cos(\omega.t). \end{aligned}$$

- Algorithms application

We fix v and ω varied. Figure 6 shows result of model fitting, using two algorithms (Levenberg-Marquardt's and Gauss-Newton algorithms). Additionally, Table 1 shows comparison of algorithms with experimental data.

Fig. 6. Different angular velocity situations $\omega_1 = 2.15 \text{ rad.s}^{-1}$ (a); $\omega_2 = 1.44 \text{ rad.s}^{-1}$ (b); $\omega_3 = 1.11 \text{ rad.s}^{-1}$ (c) for a condition of a point mass fixed

We fix w and v varied. Figure 7 shows results of model fitting using Levenberg-Marquardt's algorithm. Also, Table 2 shows comparison of algorithms with experimental data.

Table 1. Comparison of algorithms/experimental data with condition of launch a point mass is fixed

Parameters	Experimental data		Levenberg–Marquardt's	Gauss–Newton
Condition of launch a point mass (v) m.s ⁻¹	0.83		0.79	0.80
Angular velocity (ω) rad.s ⁻¹	ω_1	2.15	2.09	2.10
	ω_2	1.44	1.45	1.48
	ω_3	1.11	1.06	1.07

Fig. 7. Different conditions of launch a point mass $v_1 = 0.5$ m.s⁻¹ (a); $v_2 = 1.05$ m.s⁻¹ (b); $v_3 = 1.5$ m.s⁻¹ (c) for a angular velocity fixed

Summarizing, all motions will be reduced outward, w the right, the degree of deflection determined by the motion relative to the rotating turntable. The difference between the deflections constitutes the Coriolis effect.

Table 2. Comparison of algorithms/experimental data with angular velocity is fixed

Parameters	Experimental data		Levenberg–Marquardt’s	Gauss–Newton
Angular velocity (ω) rad.s^{-1}	1.50		1.50	1.52
Condition of launch a point mass (v) m.s^{-1}	v_1	0.50	0.50	0.48
	v_2	1.05	1.08	1.03
	v_3	1.50	1.51	1.48

5.2. Error analysis

The robustness of Levenberg-Marquardt’s and Gauss-Newton algorithms is tested using statistical indicators (standard-deviation (SD) and root mean square error (RMSE)), see Table 3 for each parameter value.

Table 3. Error statistical indicators of the algorithms

Methods	Parameters	SD (%)	RMSE
Levenberg–Marquardt’s	v	3.9	0.0460
	ω_1	0.8	0.0078
	ω_2	0.9	0.0360
	ω_3	0.6	0.0057
Gauss–Newton	v	3.7	0.0480
	ω_1	2.9	0.0375
	ω_2	0.7	0.0042
	ω_3	0.5	0.0058
Levenberg–Marquardt’s	ω	0	0.0000
	v_1	0.7	0.0070
	v_2	1.1	0.0120
	v_3	0.1	0.0022
Gauss–Newton	ω	0.2	0.0036
	v_1	2.2	0.0290
	v_2	2.2	0.0219
	v_3	1.6	0.0150

Levenberg–Marquardt’s algorithm is more robust than Gauss-Newton algorithm that means it finds a solution even if it starts very far from the final minimum. At a large distance from the function minimum, the steepest descent method is utilized to provide steady and convergent progress toward the solution. As the solution approaches the minimum, damping parameter λ is adaptively decreased, the Levenberg-Marquardt’s algorithm approaches the Gauss-Newton algorithm, and the solution typically converges rapidly to the minimum [10, 21, 22].

6. Conclusion

The aim of this paper is to provide simple explanations of different motion observed of a point mass in a non-inertial reference frame. These new forces enter the equation of motion in the rotating frame. In addition, the physical origin of the inertial forces is generally only truly understood by viewing the motion in the inertial frame and then, relating that to the non-inertial view. The results that show, all motions will be reduced outward, with the right, the degree of deflection (Coriolis effect), determined by the motion relative to the rotating turntable. Additionally, the algorithms accuracy (Levenberg-Marquardt's and Gauss-Newton) for parameter estimation (condition of launch a point mass and angular velocity) of the motion observed of a point mass in a non-inertial reference frame in Newtonian mechanics, using statistical test, has been evaluated, shows Levenberg-Marquardt's algorithm is more robust than Gauss-Newton algorithm at small error values.

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