ELECTRIC AND HEAT CONDUCTIONS ACROSS A CRACK IN A THERMOELECTRIC MATERIAL*

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ABSTRACT. The crack problem in a thermoelectric material is studied in this paper. Two kinds of crack surface conditions are discussed. The closed form solutions are derived, based on the complex variable method. The field intensity factors as well as the conversion efficiency are discussed in detail. The results show that the electric current density and thermal flux density exhibit traditional square-root singularity at the crack tip in case 1, while the electric current density has no singularity at the crack tip in case 2. It is proved due to the electric current flow and thermal flux separation, that the thermoelectric conversion efficiency can be higher than the maximum conversion efficiency of one-dimensional thermoelectric under the same temperature condition. In the numerical example, the conversion efficiency is increased by 29.6% as compared to the maximum conversion efficiency of one-dimensional thermoelectric.

KEY WORDS: Thermoelectric material, crack, field intensity factor, conversion efficiency.

1. Introduction

Thermoelectric materials are widely pursued for technological applications due to intrinsic coupling behaviour, including waste heat recovery, solid state thermal management, solar energy harvesting and carbon reduction (Disalvo [1]; Yang and Caillat [2]; Narducci [3]; Tritt and Subramanian [4]; Kraemer *et al* [5]; Bell [6]). Much effort has been devoted to developing hybrid thermoelectric materials with high conversion efficiency (Heremans *et al* [7]; Snyder

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and Toberer [8]; Gothard *et al* [9]; Vashaee and Shakouri [10]). Though continuum analysis of thermoelectrics has only recently attracted some attentions (Webman *et al* [11]; Mahan [12]), despite significant progress has been made in thermoelectric material development.

Most of the investigations on thermoelectric crack problems were experimental in nature. For example, Isoda *et al* [13] tested the thermal shock resistance and thermo electric properties of boron doped iron disilicides. The grain-boundary crack, appeared in the initial period of thermal duration test of CoSb_3 thermoelectric material, was observed by Zhao *et al* [14]. Schmidt *et al* [15] studied the room-temperature mechanical properties and slow crack growth behaviour of Mg_2Si , and Eilertsen *et al* [16] explored the fracture toughness of $\text{Co}_4\text{Sb}_{12}$ and $\text{In}_{0.1}\text{Co}_4\text{Sb}_{12}$ thermoelectric skutterudites by three methods. These studies highlighted the importance of thermoelectric reliability, yet, the analysis of the effectiveness of cracks on the conversion efficiency, is rare.

The main effort on improving the conversion efficiency is improving the thermoelectric figure of merit. For example, it was reported that the $\text{Ag}_{1-x}\text{Pb}_{18}\text{SbTe}_{20}$ showed a very high ZT value of 2.2 at 800 K [17], and with grain size less than 100 nm, the ZT value of $\text{Bi}_x\text{Sb}_{2-x}\text{Te}_3$ nano-composite is increased from 1.2 to 1.4 [18], at about 370 K. A $\text{Bi}_2\text{Te}_3/\text{Sb}_2\text{Te}_3$ bulk nanocomposites with laminated nanostructures was produced, by using a simple route involving hydrothermal synthesis and hot pressing, which has a maximum ZT value of 1.4 at 450 K [19]. Whether there is another way to improve the conversion efficiency of the thermoelectric material? This fact motivated our current work.

In the present work, the crack problem in a thermoelectric material is studied. The general solution for two-dimensional problem is derived, then the analytical solutions for a crack embedded in an infinite thermoelectric matrix are presented and finally, discussions are made about the effects of crack on the conversion efficiency of the thermoelectric material.

2. Governing equations

Since the material properties do not change in the unit cell, the coupled transports of heat and electrons in a thermoelectric material, with the respective transport equations, are given by (Harman and Honig [20]):

$$(1) \quad -\mathbf{J}_N = \left(\frac{T\sigma}{e^2} \right) \frac{1}{T} \nabla \mu - \left(\frac{T^2 \sigma \varepsilon}{e} \right) \nabla \frac{1}{T},$$

$$(2) \quad \mathbf{J}_Q = - \left(\frac{T^2 \sigma \varepsilon}{e} \right) \frac{1}{T} \nabla \mu + (T^3 \sigma \varepsilon^2 + T^2 \kappa) \nabla \frac{1}{T},$$

wherein the electron flux $\mathbf{J}_N$ is coupled with the heat flux $\mathbf{J}_Q$ through the seebeck coefficient, and T , σ , κ , e and μ are the temperature, electric conductivity, thermal conductivity, charge and electrochemical potential of the electron, respectively. Note, that:

$$(3) \quad \mu = e\varphi,$$

where φ is the electric potential. Since the electric current density can be derived from electron flux:

$$(4) \quad \mathbf{J} = e\mathbf{J}_N.$$

The transport equations governing electric current density and heat flux can be derived as follows:

$$(5) \quad -\mathbf{J} = \sigma \nabla \varphi + \sigma \varepsilon \nabla T,$$

$$(6) \quad \mathbf{J}_Q = T \varepsilon \mathbf{J} - \kappa \nabla T.$$

Since energy is transported by both electrons and heat, the energy flux $\mathbf{J}_E$ can be expressed as:

$$(7) \quad \mathbf{J}_E = \mathbf{J}_Q + \varphi \mathbf{J}.$$

In the following analysis, we limit ourselves to systems wherein both charges and energy are conserved, such that both current density and energy flux are divergence-free (Yang, [21]):

$$(8) \quad \nabla \cdot \mathbf{J} = 0,$$

$$(9) \quad \nabla \cdot (\mathbf{J}_Q + \varphi \mathbf{J}) = 0.$$

Notice, that the heat flux is not divergence free due to Joule heating.

3. General solutions of two-dimensional problems

For the case of two-dimensional problems, the fields of temperature, electric potential, heat flow and electric current can be expressed by two analytic complex functions $f(z)$ and $f_2(z)$. To this end, we substitute Eq. (5) into Eq. (8) to obtain:

$$(10) \quad \nabla^2(\varphi + \varepsilon T) = 0.$$

According to Muskhelishvili [22], let $\varphi + \varepsilon T$ be the real part of the analytic complex function $f(z)$, we have:

$$(11) \quad \operatorname{Re}[f(z)] = \varphi + \varepsilon T,$$

where $z = x + iy$.

From Eq. (5), the current density can be represented as:

$$(12) \quad \begin{aligned} J_x &= -\frac{1}{2}\sigma \frac{\partial(f(z) + \overline{f(z)})}{\partial x} = -\frac{\sigma}{2}(f'(z) + \overline{f'(z)}), \\ J_y &= -\frac{1}{2}\sigma \frac{\partial(f(z) + \overline{f(z)})}{\partial y} = -\frac{\sigma i}{2}(f'(z) - \overline{f'(z)}), \end{aligned}$$

or

$$(13) \quad J_x - iJ_y = -\sigma f'(z), \quad J_r - iJ_\theta = -\sigma e^{i\theta} f'(z).$$

Substituting Eqs (1-2) into Eq. (9), we have:

$$(14) \quad \nabla \cdot (T\varepsilon \mathbf{J} - \kappa \nabla T) + \mathbf{J} \nabla \varphi + \varphi \nabla \cdot \mathbf{J} = 0.$$

Noting, that $\nabla \cdot \mathbf{J} = 0$, Eq. (14) can be rewritten as:

$$(15) \quad \mathbf{J}^2 = -\sigma \kappa \nabla^2 T = \frac{\sigma \kappa \nabla^2 \varphi}{\varepsilon}.$$

Combining it with Eqs (5) and (11), we obtain:

$$(16) \quad \frac{\mathbf{J}^2}{\sigma^2} = [\nabla \operatorname{Re}[f(z)]]^2 = -\frac{\kappa \nabla^2 T}{\sigma} = \frac{\kappa \nabla^2 \varphi}{\sigma \varepsilon}.$$

Namely:

$$(17) \quad \begin{aligned} \nabla^2 T &= -\frac{\sigma}{\kappa} [\nabla \operatorname{Re}[f(z)]]^2 = -\frac{\sigma}{\kappa} \frac{\partial f(z)}{\partial z} \frac{\partial \overline{f(z)}}{\partial \bar{z}} \\ \nabla^2 \varphi &= \frac{\sigma \varepsilon}{\kappa} \frac{\partial f(z)}{\partial z} \frac{\partial \overline{f(z)}}{\partial \bar{z}}. \end{aligned}$$

Integrating Eq. (17), we have:

$$(18) \quad T = -\frac{\sigma}{4\kappa} f(z) \overline{f(z)} + f_2(z) + \overline{f_2(z)} + C_2,$$

$$(19) \quad \varphi = \frac{\sigma \varepsilon}{4\kappa} f(z) \overline{f(z)} + \frac{1}{2}[f(z) + \overline{f(z)}] - \varepsilon[f_2(z) + \overline{f_2(z)}] - \varepsilon C_2,$$

where C_2 is real and denotes an uniform temperature field.

The thermal flux can then be derived from Eq. (6), as:

$$(20) \quad J_{Qx} - iJ_{Qy} = \frac{\sigma^2 \varepsilon}{4\kappa} f(z) \overline{f(z)} f'(z) - \sigma \varepsilon C_2 f'(z) \\ + \frac{\sigma f'(z) \overline{f(z)}}{2} - 2\sigma \varepsilon f'(z) \operatorname{Re}[f_2(z)] - 2\kappa f_2'(z),$$

where $f_2(z)$ is any analytic complex function.

As such, if the two analytic complex functions $f(z)$ and $f_2(z)$ can be determined, the fields of temperature, electric potential, heat flow and electric current can be derived.

4. Solutions of crack problem

Now, consider an open line segment on x axis $S = (-a, a)$ embedded in an infinite thermoelectric material. The material is subjected to in-plane electric current J_x^∞, J_y^∞ and heat flow $J_{Qx}^\infty, J_{Qy}^\infty$ at infinity, as shown in Fig. 1. (a). It should be pointed out that the electric current J_x^∞, J_y^∞ also produce heat flow $J_{Qx}^{\infty'}, J_{Qy}^{\infty'}$ at infinity. Here, the heat flows J_{Qx}^∞ and J_{Qy}^∞ stand for the thermal loads, which are independent of the electric current. Therefore, the total heat flows at infinity are $J_{Qx}^{\infty'} + J_{Qx}^\infty$ and $J_{Qy}^{\infty'} + J_{Qy}^\infty$. All the material properties and field variables are assumed to be only dependent on coordinates x and y .

4.1. Crack surface conditions

Here we will discuss two crack surface conditions:

Case 1, thermally and electrically impermeable crack. Similar impermeable crack boundary conditions has been used in elastic material (Irwin [23]) and piezoelectric materials (Deeg [24] and Park [25]). In thermoelectric materials, impermeable boundary condition means heat flow and electric current can not penetrate the crack, normal component of the heat flow and electric current at the crack surface are zero. If the distance between the two surfaces of the crack is relative large, and the crack is filled with poor thermal conductivity and electric conductivity media, this boundary condition will have a good approximation.

$$(21) \quad J_y = 0, \quad \text{on } S$$

$$(22) \quad J_{Qy} = 0 \quad \text{on } S.$$

Case 2, electrically permeable while thermally impermeable crack. Like the electrically permeable crack used in piezoelectric materials (Jackson [26]), this boundary condition means solid thermal conduction can not penetrate the crack, while the electric current can penetrate the crack, normal component of the solid thermal conduction at the crack surface are zero, while the electric current function $\varphi + \varepsilon T$ and normal component of the electric current at the crack surface are continuous. If the distance between the two surfaces of the crack is small, electrons can penetrate cracks through the tunnel effect, and this boundary condition is more appropriate. In this case, it is necessary that the electric current density J_y , and electric current function $\varphi + \varepsilon T$ are continuous across the crack surface. On the other hand, the first term of the heat flux $T\varepsilon J$ can not be prevented as the electric current flow through cracks accessible therefore, the thermally impermeable condition means temperature gradient disappears at the crack surface.

$$(23) \quad J_y^+ = J_y^- \quad \text{on } S$$

$$(24) \quad (\varphi + \varepsilon T)^+ = (\varphi + \varepsilon T)^- \quad \text{on } S$$

$$(25) \quad \frac{\partial T}{\partial y} = 0 \quad \text{on } S.$$

4.2. Infinity conditions

Considering the electric current condition and the current independent thermal condition at infinity yields

$$(26) \quad \lim_{z \rightarrow \infty} -\sigma f'(z) = J_x^\infty - iJ_y^\infty,$$

$$(27) \quad \lim_{z \rightarrow \infty} -2\kappa f_2'(z) = J_{Qx}^\infty - iJ_{Qy}^\infty.$$

4.3. Solution for Case 1 – thermally and electrically impermeable crack

According to Eq. (13) and the electric current condition at infinity, analytic function $f(z)$ could be expressed, as:

$$(28) \quad f(z) = Az + f_0(z).$$

The following transform function Muskhelishvili [22]:

$$(29) \quad z = \frac{a}{2}\left(w + \frac{1}{w}\right), \quad w = \omega_1[z] = \frac{z + z\sqrt{1 - \frac{a^2}{z^2}}}{a},$$

maps the crack (with length $2a$) in the z -plane into a unit circle without points $(-1,0)$ and $(1,0)$ in the w -plane (Fig. 1.(a), (b)).

Therefore Eq. (13) can be rewrite as:

$$(30) \quad J_x - iJ_y = -\sigma f'(w) \frac{dw}{dz},$$

and analytic functions $f(w)$ and $f'(w)$ could be expressed, as:

$$(31) \quad f(w) = A_1 w + A_0 + \frac{A_{-1}}{w},$$

Fig. 1. (a) a crack embedded in a infinite thermoelectric matrix (b) the w -plane after conformal mapping

$$(32) \quad f'(w) = A_1 - \frac{A_{-1}}{w^2}.$$

Substituting Eq. (32) and Eq. (29) into boundary condition Eq.(21), we have:

$$(33) \quad f'(w)w^2 + \overline{f'(w)} = 0.$$

Solving Eq. (33) by using Eq. (32), yields:

$$(34) \quad A_1 = \overline{A}_{-1}.$$

Substituting Eqs (33–34) into boundary condition Eq. (22), we have:

$$(35) \quad \kappa[f'_2(w)\frac{2w^2}{a} + \overline{f'_2(w)}\frac{2}{a}] = 0.$$

Letting:

$$(36) \quad f_2(w) = B_2w^2 + B_1w + \frac{B_{-1}}{w} + \frac{B_{-2}}{w^2}.$$

Combining it with Eq. (35), and using series expansions, with which a comparison of the coefficients of the same power terms, yields:

$$(37) \quad B_2 = \overline{B}_{-2} = 0,$$

$$(38) \quad B_{-1} = \overline{B}_1.$$

Considering the infinity condition Eqs (26–27), yields:

$$(39) \quad A_1 = -\frac{a}{2\sigma}(J_x^\infty - iJ_y^\infty),$$

$$(40) \quad B_1 = -\frac{a}{4\kappa}(J_{Qx}^\infty - iJ_{Qy}^\infty).$$

All the coefficients $A_1, \overline{A}_{-1}, B_1, B_{-1}, B_2, B_{-2}$ can be determined so far.

Finally, the temperature field and electric potential field can be expressed as:

$$(41) \quad \begin{aligned} T &= -\frac{\sigma}{4\kappa}f(z)\overline{f(z)} + f_2(z) + \overline{f_2(z)} + N, \\ &= -\frac{\sigma}{4\kappa}(A_1w + \frac{A_{-1}}{w})(\overline{A}_1\overline{w} + \frac{\overline{A}_{-1}}{\overline{w}}) + B_1w + \frac{B_{-1}}{w} + \overline{B}_1\overline{w} + \frac{\overline{B}_{-1}}{\overline{w}} + N, \end{aligned}$$

$$(42) \quad \varphi = \frac{1}{2}[f(z) + \overline{f(z)}] - \varepsilon T = \frac{1}{2}(A_1 w + \frac{A_{-1}}{w} + \overline{A_1} \overline{w} + \frac{\overline{A_{-1}}}{\overline{w}}) - \varepsilon T.$$

4.4. Solution for Case 2– electrically permeable, while thermally impermeable crack

Using the similar method, for electrically permeable while thermally impermeable crack, we have

$$(43) \quad A_1 = -\frac{a}{2\sigma}(J_x^\infty - iJ_y^\infty),$$

$$(44) \quad B_1 = -\frac{a}{4\kappa}(J_{Qx}^\infty - iJ_{Qy}^\infty),$$

$$(45) \quad A_{-1} = -\frac{a}{2\sigma}(J_x^\infty - iJ_y^\infty),$$

$$(46) \quad B_{-1} = -\frac{a}{4\kappa}(J_{Qx}^\infty + iJ_{Qy}^\infty),$$

$$B = \frac{aiJ_y^\infty}{\sigma}.$$

5. Field Intensity Factors

The behaviour of materials near the crack tip has significant effect on the fracture process. Griffith [27] found that the stress field at a sharp crack tip has a $r^{-1/2}$ singularity, which leads to the establishment of the stress intensity factor. The elastic-electric field at crack tip in piezoelectric material also exhibit $r^{-1/2}$ singularity (Suo et. al. [28] and Zhang et. al. [29]), which greatly simplifies the elastic-electric field analysis in cracked piezoelectric material. This motivates us to discuss the thermal-electric field at the crack tip in thermoelectric materials.

For the thermally and electrically impermeable crack, the electric current density at the right tip of the crack is:

$$(47) \quad J_y = -\frac{\sigma i}{2} [f'(z) - \overline{f'(z)}] = J_y^\infty + \frac{aJ_y^\infty}{\sqrt{2a(z-a)}}.$$

It can be seen, that the electric current density at the crack tip has a $r^{-1/2}$ singularity.

The heat flux at the right tip of the crack is:

$$(48) \quad J_{Qx} - iJ_{Qy} = \frac{\sigma^2 \varepsilon}{k} f(a) \overline{f(a)} f'(a) - \sigma \varepsilon C_2 f'(a) \\ + 2\sigma f'(a) \overline{f(a)} - 2\sigma \varepsilon f'(a) \operatorname{Re}[f_2(a)] - 2\kappa f'_2(a),$$

$$\text{where: } f(a) = -\frac{1}{\sigma}(J_x^\infty - iJ_y^\infty)a - \frac{aiJ_y^\infty}{\sigma}, \quad f'(a) = -\frac{1}{\sigma}(J_x^\infty - iJ_y^\infty) + \frac{aiJ_y^\infty}{\sigma\sqrt{2a(z-a)}}, \\ f_2(a) = -\frac{aJ_{Qx}^\infty}{2\kappa}, \quad f'_2(a) = \frac{aiJ_{Qy}^\infty}{2\kappa} \frac{1}{\sqrt{2a(z-a)}}.$$

It can be seen, that the heat flux at the crack tip also has a $r^{-1/2}$ singularity.

For the thermally impermeable and electrically permeable crack, the electric current density at the right tip of the crack is:

$$(49) \quad J_y = -\frac{\sigma i}{2} [f'(z) - \overline{f'(z)}] = J_y^\infty.$$

It can be seen that the electric current density at the crack tip has no singularity.

The heat flux at the right tip of the crack is:

$$(50) \quad J_{Qx} - iJ_{Qy} = \frac{\sigma^2 \varepsilon}{k} f(a) \overline{f(a)} f'(a) - \sigma \varepsilon C_2 f'(a) \\ + 2\sigma f'(a) \overline{f(a)} - 2\sigma \varepsilon f'(a) \operatorname{Re}[f_2(a)] - 2\kappa f'_2(a),$$

$$\text{where: } f(a) = -\frac{a}{\sigma}(J_x^\infty - iJ_y^\infty), \quad f'(a) = -\frac{1}{\sigma}(J_x^\infty - iJ_y^\infty), \quad f_2(a) = -\frac{aJ_{Qx}^\infty}{2\kappa}, \\ f'_2(a) = \frac{aiJ_{Qy}^\infty}{2\kappa} \frac{1}{\sqrt{2a(z-a)}}.$$

It can be seen, that the heat flux at the crack tip has a $r^{-1/2}$ singularity.

6. Results and discussion

Numerical calculations were carried out for $\text{Ag}(\text{Pb}_{1-y}\text{Sn}_y)_m\text{SbTe}_{2+m}$. Therefore, the material constants of the matrix are: $\varepsilon_1 = 200 \times 10^{-6}$ V/K, $\sigma_1 = 110 \times 10^3$ S/m, $\kappa_1 = 1.6$ W/m/K. The calculation area is: $-0.01 \leq x \leq 0.01$, $-0.01 \leq y \leq 0.01$. The thermoelectric conversion efficiency is $\eta = \frac{J_{Q, \text{input}} - J_{Q, \text{output}}}{J_{Q, \text{input}}}$.

Fig. 2. The conversion efficiency η versus electric current density J_y^∞ in case 1

Fig. 3. The conversion efficiency η versus electric current density J_y^∞ in case 2

Figure 2 shows the thermoelectric conversion efficiency η versus electric current density J_y^∞ in case 1 (thermally and electrically impermeable crack) at $a = 0.009$ m, $J_x^\infty = 0$, $J_{Qx}^\infty = 0$ and . It is seen, that the conversion efficiency η firstly increased then decreased with the electric current density J_y^∞ increasing. Which means, there is an optimal value of electric current density. The thermoelectric conversion efficiency η in this case is always less than the maximum conversion efficiency of one-dimensional thermoelectric, under the same temperature condition.

Fig. 4. The conversion efficiency η versus crack length a in case 1 Fig. 5. The conversion efficiency η versus crack length a in case 2

The curves for variations of thermoelectric conversion efficiency η , with respect to electric current density J_y^∞ , in case 2 (thermally impermeable and electrically permeable crack) are shown in Fig. 3 for the case of $a = 0.009$ m, $J_x^\infty = 0$, $J_{Qx}^\infty = 0$ and $J_{Qy}^\infty = 200$ J/m²·s. It is seen, that the thermoelectric conversion efficiency η in this case can be higher than the maximum conversion efficiency of one-dimensional thermoelectric under the same temperature condition. This is important, as it proves the maximum conversion efficiency of one-dimensional thermoelectric may be a breakthrough in the two-dimensional condition.

The conversion efficiency improvement is due to the electric current

flow and thermal flux separation. Heat flow presence around the thermally impermeable and electrically permeable crack, is resulting in increased heat flow path. Under the same temperature difference, an increased heat flow path reduces the temperature gradient along the path, therefore reducing the heat leak and improving the conversion efficiency.

Figures 4–5 show the thermoelectric conversion efficiency η versus crack length a in case 1, (Figure 4) and case 2, (Figure 5) at $J_y^\infty = 1600 \text{ A/m}^2$, $J_x^\infty = 0$, $J_{Qx}^\infty = 0$ and $J_{Qy}^\infty = 200 \text{ J/m}^2\cdot\text{s}$. It is seen, that the thermoelectric conversion efficiency η in case 1 is always less than the maximum conversion efficiency of one-dimensional thermoelectric under the same temperature condition. While the thermoelectric conversion efficiency η in case 2 can be higher than the maximum conversion efficiency of one-dimensional thermoelectric under the same temperature condition. The conversion efficiency in case 2 increases with increasing crack length a , the maximum conversion efficiency is 0.00175, which is increased by 29.6% compared to the maximum conversion efficiency of one-dimensional thermoelectric under the same temperature condition.

7. Conclusions and future work

The crack problem in a thermoelectric material is studied in this paper. The closed form solutions are derived. The solution show, that the electric current density and thermal flux density at the crack tip have a $r^{-1/2}$ singularity in case 1, while the electric current density at the crack tip has no singularity in case 2. Due to the electric current flow and thermal flux separation, the thermoelectric conversion efficiency η in case 2 can be higher than the one-dimensional thermoelectric under the same temperature condition.

In the future, we will continue to study the coupled thermo-elasto-electric problem and inclusion problem in thermoelectric material, and we are here to give some discussions.

For thermal stress problems, stress function Φ must satisfy the compatibility equation:

$$\nabla^4 \Phi + E\alpha \nabla^2 T = 0,$$

where α is the thermal expansion coefficient. This results in:

$$\nabla^4 \Phi = -E\alpha \nabla^2 T = \frac{E\alpha}{\sigma\kappa} \mathbf{J} \bullet \mathbf{J} = \frac{E\alpha\sigma}{\kappa} f'(z) \overline{f'(z)}.$$

Therefore, stress functions Φ are composed by general solution and particular solution:

$$\Phi = \Phi_1 + \Phi_2,$$

where the general solution Φ_2 satisfies $\nabla^4\Phi_2 = 0$, and the particular solution Φ_1 satisfies $\nabla^4\Phi_1 = \frac{E\alpha\sigma}{\kappa}f'(z)\overline{f'(z)}$. The particular solution could be chosen as: $\Phi_1 = -\frac{E\alpha}{4}\iint Tdzd\bar{z}$, as the temperature can be obtained by the method developed in this paper, the subsequent task is to search the general solution Φ_2 (satisfies $\nabla^4\Phi_2 = 0$) at given stress boundary conditions, which is similar to the pure elastic problem.

For inclusion problems, there will be four unknown functions, $f(z)$, $f_2(z)$ in matrix and $f_*(z)$, $f_{2*}(z)$ in inclusion. Correspondingly, we will have four boundary conditions, such as the temperature, electric potential, electric current density, and heat flux, are continuous across the interface. The inclusion problem may be solved substituting the four unknown functions into the four boundary conditions and using the similar mathematical treatment as used in Sec. 4.

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Nomenclature

a semi-length of the crack, m
 $\mathbf{J}$ electric current density, A/m²
 $\mathbf{J}_E$ energy flux, J/m²·s
 $\mathbf{J}_Q$ heat flux, J/m²·s
 T temperature, K
 e charge, C
 u axial coordinate in the w -plane, m
 v axial coordinate in the w -plane, m
 x axial coordinate, m
 y axial coordinate, m
 z complex coordinate, m
 σ electric conductivity, S/m
 κ thermal conductivity, W/m·K
 ϕ electric potential, V
 A_k unknown constant
 B_k unknown constant
 C_2 unknown constant
 Re real part
 Im imaginary part
 i unit imaginary number
 ∇ gradient
 ∇^2 divergence of the gradient