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ON $(\mathcal{I}, \gamma)$ - g^* -CLOSED SETS VIA IDEAL TOPOLOGICAL SPACES

ABSTRACT. In this paper we introduce (I, γ) - g^* -closed sets in topological spaces and also introduce $\gamma g^* - T_I$ -spaces and investigate some of their properties.

KEY WORDS: (I, γ) - g -closed set, (I, γ) - gs -closed sets, $\gamma\alpha$ - T_I -spaces.

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1. Introduction and preliminaries

Recently Julian Dontchev et. al. [3] introduce (I, γ) -generalized closed sets and R. Devi et. al. [1] introduce (I, γ) -generalized semi-closed sets via topological ideals. In this paper we introduce (I, γ) - g^* -closed sets and investigate some of their properties.

An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following two properties:

- (i) $A \in I$ and $B \subset A$ implies $B \in I$
- (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$

For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I \text{ for each neighbourhood } U \text{ of } x\}$ is called the local function of A with respect to I and τ . Recall that $A \subseteq (X, \tau, I)$ is called τ^* -closed [5] if $A^* \subseteq A$. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(I)$, finer than τ . An operation γ [6,10] on the topology τ on a given topological space (X, τ) is a function from the topology itself into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V .

The following operators are examples of the operation γ : the closure operator γ_{cl} defined by $\gamma(U) = cl(U)$, the identity operator γ_{id} defined by $\gamma(U) = U$. Another example of the operation γ is the γ_f -operator defined by $U^{\gamma_f} = (FrU)^c = X/FrU$ [11]. Two operators γ_1 and γ_2 are called mutually dual [11] if $U^{\gamma_1} \cap U^{\gamma_2} = U$ for each $U \in \tau$. For example the identity operator

is mutually dual to any other operator, while the γ_f -operator is mutually dual to the closure operator [11].

Definition 1. A subset A of a space (X, τ) is called

- (a) an α -open set [9] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$.
- (b) a semi-open set [8] if $A \subseteq \text{cl}(\text{int}(A))$.
- (c) a generalized closed (briefly g -closed) set [7] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (d) a g^* -closed set [12] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .
- (e) an (I, γ) -generalized closed (briefly (I, γ) - g -closed) set [3] if $A^* \subseteq U^\gamma$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (f) an (I, γ) -generalized semi closed (briefly (I, γ) - gs -closed) set [1] if $A^* \subseteq U^\gamma$, whenever $A \subseteq U$ and U is semi-open in (X, τ) .

Throughout this paper the operator γ is defined as $\gamma : \tau^g \rightarrow P(X)$, where τ^g denotes the set of all g -open sets of (X, τ) .

2. Properties of $(\mathcal{I}, \gamma)$ - g^* -closed sets

Definition 2. A subset A of a topological space (X, τ) is called $(\mathcal{I}, \gamma)$ - g^* -closed if $A^* \subset U^\gamma$, whenever $A \subset U$ and U is g -open in (X, τ) .

Example 1. Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\phi\}$ and $U^\gamma = \text{int}(\text{cl}(U))$. Here $(\mathcal{I}, \gamma)$ - g^* -closed sets are $X, \phi, \{a, c\}, \{b, c\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{b, c, d\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{b, c, d, e\}$.

Theorem 1. (a) Every g^* -closed set $\mathcal{I}$ - g^* -closed.

(b) Every $(\mathcal{I}, \gamma)$ - g^* -closed set is $(\mathcal{I}, \gamma)$ - g -closed.

(c) Every $(\mathcal{I}, \gamma)$ - g^* -closed set is $(\mathcal{I}, \gamma)$ - gs -closed.

The converse of the above theorem need not be true by the following example.

Example 2. (a) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Here $A = \{a\}$ is $\mathcal{I}$ - g^* -closed but not g^* -closed.

(b) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and $\gamma = \text{id}$.

Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$ - g -closed but not $(\mathcal{I}, \gamma)$ - g^* -closed.

(c) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and $\gamma = \text{id}$.

Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$ - gs -closed but not $(\mathcal{I}, \gamma)$ - g^* -closed.

Theorem 2. If A is $\mathcal{I}$ - g^* -closed and g -open, then A is τ^* -closed.

Proof. Since A is $\mathcal{I}$ - g^* -closed, then $A^* \subset U$, U is g -open. It is given that A is g -open implies $A^* \subset A$. Hence A is τ^* -closed. ■

Lemma 1 ([4], Theorem II3). *Let $(A_i)_{i \in I}$ be a locally finite family of sets in $(X, \tau, \mathcal{I})$. Then $\cup_{i \in I} A_i^*(\mathcal{I}) = (\cup_{i \in I} A_i)^*(\mathcal{I})$.*

Theorem 3. *Let $(X, \tau, \mathcal{I}, \gamma)$ be a topological space.*

- (a) *If $(A_i)_{i \in I}$ is a locally finite family of sets and each $A_i \in IG^*(X)$, then $\cup_{i \in I} A_i \in IG^*(X)$.*
 (b) *Finite intersection of $(\mathcal{I}, \gamma)$ - g^* -closed sets need not be $(\mathcal{I}, \gamma)$ - g^* -closed.*

Proof.

- (a) Let $\cup_{i \in I} A_i \subset U$, where U is g -open. Since $A_i \in IG^*(X)$ for each $i \in I$, then $A_i^* \subset U^\gamma$. Hence $\cup_{i \in I} A_i^* \subset U^\gamma$. By Lemma 1, $(\cup_{i \in I} A_i)^* = \cup_{i \in I} A_i^*$, then $(\cup_{i \in I} A_i)^* \subset U^\gamma$. Hence $\cup_{i \in I} A_i \in IG^*(X)$.
 (b) Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\phi\}$ and $\gamma = \gamma_{ic}$. Set $A = \{a, c\}$ and $B = \{b, c\}$. Clearly $A, B \in IG^*(X)$ but $A \cap B = \{c\} \notin IG^*(X)$. ■

Lemma 2 ([4]). *If A and B are subsets of $(X, \tau, \mathcal{I})$, then $(A \cap B)^*(\mathcal{I}) \subset A^*(\mathcal{I}) \cap B^*(\mathcal{I})$.*

Theorem 4. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space. If $A \subset X$ is $\mathcal{I}$ - g^* -closed and B is closed and τ^* -closed, then $A \cap B$ is $\mathcal{I}$ - g^* -closed.*

Proof. Let $U \in \tau^g$ be such that $A \cap B \subset U$. Then $A \subset U \cap (X/B)$. Since A is $\mathcal{I}$ - g^* -closed, then $A^* \subset U \cap (X/B)$. Hence $B \cap A^* \subset U \cap B \subset U$, but we know that $B^* \subset B$, therefore $(A \cap B)^* \subset A^* \cap B^* \subset A^* \cap B \subset U$, by Lemma 2. Hence $A \cap B$ is $\mathcal{I}$ - g^* -closed. ■

Theorem 5. *Let A be a subset of $(X, \tau, \mathcal{I}, \gamma_{id})$. Then, A is $\mathcal{I}$ - g^* -closed if and only if $A^* - A$ does not contain any non-empty closed subset.*

Proof. *Necessity.* Assume that F is a closed subset of $A^* - A$. Note that clearly $A \subset X - F$, where A is $\mathcal{I}$ - g^* -closed and $X - F \in \tau$. Then $A^* \subset X - F$, that is $F \subset X - A^*$. Since due to our assumption $F \subset A^*$, $F \subset (X - A^*) \cap A^* = \phi$.

Sufficiency. Let U be an open subset and hence g -open subset containing A . Since A^* is closed [5, Theorem 2.3(c)] and $A^* \cap (X - U) \subset A^* - A$ holds, then $A^* \cap (X - U)$ is a closed set contained in $A^* - A$. By assumption, $A^* \cap (X - U) = \phi$ and hence $A^* \subset U$. ■

A subset S of a space $(X, \tau, \mathcal{I}, \gamma)$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I} : I \subset S\} = \{I \cap S : I \in \mathcal{I}\}$ on S [2].

Lemma 3 ([3]). *Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subset S \subset X$. Then, $A^*(\mathcal{I}_S, \tau/S) = A^*(\mathcal{I}, \tau) \cap S$ holds.*

Proof. First we prove the following implication: $A^*(\mathcal{I}_S, \tau/S) \subset A^*(\mathcal{I}, \tau) \cap S$. Let $x \notin A^*(\mathcal{I}, \tau) \cap S$. We consider the following two cases:

Case 1. $x \notin S$. Since $A^*(\mathcal{I}_S, \tau/S) \subset S$, then $x \notin A^*(\mathcal{I}_S, \tau/S)$.

Case 2. $x \in S$. In this case $x \notin A^*(\mathcal{I}, \tau)$. There exists a set $V \in \tau$ such that $x \in V$ and $V \cap A \in \mathcal{I}$. Since $x \in S$, we have a set $S \cap V \in \tau/S$ such that $x \in S \cap V$ and $(S \cap V) \cap A \in \mathcal{I}$ and hence $(S \cap V) \cap A \in \mathcal{I}_S$. Consequently, $x \notin A^*(\mathcal{I}_S, \tau/S)$. Both cases show the implication.

Secondly, we prove the converse implication: $A^*(\mathcal{I}, \tau) \cap S \subset A^*(\mathcal{I}_S, \tau/S)$. Let $x \notin A^*(\mathcal{I}_S, \tau/S)$. Then, for some open subset $U \cap S$ of $(S, \tau/S)$ containing x , we have $(U \cap S) \cap A \in \mathcal{I}_S$. Since $A \subset S$, then $U \cap A \in \mathcal{I}_S \subset \mathcal{I}$, i.e., $U \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x . This shows that $x \notin A^*(\mathcal{I}, \tau)$. ■

Theorem 6. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma_{id})$. If A is $\mathcal{I}_S$ - g^* -closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is closed in (X, τ) , then A is $\mathcal{I}$ - g^* -closed in $(X, \tau, \mathcal{I}, \gamma_{id})$.*

Proof. Let $A \subset U$, where $U \in \tau^g$. Let $x \notin U$. We consider the following two cases.

Case (i). $x \in S$. By assumption, $A^*(\mathcal{I}_S, \tau/S) \subset U \cap S \subset U$. We show that $A^*(\mathcal{I}) \subset A^*(\mathcal{I}_S, \tau/S)$. Let $x \notin A^*(\mathcal{I}_S, \tau/S)$. Since $x \in S$, then for some open subset V_S of $(S, \tau/S)$ containing x , we have $V_S \cap A \in \mathcal{I}_S$, since $V_S = V \cap S$ for some $V \in \mathcal{I}$, then $(S \cap V) \cap A \in \mathcal{I}_S \subset \mathcal{I}$, that is $V \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x . This shows that $x \notin A^*(\mathcal{I})$. Hence $A^*(\mathcal{I}) \subset U$.

Case (ii). $x \notin S$. Then X/S is an open neighbourhood of x disjoint from A . Hence $x \notin A^*(\mathcal{I})$. Consequently $A^*(\mathcal{I}) \subset U$.

Both cases we show that the local function of A with respect to $\mathcal{I}$ and τ is in U . Hence A is $\mathcal{I}$ - g^* -closed in $(X, \tau, \mathcal{I}, \gamma_{id})$. ■

Theorem 7. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and $A \subset S \subset X$. If A is $\mathcal{I}_S$ - g^* -closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is $\mathcal{I}$ - g^* -closed in X , then A is $\mathcal{I}$ - g^* -closed in X .*

Proof. Let $A \subset U$ and $U \in \tau^g$. By assumption and Lemma 3, $A^*(\mathcal{I}, \tau) \cap S \subset U \cap S$. Then we have $S \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Since $X/A^*(\mathcal{I}, \tau) \in \tau^g$, then $A^*(\mathcal{I}, \tau) \subset S^*(\mathcal{I}, \tau) \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Therefore, we have that $A^*(\mathcal{I}, \tau) \subset U$ and hence A is $\mathcal{I}$ - g^* -closed in X . ■

Corollary 1. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and A and F subsets of X . If A is $\mathcal{I}$ - g^* -closed and F is closed in (X, τ) , then $A \cap F$ is $\mathcal{I}$ - g^* -closed.*

Proof. Since $A \cap F$ is closed in $A, \tau/A$, then $A \cap F$ is $\mathcal{I}_A$ - g^* -closed in $(A, \tau/A, \mathcal{I}_A)$. By Theorem 7, $A \cap F$ is $\mathcal{I}$ - g^* -closed. ■

Theorem 8. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma)$. If $A \in IG^*(X)$ and $S \in \tau^g$, then $A \in IG^*(S)$.*

Proof. Let U be a g -open subset of $(S, \tau/S)$ such that $A \subset U$. Since $S \in \tau^g$, then $U \in \tau^g$. Then $A^*(\mathcal{I}) \subset U^\gamma$, since $A \in IG^*(X)$. By Lemma 3, we have $A^*(\mathcal{I}_S, \tau/S) \subset U^{\gamma/S}$, where $U^{\gamma/S}$ means the image of the operation $\gamma/S : \tau^g/S \rightarrow P(S)$, defined by $(\gamma/S)(U) = \gamma(U) \cap S$ for each $U \in \tau^g/S$. Hence $A \in IG^*(S)$. ■

Theorem 9. *If the set $A \subset (X, \tau, \mathcal{I})$ is both $(\mathcal{I}, \gamma_1)$ - g^* -closed and $(\mathcal{I}, \gamma_2)$ - g^* -closed, then it is $\mathcal{I}$ - g^* -closed, granted the operators γ_1 and γ_2 are mutually dual.*

Proof. Let $A \subset U$, where $U \in \tau^g$. Since $A^* \subset U^{\gamma_1}$ and $A^* \subset U^{\gamma_2}$, then $A^* \subset U^{\gamma_1} \cap U^{\gamma_2} = U$, since γ_1 and γ_2 are mutually dual. Hence A is $\mathcal{I}$ - g^* -closed. ■

Theorem 10. *Every set $A \subset (X, \tau, \mathcal{I})$ is $(\mathcal{I}, \gamma_{cl})$ - g^* -closed.*

Proof. Let $A \subset U$, U is g -open. We know that $A \cup A^* = \text{cl}^*(A) \subset \text{cl}(A) \subset \text{cl}(U)$. This implies that $A^* \subset \text{cl}(U)$. Hence A is $(\mathcal{I}, \gamma_{cl})$ - g^* -closed. ■

Corollary 2. *For a set $A \subset (X, \tau, \mathcal{I}, \gamma)$, the following conditions are equivalent.*

- (a) A is $(\mathcal{I}, \gamma_f)$ - g^* -closed.
- (b) A is $\mathcal{I}$ - g^* -closed.

Proof. (a) $\Rightarrow$ (b) By Theorem 10, A is $(\mathcal{I}, \gamma_{cl})$ - g^* -closed. Since γ_f and γ_{cl} are mutually dual due to [11], then $\gamma_f(U) \cap \gamma_{cl}(U) = U$. This implies that $A^* \subset U$, that is, A is $\mathcal{I}$ - g^* -closed.

(b) $\Rightarrow$ (a) Let $A \subset U$, U is g -open. Since A is $\mathcal{I}$ - g^* -closed, $A^* \subset U$. But we know that $U \subset U^{\gamma_f}$, we have $A^* \subset U^{\gamma_f}$, this implies that A is $(\mathcal{I}, \gamma_f)$ - g^* -closed. ■

3. γg^* - $T_{\mathcal{I}}$ -space

Definition 3. *A space $(X, \tau, \mathcal{I}, \gamma)$ is called an γg^* - $T_{\mathcal{I}}$ -space if every $(\mathcal{I}, \gamma)$ - g^* -closed subset of X is an τ^* -closed. We use the simple notation $g^*T_{\mathcal{I}}$ -space, in case γ is the identity operator.*

Theorem 11. *For a space $(X, \tau, \mathcal{I})$, the following conditions are equivalent.*

- (a) X is a $g^*T_{\mathcal{I}}$ -space.
- (b) Each singleton of X is either closed or τ^* -open.

Proof. (a) $\Rightarrow$ (b) Let $x \in X$. If $\{x\}$ is not closed, then $A = X/\{x\} \notin \tau$ and then A is trivially $\mathcal{I}$ - g^* -closed. By (a), A is τ^* -closed. Hence $\{x\}$ is τ^* -open.

(b) $\Rightarrow$ (a) Let A be a $\mathcal{I}$ - g^* -closed and let $x \in \text{cl}^*(A)$. We have the following two cases.

Case (i). $\{x\}$ is closed. By Theorem 2.10, $A^* - A$ does not contain any non-empty closed subset. This shows that $x \in A$.

Case (ii). $\{x\}$ is τ^* -open. Then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$.

Thus in both cases $\{x\}$ is in A and so $A = \text{cl}^*A$, that is A is τ^* -closed, which shows that X is a $g^*T_{\mathcal{I}}$ -space. ■

References

- [1] DEVI R., SELVAKUMAR A., VIGNESHWARAN M., $(\mathcal{I}, \gamma)$ -generalized semi-closed sets in topological space, *Filomat*, 24(1)(2010), 97-100.
- [2] DONTCHEV J., On Hausdorff spaces via topological ideals and $\mathcal{I}$ -irresolute functions, *Annals of the New York Academy of Sciences, Papers on General Topology and Applications*, 767(1995), 28-38.
- [3] DONTCHEV J., GANSTER M., NOIRI T., Unified operation approach of generalized closed sets via topological ideals, *Math. Japonica*, 49(1999), 395-401.
- [4] HAMLETT T.R., ROSE D., JANKOVIC D., Paracompactness with respect to an ideal, *Internat. J. Math. Math. Sci.*, 20(3)(1997), 433-442.
- [5] JANKOVIC D., HAMLETT T.R., New topologies from old via ideals, *Amer. Math. Monthly*, 97(4)(1990), 295-310.
- [6] KASAHARA S., Operation-compact spaces, *Math. Japon.*, 24(1979), 97-105.
- [7] LEVINE N., Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* (2), 19(1970), 89-96.
- [8] LEVINE N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
- [9] NJASTAD O., On some classes of nearly open sets, *Pacific J. Math.*, 15(1965), 961- 970.
- [10] OGATA H., Operations on topological spaces and associated topology, *Math. Japon.*, 36(1991), 175-184.
- [11] TONG J., Expansion of open sets and decomposition of continuous mappings, *Rend. Circ. Mat. Palermo* (2), 43(1994), 303-308.
- [12] VEERA KUMAR M.K.R.S., Between g^* -closed sets and g -closed sets, *Antarctica J. Math.*, 3(1)(2006), 43-65.

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