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HERMITE-HADAMARD TYPE INEQUALITIES FOR MT_m -PREINVEX FUNCTIONS

ABSTRACT. In the present paper, the notion of MT_m -preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving MT_m -preinvex functions along with beta function are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT_m -preinvex functions via classical integrals and Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These results not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

KEY WORDS: Hermite-Hadamard type inequality, MT-convex function, Hölder's inequality, power mean inequality, fractional integral, m -invex, P -function.

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1. Introduction and preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K . $\mathbb{R}^n$ is used to denote a generic n -dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_+ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [9], [13], [6], [21], [5], [11], [10], [4], [17], [3]) and the references cited therein. In (see [19], [14]) and the references cited therein, Tunç and Yildirim defined the following so-called MT-convex function:

Definition 1. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:

$$(2) \quad f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y).$$

Fractional calculus (see [13]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

Definition 3 (see [7]). A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 4 (see [1]). A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details please see (see [1],[20]) and the references therein.

Definition 5 (see [16]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have*

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$(3) \quad \int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|,$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^* |f|$ (see [18]).

Recently, Liu (see [12]) obtained several integral inequalities for the left-hand side of (3) under the Definition 3 of P -function. Also in (see [15]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of MT_m -preinvex function is introduced and some new integral inequalities for the left-hand side of (3) involving MT_m -preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for MT_m -preinvex functions via classical integrals are given. In Section 4, some generalizations of Hermite-Hadamard type inequalities for MT_m -preinvex functions via fractional integrals are given. In Section 5, some applications to special means are given. These results given in Sections 3-4 not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

2. New integral inequalities for MT_m -preinvex functions

Definition 6 (see [8]). *A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

We next give new definition, to be referred as MT_m -preinvex function.

Definition 7. Let $K \subseteq \mathbb{R}^n$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$. For $f : K \longrightarrow \mathbb{R}$ and any fixed $m \in (0, 1]$, if

$$(4) \quad f(my + t\eta(x, y, m)) \leq \frac{m\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y),$$

is valid for all $x, y \in K$ and $t \in (0, 1)$, then we say that $f(x)$ belong to the class of $MT_m(K)$ with respect to η .

Remark 2. In Definition 7, it is worthwhile to note that the class $MT_m(K)$ is a generalization of the class $MT(I)$ given in Definition 1 on $K = I$ with respect to $\eta(x, y, 1) = x - y$ and $m = 1$.

Example 1. $f, g : (1, \infty) \longrightarrow \mathbb{R}$, $f(x) = x^p$, $g(x) = (1 + x)^p$, $p \in (0, \frac{1}{1000})$; $h : [1, 3/2] \longrightarrow \mathbb{R}$, $h(x) = (1 + x^2)^k$, $k \in (0, \frac{1}{100})$, are simple examples of the new class of MT_m -preinvex functions with respect to $\eta(x, y, m) = x - my$ for any fixed $m \in (0, 1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving MT_m -preinvex functions along with beta function, we need the following new lemma:

Lemma 1. Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K° with $a < b$ and $ma < ma + \eta(b, a, m)$. Then for any fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b, a, m) - x)^q f(x) dx \\ &= \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(ma + t\eta(b, a, m)) dt. \end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned} & \int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b, a, m) - x)^q f(x) dx \\ &= \eta(b, a, m) \int_0^1 (ma + t\eta(b, a, m) - ma)^p (ma + \eta(b, a, m) \\ & \quad - ma - t\eta(b, a, m))^q f(ma + t\eta(b, a, m)) dt \\ &= \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(ma + t\eta(b, a, m)) dt. \end{aligned}$$

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The following definition will be used in the sequel.

Definition 8. The Euler Beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2. Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K° , $a < b$ with $ma < ma + \eta(b, a, m)$. If $|f|$ is a MT_m -preinvex function on K for any fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx \\ & \leq \frac{m}{2} \eta(b, a, m)^{p+q+1} \left[|f(a)| \beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + |f(b)| \beta\left(p + \frac{3}{2}, q + \frac{1}{2}\right) \right]. \end{aligned}$$

Proof. Since $|f|$ is a MT_m -preinvex function on K , we have

$$\left| f(ma + t\eta(b, a, m)) \right| \leq \frac{m\sqrt{t}}{2\sqrt{1-t}} |f(b)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f(a)|$$

for all $t \in (0, 1)$ and for any fixed $m \in (0, 1]$. By Lemma 1 and the fact that $|f|$ is a MT_m -preinvex function on K , we get

$$\begin{aligned} & \int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx \\ & \leq \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1-t)^q \left| f(ma + t\eta(b, a, m)) \right| dt \\ & \leq \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1-t)^q \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f(b)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f(a)| \right] dt \\ & = \frac{m}{2} \eta(b, a, m)^{p+q+1} \left[|f(a)| \beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + |f(b)| \beta\left(p + \frac{3}{2}, q + \frac{1}{2}\right) \right]. \end{aligned}$$

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Theorem 3. Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K° , $a < b$ with $ma < ma + \eta(b, a, m)$. Let $k > 1$ and $|f|^{\frac{k}{k-1}}$ be a MT_m -preinvex function on K for any fixed $m \in (0, 1]$. Then for any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx \\ & \leq \left(\frac{m\pi}{4} \right)^{\frac{k-1}{k}} \eta(b, a, m)^{p+q+1} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \\ & \quad \times \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}. \end{aligned}$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a MT_m -preinvex function on K , combining with Lemma 1 and Hölder inequality for all $t \in (0, 1)$ and for any fixed $m \in (0, 1]$, we get

$$\begin{aligned}
& \int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx \\
& \leq \eta(b,a,m)^{p+q+1} \left[\int_0^1 t^{kp} (1-t)^{kq} dt \right]^{\frac{1}{k}} \\
& \quad \times \left[\int_0^1 |f(ma+t\eta(b,a,m))|^{\frac{k}{k-1}} dt \right]^{\frac{k-1}{k}} \\
& \leq \eta(b,a,m)^{p+q+1} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \\
& \quad \times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f(b)|^{\frac{k}{k-1}} + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f(a)|^{\frac{k}{k-1}} \right) dt \right]^{\frac{k-1}{k}} \\
& = \left(\frac{m\pi}{4} \right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \\
& \quad \times \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}.
\end{aligned}$$

■

Theorem 4. Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K° , $a < b$ with $ma < ma + \eta(b, a, m)$. Let $l \geq 1$ and $|f|^l$ be a MT_m -preinvex function on K for any fixed $m \in (0, 1]$. Then for any fixed $p, q > 0$, we have

$$\begin{aligned}
& \int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx \\
& \leq \left(\frac{m}{2} \right)^{\frac{1}{l}} \eta(b,a,m)^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\
& \quad \times \left[|f(a)|^l \beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + |f(b)|^l \beta\left(p + \frac{3}{2}, q + \frac{1}{2}\right) \right]^{\frac{1}{l}}.
\end{aligned}$$

Proof. Since $|f|^l$ is a MT_m -preinvex function on K , combining with Lemma 1 and Hölder inequality for all $t \in (0, 1)$ and for any fixed $m \in (0, 1]$, we get

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p (ma+\eta(b,a,m)-x)^q f(x) dx$$

$$\begin{aligned}
&= \eta(b, a, m)^{p+q+1} \int_0^1 \left[t^p(1-t)^q \right]^{\frac{l-1}{l}} \\
&\quad \times \left[t^p(1-t)^q \right]^{\frac{1}{l}} f(ma + t\eta(b, a, m)) dt \\
&\leq \eta(b, a, m)^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt \right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 t^p(1-t)^q |f(ma + t\eta(b, a, m))|^l dt \right]^{\frac{1}{l}} \\
&\leq \eta(b, a, m)^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 t^p(1-t)^q \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f(b)|^l + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f(a)|^l \right) dt \right]^{\frac{1}{l}} \\
&= \left(\frac{m}{2} \right)^{\frac{1}{l}} \eta(b, a, m)^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\
&\quad \times \left[|f(a)|^l \beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + |f(b)|^l \beta\left(p + \frac{3}{2}, q + \frac{1}{2}\right) \right]^{\frac{1}{l}}.
\end{aligned}$$

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Remark 3. In Theorem 4, if we choose $l = 1$, we get Theorem 2.

3. Hermite-Hadamard type classical integral inequalities for MT_m -preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m -preinvex functions via classical integrals, we need the following new lemma:

Lemma 2. Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : K \rightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma + \eta(b, a, m)]$. Then, for each $x \in [ma, ma + \eta(b, a, m)]$, we have

$$\begin{aligned}
(5) \quad & \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \\
& - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u) du - \int_{mb}^{mb+\eta(x, b, m)} f(u) du \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \int_0^1 (t-1) f'(ma + t\eta(x, a, m)) dt \\
&\quad + \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \int_0^1 (1-t) f'(mb + t\eta(x, b, m)) dt.
\end{aligned}$$

Proof. Denote

$$\begin{aligned}
I &= \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \int_0^1 (t-1) f'(ma + t\eta(x, a, m)) dt \\
&\quad + \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \int_0^1 (1-t) f'(mb + t\eta(x, b, m)) dt.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
I &= \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \left[(t-1) \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} \Big|_0^1 - \int_0^1 \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} dt \right] \\
&\quad + \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \left[(1-t) \frac{f(mb + t\eta(x, b, m))}{\eta(x, b, m)} \Big|_0^1 + \int_0^1 \frac{f(mb + t\eta(x, b, m))}{\eta(x, b, m)} dt \right] \\
&= \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \\
&\quad - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u) du - \int_{mb}^{mb+\eta(x, b, m)} f(u) du \right].
\end{aligned}$$

■

Remark 4. Clearly, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ in Lemma 2, we get (see [9], Lemma 1).

Using the Lemma 2 the following results can be obtained.

Theorem 5. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have

$$\begin{aligned}
(6) \quad &\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right. \\
&\quad \left. - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u) du - \int_{mb}^{mb+\eta(x, b, m)} f(u) du \right] \right| \\
&\leq \frac{Mm\pi}{4|\eta(b, a, m)|} [\eta(x, a, m)^2 + \eta(x, b, m)^2].
\end{aligned}$$

Proof. Using Lemma 2, MT_m -preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\begin{aligned}
 & \left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right. \\
 & \quad \left. - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u)du - \int_{mb}^{mb+\eta(x, b, m)} f(u)du \right] \right| \\
 & \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 |t-1| |f'(ma + t\eta(x, a, m))| dt \\
 & \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 |1-t| |f'(mb + t\eta(x, b, m))| dt \\
 & \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 (1-t) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| \right. \\
 & \quad \left. + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)| \right] dt \\
 & \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 (1-t) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right] dt \\
 & \leq \frac{Mm\pi}{4|\eta(b, a, m)|} \left[\eta(x, a, m)^2 + \eta(x, b, m)^2 \right].
 \end{aligned}$$

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Remark 5. In Theorem 5, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 2.2).

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 6. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have

$$\begin{aligned}
 (7) \quad & \left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right. \\
 & \quad \left. - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u)du - \int_{mb}^{mb+\eta(x, b, m)} f(u)du \right] \right| \\
 & \leq \frac{M}{(p+1)^{1/p}} \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
 \end{aligned}$$

Proof. Suppose that $q > 1$. Using Lemma 2, MT_m -preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\begin{aligned}
& \left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right. \\
& \quad \left. - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u)du - \int_{mb}^{mb+\eta(x, b, m)} f(u)du \right] \right| \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 |t-1| |f'(ma + t\eta(x, a, m))| dt \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 |1-t| |f'(mb + t\eta(x, b, m))| dt \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q \right. \right. \\
& \quad \left. \left. + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q \right. \right. \\
& \quad \left. \left. + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& \leq \frac{M}{(p+1)^{1/p}} \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\end{aligned}$$

■

Remark 6. In Theorem 6, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 2.4).

Theorem 7. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have

$$(8) \quad \left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right|$$

$$\begin{aligned}
& - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u) du - \int_{mb}^{mb+\eta(x, b, m)} f(u) du \right] \Bigg| \\
& \leq M \left(\frac{1}{2} \right)^{1+\frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[\frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\end{aligned}$$

Proof. Using Lemma 2, MT_m -preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b, a, m)]$, and taking the modulus, we have

$$\begin{aligned}
& \left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right. \\
& \quad \left. - \frac{1}{\eta(b, a, m)} \left[\int_{ma}^{ma+\eta(x, a, m)} f(u) du - \int_{mb}^{mb+\eta(x, b, m)} f(u) du \right] \right| \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 |t-1| |f'(ma + t\eta(x, a, m))| dt \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 |1-t| |f'(mb + t\eta(x, b, m))| dt \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\int_0^1 (1-t) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
& \quad + \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\int_0^1 (1-t) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& \leq M \left(\frac{1}{2} \right)^{1+\frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[\frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\end{aligned}$$

■

Remark 7. In Theorem 7, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 2.6). Also, in Theorem 7, if we choose $q = 1$, we get Theorem 5.

4. Hermite-Hadamard type fractional integral inequalities for MT_m -preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m -preinvex functions via fractional integrals, we need the following new lemma:

Lemma 3. *Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : K \rightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma + \eta(b, a, m)]$. Then, for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have*

$$(9) \quad \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma + \eta(x, a, m))^-}^\alpha f(ma) - J_{(mb + \eta(x, b, m))^-}^\alpha f(mb) \right] \\ = \frac{\eta(x, a, m)^{\alpha+1}}{\eta(b, a, m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x, a, m)) dt \\ + \frac{\eta(x, b, m)^{\alpha+1}}{\eta(b, a, m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x, b, m)) dt,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler Gamma function.

Proof. Denote

$$I = \frac{\eta(x, a, m)^{\alpha+1}}{\eta(b, a, m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x, a, m)) dt \\ + \frac{\eta(x, b, m)^{\alpha+1}}{\eta(b, a, m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x, b, m)) dt.$$

Integrating by parts, we get

$$I = \frac{\eta(x, a, m)^{\alpha+1}}{\eta(b, a, m)} \left[(t^\alpha - 1) \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} \Big|_0^1 - \alpha \int_0^1 \frac{t^{\alpha-1} f(ma + t\eta(x, a, m))}{\eta(x, a, m)} dt \right] \\ + \frac{\eta(x, b, m)^{\alpha+1}}{\eta(b, a, m)} \left[(1 - t^\alpha) \frac{f(mb + t\eta(x, b, m))}{\eta(x, b, m)} \Big|_0^1 + \alpha \int_0^1 \frac{t^{\alpha-1} f(mb + t\eta(x, b, m))}{\eta(x, b, m)} dt \right]$$

$$\begin{aligned}
&= \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} \\
&\quad - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma+\eta(x, a, m))}^\alpha f(ma) - J_{(mb+\eta(x, b, m))}^\alpha f(mb) \right].
\end{aligned}$$

■

Remark 8. Clearly, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ in Lemma 3, we get (see [13], Lemma 3.1).

By using Lemma 3, one can extend to the following results.

Theorem 8. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have

$$\begin{aligned}
(10) \quad &\left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma+\eta(x, a, m))}^\alpha f(ma) - J_{(mb+\eta(x, b, m))}^\alpha f(mb) \right] \right| \\
&\leq \frac{Mm}{2} \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right] \left[\pi - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha + 1)} \right].
\end{aligned}$$

Proof. Using Lemma 3, MT_m -preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\begin{aligned}
&\left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma+\eta(x, a, m))}^\alpha f(ma) - J_{(mb+\eta(x, b, m))}^\alpha f(mb) \right] \right| \\
&\leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |t^\alpha - 1| |f'(ma + t\eta(x, a, m))| dt \\
&\quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |1 - t^\alpha| |f'(mb + t\eta(x, b, m))| dt \\
&\leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 (1 - t^\alpha) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)| \right] dt \\
&\quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 (1 - t^\alpha) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right] dt
\end{aligned}$$

$$\leq \frac{Mm}{2} \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right] \left[\pi - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha + 1)} \right].$$

■

Remark 9. In Theorem 8, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 3.2). Also, in Theorem 8, if we choose $\alpha = 1$, we get the inequality in Theorem 5.

Theorem 9. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have

$$(11) \quad \left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma + \eta(x, a, m))^-}^\alpha f(ma) - J_{(mb + \eta(x, b, m))^-}^\alpha f(mb) \right] \right| \leq M \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right] \left[\frac{\Gamma(p + 1) \Gamma(\frac{1}{\alpha})}{\alpha \Gamma(p + 1 + \frac{1}{\alpha})} \right]^{\frac{1}{p}}.$$

Proof. Suppose that $q > 1$. Using Lemma 3, MT_m -preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\begin{aligned} & \left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[J_{(ma + \eta(x, a, m))^-}^\alpha f(ma) - J_{(mb + \eta(x, b, m))^-}^\alpha f(mb) \right] \right| \\ & \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |t^\alpha - 1| |f'(ma + t\eta(x, a, m))| dt \\ & \quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |1 - t^\alpha| |f'(mb + t\eta(x, b, m))| dt \\ & \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
&\quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
&\leq M \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right] \left[\frac{\Gamma(p+1)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(p+1+\frac{1}{\alpha})} \right]^{\frac{1}{p}}.
\end{aligned}$$

■

Remark 10. In Theorem 9, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 3.5). Also, in Theorem 9, if we choose $\alpha = 1$, we get the inequality in Theorem 6.

Theorem 10. Let $A \subseteq \mathbb{R}_0$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a MT_m -preinvex function on $[ma, ma + \eta(b, a, m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have

$$\begin{aligned}
(12) \quad &\left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha+1)}{\eta(b, a, m)} \left[J_{(ma+\eta(x, a, m))^-}^\alpha f(ma) - J_{(mb+\eta(x, b, m))^-}^\alpha f(mb) \right] \right| \\
&\leq M \left(\frac{m}{2} \right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)} \right]^{\frac{1}{q}} \\
&\quad \times \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right].
\end{aligned}$$

Proof. Using Lemma 3, MT_m -preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left| \frac{\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)}{\eta(b, a, m)} \right|$$

$$\begin{aligned}
& - \frac{\Gamma(\alpha+1)}{\eta(b, a, m)} \left[J_{(ma+\eta(x, a, m))}^\alpha f(ma) - J_{(mb+\eta(x, b, m))}^\alpha f(mb) \right] \Bigg| \\
& \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |t^\alpha - 1| |f'(ma + t\eta(x, a, m))| dt \\
& \quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \int_0^1 |1 - t^\alpha| |f'(mb + t\eta(x, b, m))| dt \\
& \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 (1 - t^\alpha) |f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 (1 - t^\alpha) |f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\int_0^1 (1 - t^\alpha) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
& \quad + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left(\int_0^1 (1 - t^\alpha) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\int_0^1 (1 - t^\alpha) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& \leq M \left(\frac{m}{2} \right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\pi - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha+1)} \right]^{\frac{1}{q}} \\
& \quad \times \left[\frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right].
\end{aligned}$$

■

Remark 11. In Theorem 10, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get (see [13], Theorem 3.8). Also, in Theorem 10, if we choose $\alpha = 1$, we get Theorem 7.

5. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 9 (see [2]). A function $M : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$.
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\log(\beta) - \log(\alpha)}, \quad \alpha \neq \beta.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha \neq \beta.$$

8. The weighted p -power mean:

$$M_p \left(\begin{matrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n \\ u_1, & u_2, & \cdots, & u_n \end{matrix} \right) = \left(\sum_{i=1}^n \alpha_i u_i^p \right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \rightarrow \mathbb{R}_+$, which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Sections (-) for these means as follows:

Replace $\eta(x, y, m)$ with $\eta(x, y)$ and setting $\eta(a, x) = M(a, x)$ and $\eta(b, x) = M(b, x)$, $\forall x \in A$, for value $m = 1$ in (7), (8), (11) and (12) one can obtain the following interesting inequalities involving means:

$$(13) \quad \left| \frac{\eta(a, x)f(a) - \eta(b, x)f(b)}{M(a, b)} - \frac{1}{M(a, b)} \left[\int_a^{a+M(a, x)} f(u)du - \int_b^{b+M(b, x)} f(u)du \right] \right| \leq \frac{M}{(p+1)^{1/p}} \left(\frac{\pi}{2} \right)^{\frac{1}{q}} \left[\frac{M(a, x)^2 + M(b, x)^2}{M(a, b)} \right],$$

$$(14) \quad \left| \frac{\eta(a, x)f(a) - \eta(b, x)f(b)}{M(a, b)} - \frac{1}{M(a, b)} \left[\int_a^{a+M(a, x)} f(u)du - \int_b^{b+M(b, x)} f(u)du \right] \right| \leq M \left(\frac{1}{2} \right)^{1+\frac{1}{q}} (\pi)^{\frac{1}{q}} \left[\frac{M(a, x)^2 + M(b, x)^2}{M(a, b)} \right],$$

$$(15) \quad \left| \frac{M(a, x)^\alpha f(a) - M(b, x)^\alpha f(b)}{M(a, b)} - \frac{\Gamma(\alpha+1)}{M(a, b)} \left[J_{(a+M(a, x))}^\alpha f(a) - J_{(b+M(b, x))}^\alpha f(b) \right] \right| \leq M \left(\frac{\pi}{2} \right)^{\frac{1}{q}} \left[\frac{M(a, x)^{\alpha+1} + M(b, x)^{\alpha+1}}{M(a, b)} \right] \left[\frac{\Gamma(p+1)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(p+1+\frac{1}{\alpha})} \right]^{\frac{1}{p}},$$

$$(16) \quad \left| \frac{M(a, x)^\alpha f(a) - M(b, x)^\alpha f(b)}{M(a, b)} - \frac{\Gamma(\alpha+1)}{M(a, b)} \left[J_{(a+M(a, x))}^\alpha f(a) - J_{(b+M(b, x))}^\alpha f(b) \right] \right|$$

$$\leq M \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\pi - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha+1)} \right]^{\frac{1}{q}} \\ \times \left[\frac{M(a, x)^{\alpha+1} + M(b, x)^{\alpha+1}}{M(a, b)} \right].$$

Letting $M(a, x)$ and $M(b, x)$ equal to $A, G, H, P_r, I, L, L_p, M_p, \forall x \in A$ in (13), (14), (15) and (16), we get the inequalities involving means for a particular choices of a differentiable MT_1 -preinvex function f . The details are left to the interested reader.

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