M. A. PATHAN AND WASEEM A. KHAN

SOME NEW CLASSES OF GENERALIZED HERMITE-BASED APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

ABSTRACT. In this paper, we introduce a new class of generalized Apostol-Hermite-Euler polynomials and Apostol-Hermite-Genocchi polynomials and derive some implicit summation formulae by applying the generating functions. These results extend some known summations and identities of generalized Hermite-Euler polynomials studied by Dattoli et al, Kurt and Pathan.

KEY WORDS: Hermite polynomials, Apostol-Hermite-Bernoulli polynomials, Apostol-Hermite-Euler polynomials, Apostol-Hermite-Genocchi polynomials, summation formulae.

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1. Introduction

The 2-variable Kampe de Fériet generalization of the Hermite polynomials [2] reads

$$(1) \quad H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!}.$$

These polynomials are usually defined by the generating function

$$(2) \quad e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ when $y = -1$ and x is replaced by $2x$.

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their

familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see for details [1], [21], pp. 532-533 and [23], p. 61; see also [24] and the references cited therein):

$$(3) \quad \left(\frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!} \quad (|t| < 2\pi; 1^\alpha = 1)$$

$$(4) \quad \left(\frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)} \frac{t^n}{n!} \quad (|t| < \pi; 1^\alpha = 1)$$

and

$$(5) \quad \left(\frac{2t}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)} \frac{t^n}{n!} \quad (|t| < \pi; 1^\alpha = 1).$$

So that obviously the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given respectively by

$$B_n(x) = B_n^{(1)}(x), E_n(x) = E_n^{(1)}(x).$$

and

$$(6) \quad G_n(x) = G_n^{(1)}(x) \quad (n \in \mathbb{N}).$$

For the classical Bernoulli numbers B_n , the classical Euler numbers E_n and the classical Genocchi numbers G_n

$$B_n^1(0) = B_n(0) = B_n, \quad E_n^1(0) = E_n(0) = E_n$$

and

$$(7) \quad G_n^1(0) = G_n(0) = G_n,$$

respectively.

In particular, Luo and Srivastava [8, 9] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$; Luo [11, 12, 13] introduced the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ in [10, 15, 16, 17]. These polynomials are defined, respectively as follows.

Definition 1. The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$(8) \quad \left(\frac{t}{\lambda e^t - 1} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!},$$

$$(|t| < 2\pi, \text{ if } \lambda = 1; |t| < |\log \lambda|, \text{ if } \lambda \neq 1; 1^\alpha = 1)$$

with

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x; 1)$$

and

$$(9) \quad B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0; \lambda)$$

where $B_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Bernoulli numbers of order α .

Definition 2. The generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$(10) \quad \left(\frac{2}{\lambda e^t + 1} \right)^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < |\log(-\lambda)| < \pi, 1^\alpha = 1)$$

with

$$E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x; 1)$$

and

$$(11) \quad E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0; \lambda)$$

where $E_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Euler numbers of order α .

Definition 3. The generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$(12) \quad \left(\frac{2t}{\lambda e^t + 1} \right)^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < |\log(-\lambda)| < \pi, 1^\alpha = 1)$$

with

$$(13) \quad G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1), \quad G_n^{(\alpha)}(\lambda) = G_n^{(\alpha)}(0; \lambda)$$

where $G_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Genocchi numbers of order α .

Recently, Tremblay et al [25, 26] studied a new family of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α in the following form.

Definition 4. For arbitrary real or complex parameter α and for $a, c \in \mathbb{R}^+$, the generalized Apostol-Bernoulli polynomials $B_n^{[m-1, \alpha]}(x; a, c, \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log(a)| < 2\pi$, if $\lambda = 1$ or with $|t \log(a)| < |\log(\lambda)|$, if $\lambda \neq 1$ by means of the following generating function:

$$(14) \quad t^{m\alpha} [A(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1, \alpha]}(x; a, c, \lambda) \frac{t^n}{n!}$$

where

$$(15) \quad A(\lambda, a; t) = \left(\lambda a^t - \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!} \right)^{-1}.$$

It is easy to see that if we set $m = 1$, $a = c = e$ in (14), we arrive at the following

$$(16) \quad \left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; e, e, \lambda) \frac{t^n}{n!}, \quad |t| < 2\pi, \quad 1^\alpha = 1$$

with

$$(17) \quad B_n^{[0, \alpha]}(x, e, e; \lambda) = B_n^{(\alpha)}(x; \lambda).$$

Obviously when we set $\lambda = 1$ and $\alpha = 1$ in (17), we obtain

$$B_n^{[0, 1]}(x, e, e; 1) = B_n^{(\alpha)}(x)$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Definition 5. For arbitrary real or complex parameter α and for the $a, c \in \mathbb{R}^+$, the Apostol-Euler polynomials $E_n^{[m-1, \alpha]}(x; a, c, \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log a| < |t \log(-\lambda)|$ by means of the generating function

$$(18) \quad 2^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1, \alpha]}(x; a, c, \lambda) \frac{t^n}{n!}$$

where

$$(19) \quad B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!} \right)^{-1}.$$

It is easy to see that if we set $m = 1$, $a = c = e$ in (18), we arrive at the following

$$(20) \quad \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{[0,\alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad |t| < \pi, 1^\alpha = 1$$

with

$$E_n^{[0,\alpha]}(x, e, e; \lambda) = E_n^{(\alpha)}(x; \lambda).$$

Definition 6. For arbitrary real or complex parameter α and for the $a, c \in R^+$, the Apostol-Genocchi polynomials $G_n^{[m-1,\alpha]}(x; a, c, \lambda)$, $m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log a| < |t \log(-\lambda)|$ by means of the generating function

$$(21) \quad 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!}$$

where $B(\lambda, a; t)$ is given by equation (19). Obviously if we set $m = 1$, $a = c = e$ in (21), we obtain

$$(22) \quad \left(\frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{[0,\alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad |t| < \pi, 1^\alpha = 1$$

with

$$(23) \quad G_n^{[0,\alpha]}(x, e, e; \lambda) = G_n^{(\alpha)}(x; \lambda).$$

The popularity of Hermite, Bernoulli and Euler polynomials in number theory, combinatorics and mathematical physics is due in part to the papers of researchers in [3] to [5], [9] to [14], [18], [19], [20], [22] and their generalizations and various extensions which appeared in the literature. In this paper, we propose a further generalization of Apostol-Euler polynomials and Apostol-Genocchi polynomials and we give some properties involving them. For the new class of Apostol-Hermite-Euler polynomials ${}_H E_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ and Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$, we modify generating functions given by Tremblay et al [26] and derive some identities.

2. New classes of generalized Hermite-Based, Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

The following definitions provide a natural generalization of the Apostol-Bernoulli polynomials $B_n^{[m-1, \alpha]}(x; \lambda)$, $m \in N$ of order $\alpha \in \mathbb{C}$, Apostol-Euler polynomials $E_n^{[m-1, \alpha]}(x; \lambda)$, $m \in N$ of order $\alpha \in \mathbb{C}$ and Apostol-Genocchi polynomials $G_n^{[m-1, \alpha]}(x; \lambda)$, $m \in N$ of order $\alpha \in \mathbb{C}$.

Definition 7. For arbitrary real or complex parameter α and for $a, c \in \mathbb{R}^+$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$ $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log(a)| < |\log(-\lambda)|$, by means of the following generating function:

$$(24) \quad t^{m\alpha} [A(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

where $A(\lambda, a; t)$ is given by equation (15). It is easy to see that if we set $y=0$ in (24), we arrive at a recent result of Tremblay et al [26, p. 3, Eq. (1.8)] involving the generalized Apostol-Bernoulli polynomials

$$(25) \quad t^{m\alpha} [A(\lambda, a; t)]^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1, \alpha]}(x; a, c, \lambda) \frac{t^n}{n!}.$$

For $c = e$ in (24) gives

$$(26) \quad t^{m\alpha} [A(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1, \alpha]}(x, y; a, e, \lambda) \frac{t^n}{n!}.$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (24), we arrive at the following result

$$(27) \quad \left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{[0, \alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (|t| < 2\pi, 1^\alpha = 1)$$

which is a generating function for the generalized Apostol-Bernoulli polynomials of order α . Thus we have

$$(28) \quad B_n^{[0, \alpha]}(x; e, e, \lambda) = B_n^{[\alpha]}(x; \lambda).$$

Definition 8. For arbitrary real or complex parameter α and $a, c \in \mathbb{R}^+$, the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$,

$m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log a| < |\log(-\lambda)|$ by means of generating function

$$(29) \quad 2^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

where $B(\lambda, a; t)$ is given by equation (19). It is easy to see that if we set $y=0$ in (29), we arrive at a recent result of Tremblay et al [26, p.3, Eq.(2.1)] involving the generalized Apostol-Euler polynomials

$$(30) \quad 2^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1, \alpha]}(x; a, c, \lambda) \frac{t^n}{n!}.$$

For $c = e$ in (29) gives

$$(31) \quad 2^{m\alpha} [B(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, e, \lambda) \frac{t^n}{n!}.$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (29), we arrive at the following result

$$(32) \quad \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{[0, \alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (|t| < \pi, 1^\alpha = 1)$$

which is a generating function for the generalized Apostol-Euler polynomials of order α . Thus we have

$$(33) \quad E_n^{[0, \alpha]}(x; e, e, \lambda) = E_n^{[\alpha]}(x; \lambda).$$

Definition 9. For arbitrary real or complex parameter α and $a, c \in \mathbb{R}^+$, the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t \log a| < |\log(-\lambda)|$ by means of generating function

$$(34) \quad 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

where $B(\lambda, a; t)$ is given by equation (19). It is easy to see that if we set $y = 0$ in (34), we arrive at a recent result of Tremblay et al [26, p.5, Eq.(2.4)] involving the generalized Apostol-Genocchi polynomials

$$(35) \quad 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1, \alpha]}(x; a, c, \lambda) \frac{t^n}{n!}.$$

For $c = e$ in (34) gives

$$(36) \quad 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1, \alpha]}(x, y; a, e, \lambda) \frac{t^n}{n!}.$$

Obviously if we set $y = 0$, $m = 1$, $a = c = e$ in (34), we arrive at the following result

$$(37) \quad \left(\frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{[0, \alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (|t| < \pi, 1^\alpha = 1)$$

which is a generating function for the generalized Apostol-Genocchi polynomials of order α . Thus we have

$$(38) \quad G_n^{[0, \alpha]}(x; e, e, \lambda) = G_n^{[\alpha]}(x; \lambda).$$

The generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$ defined by (25) possess the following interesting properties. These are stated as Theorems 1 to 4 below:

Theorem 1. The generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$ and Apostol-Hermite-Bernoulli polynomials ${}_H B_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$, $\alpha \in N_0$ are related by

$$(39) \quad {}_H B_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) = \frac{(-1)^\alpha n!}{2^{m\alpha} (n - m\alpha)!} {}_H E_{n-m\alpha}^{[m-1, \alpha]}(x, y; a, c, \lambda)$$

or equivalently by

$$(40) \quad {}_H E_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) = \frac{(-2^m)^\alpha n!}{(n + m\alpha)!} {}_H B_{n+m\alpha}^{[m-1, \alpha]}(x, y; a, c, \lambda)$$

Proof. Considering the generating function (24)

$$\begin{aligned} t^{m\alpha} [A(-\lambda, a; t)]^\alpha e^{xt+yt^2} &= \sum_{n=0}^{\infty} {}_H B_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!} \\ \frac{(-1)^\alpha t^{m\alpha}}{2^{m\alpha}} t^{m\alpha} [B(\lambda, a; t)]^\alpha e^{xt+yt^2} &= \sum_{n=0}^{\infty} {}_H B_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!} \\ \sum_{n=0}^{\infty} {}_H B_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!} &= \frac{(-1)^\alpha}{2^{m\alpha}} \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^{n+m\alpha}}{n!} \end{aligned}$$

Replacing n by $n - m\alpha$ in R.H.S of above equation, we get

$$\sum_{n=0}^{\infty} {}_H B_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!} = \frac{(-1)^\alpha}{2^{m\alpha}} \sum_{n=0}^{\infty} {}_H E_{n-m\alpha}^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{(n - m\alpha)!}$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the result (38). Next consider the generating function (25)

$$2^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

$$\frac{(-1)^\alpha 2^{m\alpha}}{t^{m\alpha}} t^{m\alpha} [A(\lambda, a; t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!}$$

$$(-2^m)^\alpha \sum_{n=0}^{\infty} {}_H B_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^{n-m\alpha}}{n!} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!}$$

Replacing n by $n + m\alpha$ in L.H.S of above equation, we get

$$(-2^m)^\alpha \sum_{n=0}^{\infty} {}_H B_{n+m\alpha}^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^n}{(n + m\alpha)!}$$

$$= \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha]}(x, y; a, c, -\lambda) \frac{t^n}{n!}$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the result (40). ■

For $y = 0$ in equation (39) and (40), the result reduces to known result Tremblay et al [26](see also [6]).

Theorem 2. Let $a, b, c \in \mathbb{R}^+$, α an arbitrary complex number and $m \in \mathbb{N}$. Then the generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$ satisfy the following relations

$$(41) \quad {}_H E_n^{[m-1, \alpha+\beta]}(x+u, y; a, c, \lambda)$$

$$= \sum_{k=0}^n \binom{n}{k} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_{n-k}^{[m-1, \beta]}(u, a, c, \lambda)$$

Proof. Considering the generating function (29) as

$$2^{m\alpha} [B(\lambda, a; t)]^{\alpha+\beta} c^{(x+u)t+yt^2}$$

$$= \sum_{k=0}^{\infty} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^k}{k!} \sum_{n=0}^{\infty} E_n^{[m-1, \beta]}(u, a, c, \lambda) \frac{t^n}{n!}$$

$$(42) \quad \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha+\beta]}(x+u, y; a, c, \lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_n^{[m-1, \beta]}(u, a, c, \lambda) \frac{t^{n+k}}{n!k!}$$

Replacing n by $n - k$ in R.H.S of above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_H E_n^{[m-1, \alpha+\beta]}(x+u, y; a, c, \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_{n-k}^{[m-1, \beta]}(u, a, c, \lambda) \frac{t^n}{n!} \end{aligned}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (41).

For $y = 0$ in equation (41), the result reduces to known result of Tremblay et al [26]. ■

Theorem 3. *The generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1, \alpha]}$ ($x, y; a, c, \lambda$) satisfies the following recurrence relation*

$$\begin{aligned} (43) \quad & \lambda {}_H E_n^{[m-1, \alpha]}(x+1, y; a, c, \lambda) + {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \\ &= 2 \sum_{k=0}^n \binom{n}{k} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_{n-k}^{(-1)}(0, a, \lambda). \end{aligned}$$

Proof. Let

$$\begin{aligned} & \lambda {}_H E_n^{[m-1, \alpha]}(x+1, y; a, c, \lambda) + {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \\ &= (2^m a^t)^\alpha c^{xt+yt^2} (\lambda c^t + 1) \\ &= 22^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} \left(\frac{2}{\lambda a^t + 1} \right)^{(-1)} \\ &= 2 \sum_{k=0}^{\infty} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^k}{k!} \sum_{n=0}^{\infty} E_n^{(-1)}(0, a; \lambda) \frac{t^n}{n!} \\ &= 2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_n^{(-1)}(0, a; \lambda) \frac{t^{n+k}}{n!k!}. \end{aligned}$$

Replacing n by $n - k$ in R.H.S of above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\lambda {}_H E_n^{[m-1, \alpha]}(x+1, y; a, c, \lambda) + {}_H E_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2 \sum_{k=0}^n {}_H E_k^{[m-1, \alpha]}(x, y; a, c, \lambda) E_{n-k}^{(-1)}(0, a; \lambda) \right) \frac{t^n}{(n-k)!k!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we get the result (43).

For $y = 0$ in equation (43), the result reduces to known result Tremblay et al [26]. ■

Remark 1. Setting $y = 0$, $m = 1$ and $b = c = e$ in (43) and using (29), we find

$$(44) \quad \lambda E_n^\alpha(x+1; \lambda) + E_n^\alpha(x; \lambda) = 2 \sum_{k=0}^n \binom{n}{k} E_k^{(\alpha)}(x; \lambda) E_{n-k}^{(-1)}(0; \lambda)$$

Using the well known result (see [9])

$$(45) \quad E_n^{\alpha+\beta}(x+y; \lambda) = 2 \sum_{k=0}^n \binom{n}{k} E_k^{(\alpha)}(x; \lambda) E_{n-k}^{(\beta)}(y; \lambda)$$

equation (44) becomes the familiar relation for the generalized Apostol-Euler polynomials (see [9])

$$(46) \quad \lambda E_n^\alpha(x+1; \lambda) + E_n^\alpha(x; \lambda) = 2E_n^{(\alpha-1)}(x; \lambda).$$

Theorem 4. Let $a, b \in R$, α and β arbitrary complex numbers, $m \in N$. Then the generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ satisfy the following relation

$$(47) \quad {}_H E_n^{[\alpha+\beta, m-1]}(x_1+x_2, y_1+y_2; a, c, \lambda) \\ = \sum_{k=0}^n \binom{n}{k} {}_H E_{n-k}^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) {}_H E_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda).$$

Proof. Use definition (25) to get

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_H E_n^{[\alpha+\beta, m-1]}(x_1+x_2, y_1+y_2; a, c, \lambda) \frac{t^n}{n!} \\ &= 2^{m\alpha} [B(\lambda, a; t)]^{\alpha+\beta} c^{(x_1+x_2)t+(y_1+y_2)t^2} \\ &= \left(\sum_{n=0}^{\infty} {}_H E_n^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} {}_H E_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^k}{k!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H E_n^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) {}_H E_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^{n+k}}{n!k!}. \end{aligned}$$

Replacing n by $n-k$ in R.H.S of above equation, we get

$$\begin{aligned} L.H.S. &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \right) {}_H E_{n-k}^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) \\ &\quad \times {}_H E_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^n}{(n-k)!k!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the desired result (47). For $m=1$ in equation (47), the result reduces to a known result of Gaboury et al [6., p.7, Eq. 3.6]. ■

3. New classes of Apostol-Hermite-Genocchi polynomials

Now let us shift our focus on some interesting properties for the generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ defined by (35). These are stated as Theorem 5 to Theorem 9 below:

Theorem 5. *The generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ and the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ are related by*

$$(48) \quad {}_HG_n^{[\alpha, m-1]}(x, y; a, c, -\lambda) = (-2^m)^\alpha {}_HB_n^{[\alpha, m-1]}(x, y; a, c, \lambda), \quad (\alpha \in \mathbb{C})$$

or equivalently

$$(49) \quad {}_HG_n^{[\alpha, m-1]}(x, y; a, c, \lambda) = \frac{n!}{(n - m\alpha)!} {}_HE_{n-m\alpha}^{[\alpha, m-1]}(x, y; a, c, \lambda),$$

$n, \alpha, m \in \mathbb{N}, n \geq m\alpha$.

Proof. Using definition (24)

$$\begin{aligned} t^{m\alpha} [A(\lambda, a; t)]^\alpha c^{xt+yt^2} &= \sum_{n=0}^{\infty} {}_HB_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \\ t^{m\alpha} [B(-\lambda, a; t)]^\alpha c^{xt+yt^2} &= (-2^m)^\alpha \sum_{n=0}^{\infty} {}_HB_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \\ \sum_{n=0}^{\infty} {}_HG_n^{[\alpha, m-1]}(x, y; a, c, -\lambda) \frac{t^n}{n!} &= (-2^m)^\alpha \sum_{n=0}^{\infty} {}_HB_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the desired result (48). Next using definition (26)

$$\begin{aligned} 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} &= \sum_{n=0}^{\infty} {}_HG_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \\ 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} &= \sum_{n=0}^{\infty} {}_HG_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \end{aligned}$$

$$\sum_{n=0}^{\infty} {}_H E_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^{n+m\alpha}}{n!} = \sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

Replace n by $n - m\alpha$ in L.H.S of the above equation, we get

$$\sum_{n=m\alpha}^{\infty} {}_H E_{n-m\alpha}^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{(n - m\alpha)!} = \sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!}$$

Comparing the coefficients of t on both sides, we get the result (49).

For $y = 0$ in equation (48) and (49), the result reduces to known result of Tremblay et al [26]. ■

Theorem 6. Let $a, c \in \mathbb{R}$, α an arbitrary complex number and $m \in \mathbb{N}$, then the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ satisfy the following relations

$$(50) \quad {}_H G_n^{[\alpha+\beta, m-1]}(x+u, y; a, c, \lambda) = \sum_{k=0}^n \binom{n}{k} {}_H G_k^{[m-1, \alpha]}(x, y; a, c, \lambda) G_{n-k}^{[m-1, \beta]}(u, a, c, \lambda).$$

Proof. Using definition (26)

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_H G_n^{[\alpha+\beta, m-1]}(x+u, y; a, c, \lambda) \frac{t^n}{n!} \\ &= 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\beta c^{ut} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H G_k^{[\alpha, m-1]}(x, y; a, c, \lambda) G_n^{[\beta, m-1]}(u, a, c, \lambda) \frac{t^{n+k}}{n!}. \end{aligned}$$

Replacing n by $n - k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n {}_H G_k^{[\alpha, m-1]}(x, y; a, c, \lambda) G_{n-k}^{[\beta, m-1]}(u, a, c, \lambda) \right) \frac{t^n}{(n-k)!k!}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (50).

For $y = 0$ in equation (50), the result reduces to known result of Tremblay et al [26]. ■

Theorem 7. The generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[m-1, \alpha]}(x, y; a, c, \lambda)$ satisfy the following recurrence relation

$$(51) \quad \lambda {}_H G_n^{[m-1, \alpha]}(x+1, y; a, c, \lambda) + {}_H G_n^{[m-1, \alpha]}(x, y; a, c, \lambda) = 2n \sum_{k=0}^n \binom{n-1}{k} {}_H G_k^{[m-1, \alpha]}(x, y; a, c, \lambda) G_{n-1-k}^{(-1)}(0, a, \lambda)$$

Proof. Let us write

$$\begin{aligned}
 L.H.S &= 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} (\lambda a^t + 1) \\
 &= 2t 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} \left(\frac{2t}{\lambda a^t + 1} \right)^{(-1)} \\
 &= 2t \sum_{k=0}^{\infty} {}_H G_k^{[m-1, \alpha]}(x, y; a, c, \lambda) \frac{t^k}{k!} \sum_{n=0}^{\infty} G_n^{(-1)}(0, a; \lambda) \frac{t^n}{n!} \\
 &= 2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H G_k^{[m-1, \alpha]}(x, y; a, c, \lambda) G_n^{(-1)}(0, a; \lambda) \frac{t^{n+k+1}}{n!k!}.
 \end{aligned}$$

Replacing n by $n - k - 1$ in R.H.S of above equation, we get

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \left(\lambda {}_H G_n^{[m-1, \alpha]}(x+1, y; a, c, \lambda) + {}_H G_n^{[m-1, \alpha]}(x, y; a, c, \lambda) \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(2n \sum_{k=0}^n {}_H G_k^{[m-1, \alpha]}(x, y; a, c, \lambda) G_{n-1-k}^{(-1)}(0, a; \lambda) \right) \frac{t^n}{(n-1-k)!k!}.
 \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we get the result (51).

For $y = 0$ in equation (51), the result reduces to known result of Tremblay et al [26]. ■

Remark 2. Setting $y = 0$, $m = 1$ and $b=c=e$ in (51) and using (34), we find

$$(52) \quad \lambda G_n^\alpha(x+1; \lambda) + G_n^\alpha(x; \lambda) = 2n \sum_{k=0}^n \binom{n-1}{k} G_k^{(\alpha)}(x; \lambda) E_{n-1-k}^{(-1)}(0; \lambda)$$

Using the well known result (see [9])

$$(53) \quad G_n^{\alpha+\beta}(x+y; \lambda) = \sum_{k=0}^n \binom{n}{k} G_k^{(\alpha)}(x; \lambda) G_{n-k}^{(\beta)}(y; \lambda)$$

equation (52) becomes the familiar relation for the generalized Apostol-Genocchi polynomials (see [9])

$$(54) \quad \lambda G_n^\alpha(x+1; \lambda) + G_n^\alpha(x; \lambda) = 2n G_{n-1}^{(\alpha-1)}(x; \lambda).$$

Theorem 8. Let $a, b, c, p, q \in R$, α an arbitrary complex number and $m \in N$, then the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ satisfy the following relation

$$\begin{aligned}
 (55) \quad & {}_H G_n^{[\alpha+\beta, m-1]}(px, qy; a, c, \lambda) \\
 &= n! \sum_{k=0}^n \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} {}_H G_{n-k}^{[m-1, \alpha]}(x, y; a, c, \lambda) ((p-1)x \ln c)^k \\
 &\quad \times ((q-1)y \ln c)^j \frac{1}{(n-k-2j)!j!}
 \end{aligned}$$

Proof. Using definition (26)

$$\begin{aligned}
 & \sum_{n=0}^{\infty} {}_H G_n^{[\alpha+\beta, m-1]}(px, qy; a, c, \lambda) \frac{t^n}{n!} \\
 &= 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{xt+yt^2} c^{(p-1)xt} c^{(q-1)yt^2} \\
 &= \left(\sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \right) \\
 &\quad \times \left(\sum_{k=0}^{\infty} ((p-1)x \ln c)^k \frac{t^k}{k!} \right) \left(\sum_{j=0}^{\infty} ((q-1)y \ln c)^j \frac{t^{2j}}{j!} \right) \\
 &= \left(\sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \right) \\
 &\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} ((p-1)x \ln c)^k ((q-1)y \ln c)^j \frac{t^{k+2j}}{k!j!}
 \end{aligned}$$

Replacing k by $k-2j$ in above equation, we have

$$\begin{aligned}
 L.H.S. &= \left(\sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) \frac{t^n}{n!} \right) \\
 &\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} ((p-1)x \ln c)^{k-2j} ((q-1)y \ln c)^j \frac{t^k}{(k-2j)!j!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} {}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda) ((p-1)x \ln c)^{k-2j} \\
 &\quad \times ((q-1)y \ln c)^j \frac{t^{n+k}}{(k-2j)!j!n!}
 \end{aligned}$$

Replacing n by $n - k$ in above equation, we have

$$\begin{aligned} L.H.S. &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} {}_H G_{n-k}^{[\alpha, m-1]}(x, y; a, c, \lambda) ((p-1)x \ln c)^{k-2j} \\ &\quad \times ((q-1)y \ln c)^j \frac{t^n}{(n-k-2j)!j!k!} \end{aligned}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (3.8). For $m = 1$ in equation (3.8), the result reduces to a known result of Gaboury et al [6, p.10., Eq.3.16]. ■

Theorem 9. Let $a, b \in R$, α and β arbitrary complex number $m \in N$ then the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[\alpha, m-1]}(x, y; a, c, \lambda)$ satisfy the following relation

$$\begin{aligned} &{}_H G_n^{[\alpha+\beta, m-1]}(x_1 + x_2, y_1 + y_2; a, c, \lambda) \\ &= \sum_{k=0}^n \binom{n}{k} {}_H G_{n-k}^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) {}_H G_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \end{aligned}$$

Proof. Use definition (25) to get

$$\begin{aligned} L.H.S &= 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^{\alpha+\beta} c^{(x_1+x_2)t+(y_1+y_2)t^2} \\ &= \left(\sum_{n=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) \frac{t^n}{n!} \right) \\ &\quad \times \left(\sum_{k=0}^{\infty} {}_H G_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^k}{k!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_H G_n^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) {}_H G_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^{n+k}}{n!k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \\ &\quad \times {}_H G_{n-k}^{[\alpha, m-1]}(x_1, y_1; a, c, \lambda) {}_H G_k^{[\beta, m-1]}(x_2, y_2; a, c, \lambda) \frac{t^n}{(n-k)!k!} \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the desired result (56). For $m = 1$ in equation (56), the result reduces to a known result of Gaboury et al [6, p.7, Eq. 3.6]. ■

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M. A. PATHAN

CENTRE FOR MATHEMATICAL AND STATISTICAL SCIENCES (CMSS)

KFRI, PEECHI P.O., THRISSUR, KERALA-680653, INDIA

e-mail: mapathan@gmail.com

WASEEM A. KHAN

DEPARTMENT OF MATHEMATICS

INTEGRAL UNIVERSITY

LUCKNOW-226026, INDIA

e-mail: waseem08_khan@rediffmail.com

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