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SOLVABILITY OF SEQUENCE SPACES EQUATIONS OF THE FORM $(E_a)_\Delta + F_x = F_b$

ABSTRACT. Given any sequence $a = (a_n)_{n \geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n \geq 1}$ such that $y/a = (y_n/a_n)_{n \geq 1} \in E$; in particular, $s_a^{(c)}$ denotes the set of all sequences y such that y/a converges. For any linear space F of sequences, we have $F_x = F_b$ if and only if x/b and $b/x \in M(F, F)$. The question is: what happens when we consider the perturbed equation $\mathcal{E} + F_x = F_b$ where $\mathcal{E}$ is a special linear space of sequences? In this paper we deal with the perturbed sequence spaces equations (SSE), defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ_p , ($p > 1$) and Δ is the operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$ with the convention $y_0 = 0$. For $E = c_0$ the previous perturbed equation consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the next statement. The condition $y_n/b_n \rightarrow L_1$ holds if and only if there are two sequences u, v with $y = u + v$ such that $\Delta_n u/a_n \rightarrow 0$ and $v_n/x_n \rightarrow L_2$ ($n \rightarrow \infty$) for all y and for some scalars L_1 and L_2 . Then we deal with the resolution of the equation $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$, or s_1 , and give applications to particular classes of (SSE).

KEY WORDS: BK space, spaces of strongly bounded sequences, sequence spaces equations, sequence spaces equations with operator.

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1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n \geq 1}$, ℓ_∞ , c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also $\ell^p = \{y \in \omega : \sum_{k=1}^\infty |y_k|^p < \infty\}$ for $1 \leq p < \infty$. If $y, z \in \omega$, then we write $yz = (y_n z_n)_{n \geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n \geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular

$1/u = e/u$, where $e = \mathbf{1}$ is the sequence with $e_n = 1$ for all n . Finally, if $a \in U^+$ and E is any subset of ω , then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω . Then the set $M(E, F) = \{y \in \omega : yz \in F \text{ for all } z \in E\}$ is called the *multiplier space of E and F* . In [1], the sets s_a , s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty, c_0, c$, respectively. In [3] the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0 , or $s^{(c)}$. Then in [6] the solvability was determined of sequences spaces inclusion equations $G_b \subset E_a + F_b$ where $E, F, G \in \{s^0, s^{(c)}, s\}$ and some applications were given to sequence spaces inclusions with operators.

In this paper we deal with the solvability of perturbed equations defined as follows. Let F be any linear space of sequences, and b be a positive sequence. It is known that the solutions of the equation $F_x = F_b$ where x is the unknown, are determined by $x \in cl^{M(F,F)}(b)$. Then we consider the perturbed equation $\mathcal{E} + F_x = F_b$, where $\mathcal{E}$ is a particular linear space of sequences. For example, the solutions of the equation $c_x = c$ are determined by $\lim_{n \rightarrow \infty} x_n = L > 0$. Then the perturbed equation defined by $c_a + c_x = c$, has the same solutions if and only if $a_n \rightarrow 0$ as n tends to infinity; then if $a_n \rightarrow l > 0$ as n tends to infinity the set of all its solutions is equal to c ; finally, if $a \notin c$ the perturbed equation has no solutions, (cf. [7]). Here we extend some results given in [12], [6], [4], [5], [11], [7], [9], [10]. In [11] for given sequences a and b was determined the set of all positive sequences x for which $y_n/b_n \rightarrow l$ if and only if there are sequences u and v for which $y = u + v$ and $u_n/a_n \rightarrow 0$, $v_n/x_n \rightarrow l'$ ($n \rightarrow \infty$) for all y and for some scalars l and l' . This statement is equivalent to the sequence spaces equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In [7] we determined the set of all $x \in U^+$ such that for every sequence y , we have $y_n/b_n \rightarrow l$ if and only if there are sequences u and v with $y = u + v$ and $|u_n/a_n|^{1/n} \rightarrow 0$ and $v_n/x_n \rightarrow l'$ ($n \rightarrow \infty$) for some scalars l and l' . This statement means $\Gamma_a + s_x^{(c)} = s_b^{(c)}$, where Γ is the set of all entire sequences. So we are led to deal with *special sequence spaces equations (SSE)*, (*resp. sequence spaces inclusion equations (SSIE)*), which are determined by an identity, (*resp. inclusion*), for which each term is a *sum* or a *sum of products of sets of the form* $(E_a)_T$ and $(E_{f(x)})_T$ where f maps U^+ to itself, E is a linear space of sequences, x is the unknown and T is a triangle. It can be found in [5] a solvability of the (SSE) $E_a + \left(s_x^{(c)}\right)_{B(r,s)} = s_x^{(c)}$ where $E = s, s^0$, or $s^{(c)}$ and x is the unknown. In [11] we determined the sets of all positive sequences x that satisfy each of the systems $s_a^0 + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + (c_x)_\Delta = c_b$, $c_x \supset c_b$. Then a resolution can be found of the (SSE) with operators defined by $(E_a)_{C(\lambda)D_\tau} + (c_x)_{C(\mu)D_\tau} = c_b$ with $E = c_0$, or ℓ_∞ . Recently in [8] a study can be found on the (SSE) with operator

$(E_a)_{C(\lambda)C(\mu)} + (E_x)_{C(\lambda\sigma)C(\mu)} = E_b$, where $b \in \widehat{C}_1$ and E is any of the sets ℓ_∞ , or c_0 . For $E = c_0$ the resolution of this equation consists in determining the set of all $x \in U^+$ such that for every sequence y the condition $y_n/b_n \rightarrow 0$ ($n \rightarrow \infty$) holds if and only if there are $u, v \in \omega$ such that $y = u + v$ and

$$(1) \quad \frac{1}{\lambda_n a_n} \sum_{k=1}^n \left(\frac{1}{\mu_k} \sum_{i=1}^k u_i \right) \rightarrow 0 \quad \text{and} \\ \frac{1}{\lambda_n \sigma_n x_n} \sum_{k=1}^n \left(\frac{1}{\mu_k} \sum_{i=1}^k v_i \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

There is also a resolution of the (SSE) $(s_a)_{(C(\lambda)D_\tau)} + (s_x^0)_{(C(\mu)D_\tau)} = s_b^0$.

In this paper we deal with some classes of (SSE) with the operators of the form $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ^p , ($p > 1$) and of the form $(E_a)_\Delta + s_x^0 = s_b^0$.

This paper is organized as follows. In Section 2 we recall some definitions and results on sequence spaces and matrix transformations. In Section 3 we recall general results on the multiplier $M(E, F)$ of some sequence spaces. In Section 4 we recall some results on the solvability of the equation $E + Fx = F_b$ in the general case. In Section 5 we determine the sets $M((E_a)_\Delta, F)$ and we deal with the (SSIE) $F_b \subset (E_a)_\Delta + Fx$. In Section 6 we apply the previous results to solve the (SSE) $(E_a)_\Delta + c_x = c_b$ where $E = c_0$, or ℓ^p , ($p > 1$). In Section 7 we apply the results of Section 6 to solve special (SSE) of the form $(E_a)_\Delta + c_x = c_b$. Finally in Section 8 we deal with the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$, or s_1 .

2. Preliminaries and notations

An FK space is a *complete metric space*, for which convergence implies *coordinatewise convergence*. A BK space is a Banach space of sequences that is an FK space. A BK space E is said to have AK if for every sequence $y = (y_n)_n \in E$, then $y = \lim_{p \rightarrow \infty} \sum_{k=1}^p y_k e^{(k)}$, where $e^{(k)} = (0, \dots, 1, \dots)$, 1 being in the k -th position.

For any given infinite matrix $A = (a_{nk})_{n,k}$ we define the operators $A_n = (a_{nk})_{k \geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^\infty a_{nk} y_k$, where $y = (y_k)_{k \geq 1}$, and the series are assumed convergent for all n . So we are led to the study of the operator A defined by $Ay = (A_n y)_n$ mapping between sequence spaces. When A maps E into F , where E and F are subsets of ω , we write $A \in (E, F)$, (cf. [13]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all bounded linear operators L mapping in E , with norm $\|L\| = \sup_{y \neq 0} (\|L(y)\|_E / \|y\|_E)$ satisfies the identity $\mathcal{B}(E) = (E, E)$. We will use the operator Δ of the first difference defined by $\Delta_n y = y_n - y_{n-1}$

for $n \geq 1$ with the convention $y_0 = 0$. It is well known that the operator Σ defined by $\Sigma_n y = \sum_{k=1}^n y_k$ for all n , is the inverse of Δ , that is, $\Delta(\Sigma y) = \Sigma(\Delta y) = y$ for all y . Let $U^+ \subset \omega$ be the set of all sequences $\mathbf{u} = (u_n)_n$ with $u_n > 0$ for all n . Then for any given sequence $u = (u_n)_n \in \omega$ we define the infinite diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n . It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have

$$E_u = D_u E = \{y = (y_n)_{n \geq 1} \in \omega : y/u \in E\}.$$

We will use the sets s_a^0 , $s_a^{(c)}$, s_a and ℓ_a^p defined as follows (cf. [1]). For given $a \in U^+$ and $p \geq 1$ we put $D_a c_0 = s_a^0$, $D_a c = s_a^{(c)}$, $D_a \ell_\infty = s_a$, and $D_a \ell_p = \ell_a^p$. Each of the spaces $D_a E$, where $E \in \{c_0, c, \ell_\infty, \ell^p\}$ with $p > 1$, is a *BK space*, and we have $\|y\|_{s_a} = \sup_{n \geq 1} (|y_n|/a_n)$, and $\|y\|_{\ell_a^p} = (\sum_{k=1}^\infty (|y_k|/a_k)^p)^{1/p}$. Then s_a^0 and ℓ_a^p have *AK*. If $a = (r^n)_n$ with $r > 0$, we write s_r , s_r^0 and $s_r^{(c)}$ for the sets s_a , s_a^0 and $s_a^{(c)}$ respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if

$$(2) \quad \sup_n \left(\sum_{k=1}^\infty |\mathbf{a}_{nk}| \right) < \infty.$$

We will also use the well-known characterizations of (c_0, c_0) and (c_0, c) . We have $A \in (c_0, c_0)$ if and only if (2) holds and $\lim_{n \rightarrow \infty} \mathbf{a}_{nk} = 0$ for all k ; and we have $A \in (c_0, c)$ if and only if (2) holds and $\lim_{n \rightarrow \infty} \mathbf{a}_{nk} = l_k$ for some scalar l_k and for all k . For any subset F of ω , we write $F_A = \{y \in \omega : Ay \in F\}$. Let $cs = c_\Sigma$ denote the set of all convergent series. For any subset E of ω we write $AE = \{y \in \omega : y = Ax \text{ for some } x \in \omega\}$. We will use the well-known property, stated as follows. For any given triangles T and T' , we have $T' \in (E_T, F)$ if and only if $T'T^{-1} \in (E, F)$ for any subsets $E, F \subset \omega$. It is also well known that $A \in (E, F_T)$ if and only if $TA \in (E, F)$.

3. The multipliers of some sets and matrix transformations

3.1. The multipliers of classical sets

First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω , we then write $yz = (y_n z_n)_{n \geq 1}$ and

$$M(E, F) = \{y \in \omega : yz \in F \text{ for all } z \in E\},$$

$M(E, F)$ is called the *multiplier space of E and F* . In the following we will use the next elementary results.

Lemma 1. *Let $E, \tilde{E}, F$ and $\tilde{F}$ be arbitrary subsets of ω . Then*

- (i) $M(E, F) \subset M(\tilde{E}, F)$ for all $\tilde{E} \subset E$,
- (ii) $M(E, F) \subset M(E, \tilde{F})$ for all $F \subset \tilde{F}$.

Lemma 2. *Let $a, b \in U^+$ and let E and F be two subsets of ω . Then $D_a E \subset D_b F$ if and only if $a/b \in M(E, F)$.*

Lemma 3. *Let $a, b \in U^+$ and let $E, F \subset \omega$. Then $A \in (D_a E, D_b F)$ if and only if $D_{1/b} A D_a \in (E, F)$.*

By [14, Lemma 3.1, p. 648] and [16, Example 1.28, p. 157], we obtain the next result.

Lemma 4. *We have*

- (i) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ and $M(c, c) = c$;
- (ii) $M(\chi, \ell_\infty) = M(c_0, \chi') = \ell_\infty$ for $\chi, \chi' = c_0, c$, or ℓ_∞ .

4. On the (SSE) $E + F_x = F_b$

In this section we apply the previous results to the solvability of the (SSE) $E + F_x = F_b$ with $\mathbf{1} \in F$.

4.1. Regular sequence spaces equations

For $b \in U^+$ and for any subset F of ω , we denote by $cl^F(b)$ the equivalent class for the equivalence relation R_F defined by

$$x R_F y \text{ if } D_x F = D_y F \text{ for } x, y \in U^+.$$

It can easily be seen that $cl^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F, F)$ and $b/x \in M(F, F)$, (cf. [11]). We then have $cl^F(b) = cl^{M(F, F)}(b)$. For instance $cl^c(b)$ is the set of all $x \in U^+$ such that $D_x c = D_b c$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim C b_n$ ($n \rightarrow \infty$) for some $C > 0$. In [11] we denote by $cl^\infty(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^\infty(b)$ is the set of all $x \in U^+$ such that $K_1 \leq x_n/b_n \leq K_2$ for all n and for some $K_1, K_2 > 0$.

Let X and Y be two linear spaces of sequences. Then the sum of X and Y defined by $Z = X + Y = \{x + y : x \in X \text{ and } y \in Y\}$, is a linear space of sequences. Let b be a positive sequence and F be a linear subspace of ω . As we have seen above the solutions of the equation $F_x = F_b$ are defined by $x \in cl^F(b)$. Then the question is: what are the solutions of the perturbed equation $E + F_x = F_b$, where E is a linear space of sequences? In this way we are led to consider the set $\mathcal{S}(E, F) = \{x \in U^+ : E + F_x = F_b\}$, where $b \in U^+$, and E is a linear subspace of ω .

Definition 1. We say that $\mathcal{S}(E, F)$, (or the equation $E + F_x = F_b$), is regular if

$$\mathcal{S}(E, F) = \begin{cases} cl^{M(F, F)}(b), & \text{if } 1/b \in M(E, F), \\ \emptyset, & \text{if } 1/b \notin M(E, F). \end{cases}$$

Note that $E + F_x = F_b$ is not regular in general. Indeed for $E = F = \ell_\infty$ we have $M(\ell_\infty, \ell_\infty) = \ell_\infty$ and if $1/b \in \ell_\infty \setminus c_0$ and $s_a = s_1$ we have $\mathcal{S}(\ell_\infty, \ell_\infty) = s_b \cap U^+ \neq cl^\infty(b)$, (cf. [12, Theorem 11, pp. 916-917]). In particular the solutions of the (SSE) $\ell_\infty + s_x = \ell_\infty$ are determined by $0 < x_n \leq M$ for all n and for some $M > 0$. It is interesting to notice that by [7, Theorem 5.2, p. 108], the (SSE) $c + c_x = c_b$ is not regular, since $1/b \in c \setminus c_0$ implies $\mathcal{S}(c, c) = c_b$.

In the following we will use the condition

$$(3) \quad \chi \subset \chi(D_\alpha) \quad \text{for all } \alpha \in c(1),$$

where $\chi \subset \omega$ is any linear space, and $c(1)$ is the set of all sequences that tend to 1. It can easily be seen that this condition is true for any of the spaces $F = c, s_1$. To state the next results we also need the next conditions:

$$(4) \quad \mathbf{1} \in F,$$

$$(5) \quad F \subset M(F, F).$$

We then recall the next result which is a direct consequence of [7, Theorem 5.1, pp. 106-107].

Lemma 5. Let $b \in U^+$ and let E, F be two linear subspaces of ω . We assume F satisfies the conditions in (3), (4), (5), and that $M(E, F) \subset M(E, c_0)$. Then the set $\mathcal{S}(E, F)$ is regular.

5. Some results on the multiplier $M((E_a)_\Delta, F)$ and on the (SSIE) $F_b \subset (E_a)_\Delta + F_x$

In this section we explicitly calculate the multiplier $M((E_a)_\Delta, F)$ where $E = c_0$, or ℓ_p and $F = c_0, c$, or s_1 . Then we deal with the (SSIE) defined by $F_b \subset (E_a)_\Delta + F_x$ where E and F are linear spaces of sequences, and $E \subset s_1$ and $c_0 \subset F \subset s_1$.

In the following we will use the factorable matrix $D_\alpha \Sigma D_\beta$, with α and $\beta \in \omega$ defined by $(D_\alpha \Sigma D_\beta)_{nk} = \alpha_n \beta_k$ for $k \leq n$ for all n , the other entries being equal to zero.

5.1. The mutipliers $M((E_a)_\Delta, F)$ where $E = c_0$, or ℓ_p and $F = c_0$, c , or s_1

Lemma 6. *Let $a \in U^+$ and let $p > 1$. Then*

(i) *the condition $a \notin cs$ implies*

$$(6) \quad M((s_a^0)_\Delta, F) = s\left(\frac{1}{\sum_{k=1}^n a_k}\right)_n \quad \text{for } F = c_0, c, \text{ or } s_1.$$

(ii) *The condition $a^q \notin cs$ where $q = p/(p-1)$ implies*

$$(7) \quad M((\ell_a^p)_\Delta, F) = s\left((\sum_{k=1}^n a_k^q)^{-1/q}\right)_n \quad \text{for } F = c_0, c, \text{ or } s_1.$$

Proof. (i) We have $\alpha \in M((s_a^0)_\Delta, c_0)$ if and only if $D_\alpha \Sigma D_a \in (c_0, c_0)$. By the characterization of (c_0, c_0) we have

$$(8) \quad |\alpha_n| \sum_{k=1}^n a_k \leq K \text{ for all } n \text{ and for some } K > 0$$

and

$$(9) \quad \alpha \in c_0.$$

But since $a \notin cs$ the condition in (8) implies (9) and we have $\alpha \in M((s_a^0)_\Delta, c_0)$ if and only if (8) holds. This shows the identity in (6) for $F = c_0$. In a similar way the identity (6) for $F = s_1$ can easily be shown. From the inclusions $M((s_a^0)_\Delta, c_0) \subset M((s_a^0)_\Delta, c) \subset M((s_a^0)_\Delta, s_1)$, we conclude that the identity in (6) holds for $F = c$.

(ii) We have $\alpha \in M((\ell_a^p)_\Delta, c_0)$ if and only if $D_\alpha \Sigma D_a \in (\ell^p, c_0)$. By the characterization of (ℓ^p, c_0) , (see for instance [16, Theorem 1.37, pp. 160-161]), we have

$$(10) \quad |\alpha_n|^q \sum_{k=1}^n a_k^q \leq K \text{ for all } n \text{ and for some } K > 0$$

and (9) holds. But since $a^q \notin cs$ the condition in (10) implies (9) and we have $\alpha \in M((\ell_a^p)_\Delta, c_0)$ if and only if (10) holds. So we have shown that the identity in (7) holds for $F = c_0$. In a similar way the identity in (7) with $F = s_1$ can easily be shown. We conclude the proof using the inclusions $M((\ell_a^p)_\Delta, c_0) \subset M((\ell_a^p)_\Delta, c) \subset M((\ell_a^p)_\Delta, s_1)$. ■

5.2. Some properties of the (SSIE) $F_b \subset (E_a)_\Delta + F_x$

Let E and F be two linear subspaces of ω . We define by $\mathcal{I}((E_a)_\Delta, F)$ the set of all $x \in U^+$ such that $F_b \subset (E_a)_\Delta + F_x$. It can easily be seen that the

sets $(E_a)_\Delta$ and F_x are linear spaces of sequences, and we have $z \in (E_a)_\Delta + F_x$ if and only if there are $\xi \in E$ and $f \in F$ such that $z_n = \sum_{k=1}^n a_k \xi_k + f_n x_n$. To simplify we will denote by $\mathcal{I}_E^F$ the set $\mathcal{I}((E_a)_\Delta, F)$.

In the following we will use the sequence $\sigma = (\sigma_n)_n$, defined for $a, b \in U^+$ by

$$\sigma_n = \frac{1}{b_n} \sum_{k=1}^n a_k.$$

For any given $b \in U^+$ we write $s_b^\bullet$ for the set of all sequences x such that $x_n \geq K b_n$ for all n and for some $K > 0$. Notice that we have $s_b \cap s_b^\bullet = cl^\infty(b)$. First we state the next lemma.

Lemma 7. *Let $a, b \in U^+$, and let E and F be two linear subspaces of s , that satisfy $E, F \subset s_1$ and $F \supset c_0$. Then we have*

(i) *Assume $\sigma \in c_0$. Then*

a) $\mathcal{I}_E^F \subset \mathcal{I}_{s_1}^{s_1}$, b) $\mathcal{I}_E^F \subset s_b^\bullet$.

(ii) *Assume $a \in c_0$. Then we have $\mathcal{I}_E^F \subset s_1^\bullet$ for $b = e$.*

Proof. (i) a) Let $x \in \mathcal{I}_E^F$. Then we have $F_b \subset (E_a)_\Delta + F_x$ and since $E, F \subset s_1$ we obtain

$$(E_a)_\Delta + F_x = \Sigma D_a E + D_x F \subset (\Sigma D_a + D_x) s_1,$$

where $\Sigma D_a + D_x$ is a triangle, and

$$(11) \quad F_b \subset (s_1)_T,$$

with $T = (\Sigma D_a + D_x)^{-1}$. Now the condition in (11) implies $T \in (F_b, s_1)$, but we have, since $F \supset c_0$

$$(F_b, s_1) \subset (s_b^0, s_1) = (s_b, s_1)$$

and then $T \in (s_b, s_1)$. Finally we obtain $s_b \subset (s_a)_\Delta + s_x$. This shows the inclusion $\mathcal{I}_E^F \subset \mathcal{I}_{s_1}^{s_1}$.

(i) b) As we have just seen $x \in \mathcal{I}_E^F$ implies $s_b \subset (s_a)_\Delta + s_x$ and there are $u \in (s_a)_\Delta$ and $v \in s_x$ such that $b = u + v$. Since $(s_a)_\Delta = (\Sigma D_a) s_1$ and $b \in s_b$, there are two sequences $h, k \in s_1$ such that $b_n = \sum_{k=1}^n a_k h_k + x_n k_n$ and

$$\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^n a_k h_k \right) = k_n \text{ for all } n.$$

Then we have

$$\frac{1}{b_n} \left| \sum_{k=1}^n a_k h_k \right| \leq K \sigma_n \text{ for all } n \text{ and for some } K > 0,$$

and since $\sigma \in c_0$, we conclude $b/x \in s_1$, that is, $x \in s_b^\bullet$.

(ii) Let $x \in \mathcal{I}_E^F$. Then from (i) we obtain $s_1 \subset (s_a)_\Delta + s_x$. So the sequence $\xi = ((-1)^n)_n \in s_1$ can be written as $\xi = u + v$, where $u \in (s_a)_\Delta$ and $v \in s_x$. There are K_1 and $K_2 > 0$ such that $|\Delta_n u| = |u_n - u_{n-1}| \leq K_1 a_n$, $|\Delta_n v| = |v_n - v_{n-1}| \leq K_2 (x_n + x_{n-1})$ and

$$|(\Delta \xi)_n| = 2 = |\Delta_n u + \Delta_n v| \leq K_1 a_n + K_2 (x_n + x_{n-1}) \quad \text{for all } n \geq 2.$$

Then we have

$$x_n + x_{n-1} \geq \frac{1}{K_2} (2 - K_1 a_n),$$

and since $a \in c_0$, there is $K_3 > 0$ such that $x_n + x_{n-1} \geq K_3$ for all sufficiently large n , and it can easily be shown $x \in s_1^\bullet$. We conclude $\mathcal{I}_E^F \subset s_1^\bullet$. This completes the proof. $\blacksquare$

6. Solvability of sequence spaces equations of the form

$$(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$$

In this section we solve the (SSE) $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ^p with $p > 1$. For instance, the (SSE) defined by $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ is equivalent to the statement: $y_n/b_n \rightarrow l_1$ ($n \rightarrow \infty$) if and only if there are two sequences u, v with $y = u + v$ such that $(\Delta_n u)/a_n \rightarrow 0$ and $v_n/x_n \rightarrow l_2$ ($n \rightarrow \infty$) for all y and for some scalars l_1 and l_2 .

6.1. Solvability of the (SSE) $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ and

$$(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)} \quad \text{in the general case}$$

For any given $a, b \in U^+$ we denote by $S((E_a)_\Delta, F)$ the set of all the solutions of the (SSE) defined by $(E_a)_\Delta + F_x = F_b$ where E and F are linear spaces.

Theorem 1. *Let $a, b \in U^+$. Then we have:*

(i) *The set $S_0^c = S((s_a^0)_\Delta, c)$ of all the solutions of the (SSE) $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.*

a) *If $a \notin cs$, (that is, $\sum_k a_k = \infty$), then we have*

$$S_0^c = \begin{cases} cl^c(b), & \text{if } \sigma \in s_1, \\ \emptyset, & \text{if } \sigma \notin s_1. \end{cases}$$

b) *If $a \in cs$, then we have*

$$(12) \quad S_0^c = \begin{cases} cl^c(b), & \text{if } \frac{1}{b} \in c_0, \\ cl^c(e), & \text{if } \frac{1}{b} \in c \setminus c_0, \\ \emptyset, & \text{if } \frac{1}{b} \notin c. \end{cases}$$

(ii) The set $S_p^c = S((\ell_a^p)_\Delta, c)$ with $p > 1$, of all the solutions of the (SSE) $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.

a) If $a^q \notin cs$, then

$$S_p^c = \begin{cases} cl^c(b), & \text{if } \left(\frac{a_1^q + \dots + a_n^q}{b_n^q} \right)_n \in s_1, \\ \emptyset, & \text{if } \left(\frac{a_1^q + \dots + a_n^q}{b_n^q} \right)_n \notin s_1. \end{cases}$$

b) If $a^q \in cs$, then $S_p^c = S_0^c$ defined by (12).

Proof. (i) a) First consider the case $a \notin cs$. By Lemma 6 we have $M((s_a^0)_\Delta, c) = M((s_a^0)_\Delta, c_0)$ and we can apply Lemma 5 where $1/b \in M((s_a^0)_\Delta, c)$ if and only if $\sigma \in s_1$.

(i) b) Case $a \in cs$. We deal with the 3 cases $\alpha)$ $1/b \notin c$, $\beta)$ $1/b \in c_0$ and $\gamma)$ $1/b \in c \setminus c_0$.

Case $\alpha)$. We have $S_0^c = \emptyset$. Indeed, assume there is $x \in S_0^c$, then we have $(s_a^0)_\Delta \subset s_b^{(c)}$ and $D_{1/b}\Sigma D_a \in (c_0, c)$. From the characterization of (c_0, c) we deduce $1/b \in c$, which is a contradiction. We conclude $S_0^c = \emptyset$.

Case $\beta)$. Let $1/b \in c_0$. Then $x \in S_0^c$ implies

$$(13) \quad x \in s_b^{(c)}$$

and $s_b^{(c)} \subset (s_a^0)_\Delta + s_x^{(c)}$. Using similar arguments as those in Lemma 7, we easily see that since $b \in s_b^{(c)}$ there are $\varepsilon \in c_0$ and $\varphi \in c$ such that

$$\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^n a_k \varepsilon_k \right) = \varphi_n \text{ for all } n.$$

We deduce $b/x \in c$ since $\sigma \in c_0$. Using the condition in (13) we conclude that $x \in S_0^c$ implies $s_x^{(c)} = s_b^{(c)}$. Conversely, assume $s_x^{(c)} = s_b^{(c)}$. Then we have

$$(s_a^0)_\Delta + s_x^{(c)} = (s_a^0)_\Delta + s_b^{(c)} = s_b^{(c)}$$

since we have $\sigma \in s_1$ and $1/b \in c$. We conclude $S_0^c = cl^c(b)$.

Case $\gamma)$. Here we have $\lim_{n \rightarrow \infty} b_n = L > 0$ and $s_b^{(c)} = c$ and we are led to study the (SSE)

$$(14) \quad (s_a^0)_\Delta + s_x^{(c)} = c.$$

We have $x \in S_0^c$ implies $s_x^{(c)} \subset c$, that is, $x_n \rightarrow l$ ($n \rightarrow \infty$). Then by Lemma 7 (ii) with $E = s_1$ and $F = c$, the condition $x \in S_0^c$ implies $x \in s_1^\bullet$. This means $l > 0$ and $S_0^c = cl^c(e)$. This completes the proof.

(ii) a) Case $a^q \notin cs$. By Lemma 6 we have $M((\ell_a^p)_\Delta, c) = M((\ell_a^p)_\Delta, c_0)$. Let $\alpha \in M((\ell_a^p)_\Delta, c)$. Then we can apply Lemma 5 where $1/b \in M((\ell_a^p)_\Delta, c)$ if and only if $((\sum_{k=1}^n a_k^q)/b_n^q)_n \in s_1$.

(ii) b) Case $a^q \in cs$. As above we deal with the 3 cases $\alpha) 1/b \notin c$, $\beta) 1/b \in c_0$ and $\gamma) 1/b \in c \setminus c_0$. Case $\alpha)$. We have $x \in S_p^c$ implies $(\ell_a^p)_\Delta \subset s_b^{(c)}$ and $D_{1/b}\Sigma D_a \in (\ell^p, c)$. From the characterization of (ℓ^p, c) we deduce $1/b \in c$. We conclude that if $1/b \notin c$, then $S_p^c = \emptyset$. Case $\beta)$. We have $x \in S_p^c$ implies

$$(15) \quad x \in s_b^{(c)}$$

and

$$(16) \quad s_b^{(c)} \subset (\ell_a^p)_\Delta + s_x^{(c)}.$$

Again using similar arguments that as those in Lemma 7, we easily see that since $b \in s_b^{(c)}$ there are $\lambda \in \ell^p$ and $\varphi \in c$ such that

$$\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^n a_k \lambda_k \right) = \varphi_n \text{ for all } n.$$

From the characterization of (ℓ^p, c_0) , (cf. [16, Theorem 1.37, pp. 160-161]) we have $D_{1/b}\Sigma D_a \in (\ell^p, c_0)$ since $1/b \in c_0$ and $a^q \in cs$ together imply $(b_n^{-q}(a_1^q + \dots + a_n^q))_n \in s_1$. We deduce

$$(D_{1/b}\Sigma D_a)_n \lambda = \frac{1}{b_n} \sum_{k=1}^n a_k \lambda_k \rightarrow 0 \quad (n \rightarrow \infty),$$

and $b/x \in c$. Using the condition in (15) we conclude $x \in S_p^c$ implies $s_x^{(c)} = s_b^{(c)}$. Conversely, assume $s_x^{(c)} = s_b^{(c)}$. Since $1/b \in c_0$ and $a^q \in cs$ together imply $D_{1/b}\Sigma D_a \in (\ell^p, c)$, (cf. [16, Theorem 1.37]), we successively obtain $(\ell_a^p)_\Delta \subset s_b^{(c)}$, $(\ell_a^p)_\Delta + s_x^{(c)} = (\ell_a^p)_\Delta + s_b^{(c)} = s_b^{(c)}$ and $x \in S_p^c$. We conclude $S_p^c = cl^c(b)$.

Case $\gamma)$. Here we have $s_b^{(c)} = c$ and we are led to study the (SSE)

$$(17) \quad (\ell_a^p)_\Delta + s_x^{(c)} = c.$$

We have $x \in S_p^c$ implies $s_x^{(c)} \subset c$, that is, $x_n \rightarrow l$ ($n \rightarrow \infty$). Then by Lemma 7 (ii) with $E = \ell^p$ and $F = c$, the condition $x \in S_p^c$ implies $x \in \mathcal{I}_{\ell^p}^c$ and $x \in s_1^\bullet$. This means $l > 0$ and $S_p^c = cl^c(e)$. This completes the proof. $\blacksquare$

From Theorem 1 we immediately obtain the following.

Corollary 1. *Let $b \in U^+$ and let $\mathcal{S}$ be the set of all positive sequences x that satisfy the (SSE) $(c_0)_\Delta + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,*

- (i) $\mathcal{S} \neq \emptyset$,
- (ii) $\mathcal{S} = cl^c(b)$,
- (iii) $1/b \in s_{(1/n)_n}$.

We also obtain the following corollary, where $bv_p = \ell_\Delta^p$ is the set of all sequences of p -bounded variation.

Corollary 2. *Let $b \in U^+$ and $p > 1$, and denote by $\mathcal{S}_p$ the set of all positive sequences x that satisfy the (SSE) $bv_p + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,*

- (i) $\mathcal{S}_p \neq \emptyset$,
- (ii) $\mathcal{S}_p = cl^c(b)$,
- (iii) $1/b \in s_{(1/n^q)_n}$ with $q = p/(p-1)$.

6.2. The equation $s_x^{(c)} = s_b^{(c)}$ and the perturbed equation

$$(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$$

In view of perturbed equations we can state the following. Let b be a positive sequence. The equation

$$(18) \quad s_x^{(c)} = s_b^{(c)}$$

is equivalent to $x_n/b_n \rightarrow l$ ($n \rightarrow \infty$) for some $l > 0$. Then the (SSE)

$$(19) \quad (s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$$

can be considered as a perturbed equation of (18), and the question is: what are the conditions on a for which the perturbed equation and the (SSE) defined by (18) have the same solutions. As a direct consequence of Theorem 1 we obtain the next corollary, where $\overline{cs}$ is the complement of cs .

Corollary 3. *Let $a, b \in U^+$. Then we have*

- (i) *if $1/b \in c$, then the equations in (18) and (19) are equivalent if and only if $a \in cs \cup (\overline{cs} \cap (s_b)_\Sigma)$.*
- (ii) *If $1/b \notin c$, the perturbed equation in (19) has no solutions.*

Proof. (i) is an immediate consequence of Theorem 1. (ii) Let $a \notin cs$. The condition $\sigma \in s_1$ should imply $1/b_n \leq K (\sum_{k=1}^n a_k)^{-1}$ for all n and for some $K > 0$ and $1/b \in c_0$, which is contradictory. So the perturbed equation in (19) has no solutions. The case $a \in cs$ is a direct consequence of Theorem 1 (i) b). ■

Remark 1. We may state a similar result for the perturbed equation $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$.

6.3. Cases when b , or b^q is in $\widehat{C_1}$.

Now we state the next elementary results, where $\widehat{C_1}$ is the set of all positive sequences x that satisfy $(x_n^{-1} \sum_{k=1}^n x_k)_n \in \ell_\infty$, (cf. [1]).

Corollary 4. Let $a, b \in U^+$. Then we have

(i) Let $b \in \widehat{C_1}$. Then the set S_0^c of all positive $x \in U^+$ such that $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.

a) Let $a \notin cs$. Then we have

$$(20) \quad S_0^c = \begin{cases} cl^c(b), & \text{if } a/b \in s_1, \\ \emptyset, & \text{if } a/b \notin s_1. \end{cases}$$

b) Let $a \in cs$. Then we have $S_0^c = cl^c(b)$.

(ii) Let $p > 1$ and let $b^q \in \widehat{C_1}$ with $q = p/(p-1)$. Then the set S_p^c of all $x \in U^+$ such that $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.

a) Let $a^q \notin cs$. Then $S_p^c = S_0^c$ defined by (20).

b) Let $a^q \in cs$. Then $S_p^c = S_0^c = cl^c(b)$.

Proof. (i) a) We have $\sigma \in s_1$ if and only if $a \in s_b(\Sigma)$. But by [1, Theorem 2.6, p. 1789] we have $b \in \widehat{C_1}$ if and only if $s_b(\Delta) = s_b$. This implies that $\Delta \in (s_b, s_b)$ is bijective and so is for $\Sigma = \Delta^{-1}$. So we have $s_b(\Sigma) = s_b$. We have $\sigma \in s_1$ if and only if $a/b \in s_1$, and we conclude by Theorem 1. This completes the proof of (i) a).

(i) b) comes from that fact that $b \in \widehat{C_1}$ implies $1/b \in c_0$, (see [1, Proposition 2.1, p. 1786]).

(ii) a) Here we have

$$\left(\frac{a_1^q + \dots + a_n^q}{b_n^q} \right)_n \in s_1 \text{ if and only if } a^q \in s_{b^q}(\Sigma),$$

and as we have just seen we have $s_{b^q}(\Sigma) = s_{b^q}$ since $b^q \in \widehat{C_1}$. So we obtain

(ii) a). (ii) b) we have $b^q \in \widehat{C_1}$ implies that there are $C > 0$ and $\gamma > 1$ such that $b_n^q \geq C\gamma^n$, for all n , (cf. [1, Proposition 2.1, p. 1786]). So we have $b_n \geq C^{1/q}\gamma^{n/q}$ for all n , and $1/b \in c_0$. We conclude by Theorem 1. This completes the proof. $\blacksquare$

Remark 2. Notice that for $b \in \widehat{C_1}$ we have $S_0^c \neq \emptyset$ if and only if $a \in (cs \cup (\overline{cs} \cap s_b)) \cap U^+$.

Example 1. Consider the (SSE) with operator defined by

$$(21) \quad \left(s_{(n^{-\alpha})}^0 \right)_{\Delta} + s_x^{(c)} = s_b^{(c)},$$

with $0 < \alpha \leq 1$ and $b \in \widehat{C}_1$. We have $a/b = (n^{-\alpha}/b_n)_n$. By [1, Proposition 2.1, p 1786], $b \in \widehat{C}_1$ implies that there are $K > 0$ and $\gamma > 1$ such that $b_n \geq K\gamma^n$ for all n . This implies $a/b \in c_0$. We may apply Corollary 4 and conclude that the solutions of the (SSE) in (21) satisfy the condition $x_n \sim Cb_n$ ($n \rightarrow \infty$) for some $C > 0$.

Example 2. Let $b^q \in \widehat{C}_1$. It can easily be shown that the solutions of the (SSE) $\left(\ell_{(n^\alpha)}^p \right)_{\Delta} + s_x^{(c)} = s_b^{(c)}$ are defined by $x_n \sim Cb_n$ ($n \rightarrow \infty$) for some $C > 0$ and for all reals α .

Remark 3. Notice that if $a \in \widehat{C}_1$, the set $S_0^c = S(s_a^0, c)$ is determined by Corollary 4 (i). Indeed, $a \in \widehat{C}_1$ implies $(s_a^0)_{\Delta} = s_a^0$, (cf. [1, Theorem 2.6, p. 1789]) and we conclude from the solvability of the (SSE) $s_a^0 + s_x^{(c)} = s_b^{(c)}$, (cf. [11, Theorem 4.4, p. 7]).

Remark 4. If $\overline{\lim}_{n \rightarrow \infty} (a_{n-1}/a_n) < 1$, then $(\ell_a^p)_{\Delta} = \ell_a^p$, (cf. [2, Theorem 6.5 p. 3200]). So we have $S_p^c = cl^c(b)$ if $a/b \in s_1$, and $S_p^c = \emptyset$ if $a/b \notin s_1$.

7. Applications to particular (SSE) where a and b are classical sequences

7.1. On the (SSE) $(s_R^0)_{\Delta} + s_x^{(c)} = s_{\overline{R}}^{(c)}$

We obtain the next corollary whose the proof is elementary and is left to the reader.

Corollary 5. Let $R, \overline{R} > 0$, and denote by $S_{R, \overline{R}}$ the set of all positive sequences x that satisfy the (SSE) $(s_R^0)_{\Delta} + s_x^{(c)} = s_{\overline{R}}^{(c)}$. Then we obtain
(i) Case $R < 1$. We have

$$S_{R, \overline{R}} = \begin{cases} cl^c(\overline{R}), & \text{if } \overline{R} \geq 1, \\ \emptyset, & \text{if } \overline{R} < 1. \end{cases}$$

(ii) Case $R = 1$. We have

$$S_{R, \overline{R}} = \begin{cases} cl^c(\overline{R}), & \text{if } \overline{R} > 1, \\ \emptyset, & \text{if } \overline{R} \leq 1. \end{cases}$$

(iii) Case $R > 1$. We have

$$S_{R,\bar{R}} = \begin{cases} cl^c(\bar{R}), & \text{if } R \leq \bar{R}, \\ \emptyset, & \text{if } R > \bar{R}. \end{cases}$$

As a direct consequence of the preceding we can state the next remark.

Remark 5. Let $R, \bar{R} > 0$. We have $S_{R,\bar{R}} \neq \emptyset$ if and only if $R = 1 < \bar{R}$, or $1 < R \leq \bar{R}$, or $R < 1 \leq \bar{R}$. For instance the set of all positive sequences that satisfy the (SSE) $(s_R^0)_\Delta + s_x^{(c)} = s_2^{(c)}$ is non empty if and only if $R \leq 2$.

7.2. On the (SSE) $(s_{1/r}^0)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}$ and $(s_{(n^{-\alpha})_n}^0)_\Delta + s_{1/x}^{(c)} = s_{1/r}^{(c)}$

7.2.1. The (SSE) $(s_{1/r}^0)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}$

Now we consider the next statement: the condition $n^\alpha y_n \rightarrow l_1$ ($n \rightarrow \infty$) holds if and only if there are two sequences u, v , with $y = u + v$ such that

$$r^n (u_n - u_{n-1}) \rightarrow 0 \text{ and } x_n v_n \rightarrow l_2 \text{ } (n \rightarrow \infty)$$

for some scalars l_1, l_2 and for all $y \in \omega$. The set of all x that satisfy the previous statement is equivalent to the (SSE)

$$(22) \quad (s_{1/r}^0)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}.$$

We obtain the following.

Corollary 6. Let $r > 0$ and α be a real and let $\bar{S}_{r,\alpha}$ be the set of all positive sequences x that satisfy the (SSE) defined by (22). Then we obtain

(i) if $r < 1$, then $\bar{S}_{r,\alpha} = \emptyset$.

(ii) If $r = 1$, then we have

$$\bar{S}_{r,\alpha} = \begin{cases} cl^c((n^\alpha)_n), & \text{if } \alpha \leq -1, \\ \emptyset, & \text{if } \alpha > -1. \end{cases}$$

(iii) If $r > 1$, then we have

$$\bar{S}_{r,\alpha} = \begin{cases} cl^c((n^\alpha)_n), & \text{if } \alpha \leq 0, \\ \emptyset, & \text{if } \alpha > 0. \end{cases}$$

Proof. Notice that $r < 1$ implies $a = (r^{-n})_n \notin cs$. So the statement in (i) comes from the equivalence $\sigma_n \sim (1-r)^{-1} n^\alpha r^{-n}$ ($n \rightarrow \infty$) and $\sigma \notin s_1$, for $r < 1$. Let $r = 1$. Then we have $\sigma_n \sim n^{\alpha+1}$ ($n \rightarrow \infty$) and $\sigma \in s_1$ if and only if $\alpha \leq -1$, and we conclude by Theorem 1. This shows ii). Finally for $r > 1$, we have $a \in cs$ and $1/b = (n^\alpha)_n \in c$ if and only if $\alpha \leq 0$, and we conclude by Theorem 1. This completes the proof. ■

We immediately deduce the next remark.

Remark 6. We have $\overline{S}_{r,\alpha} \neq \emptyset$ if and only if $r = 1 \leq \alpha$, or $r > 1$ and $\alpha \leq 0$. We also have $\overline{S}_{r,0} \neq \emptyset$ if and only if $r > 1$.

Example 3. Consider the statement: $y_n/n \rightarrow l_1$ ($n \rightarrow \infty$) holds if and only if there are two sequences u, v , with $y = u + v$ such that $u_n - u_{n-1} \rightarrow 0$ and $x_n v_n \rightarrow l_2$ ($n \rightarrow \infty$) for some scalars l_1, l_2 and for all $y \in \omega$. This statement holds if and only if $x \in \overline{S}_{1,-1}$, that is, $x_n/n \rightarrow L$ ($n \rightarrow \infty$) with $L > 0$.

7.2.2. On the (SSE) $\left(s_{(1/n^\alpha)}^0\right)_\Delta + s_{1/x}^{(c)} = s_{1/r}^{(c)}$

As an application of Theorem 1 the following can easily be shown.

Corollary 7. Let $r > 0$, α be a real and $\overline{\overline{S}}_{\alpha,r}$ be the set of all $x \in U^+$ such that $\left(s_{(1/n^\alpha)}^0\right)_\Delta + s_{1/x}^{(c)} = s_{1/r}^{(c)}$. The next statements are equivalent.

- (i) $\overline{\overline{S}}_{\alpha,r} \neq \emptyset$,
- (ii) $\overline{\overline{S}}_{\alpha,r} = cl^c((r^n)_n)$,
- (iii) $r \leq 1 < \alpha$, or $\alpha \leq 1$ and $r < 1$.

Proof. This result is a direct consequence of the equivalences

$$f_n = \sum_{k=1}^n k^{-\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha} \quad (n \rightarrow \infty) \text{ if } \alpha \neq 1; \text{ and}$$

$$f_n \sim \ln n \quad (n \rightarrow \infty) \text{ if } \alpha = 1.$$

Then if $\alpha \neq 1$, we have $(r^n n^{1-\alpha})_n \in \ell_\infty$ if and only if $r \leq 1 < \alpha$, or α and $r < 1$; and if $\alpha = 1$ we have $(r^n \ln n)_n \in \ell_\infty$ if and only if $r < 1$. This concludes the proof. ■

Example 4. For $r = 1/2$ we have $\overline{\overline{S}}_{\alpha,1/2} = cl^c((2^{-n})_n)$ for all reals α .

7.3. On the (SSE) $\left(s_{(1/n^\alpha)_n}^0\right)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\beta)_n}^{(c)}$

Now let $\mathcal{S}_{\alpha,\beta}$ for all reals α and β , be the set of all positive sequences $x = (x_n)_n$ that satisfy the following statement. For every y the condition

$n^\beta y_n \rightarrow l_1$ ($n \rightarrow \infty$) holds if and only if there are two sequences u, v , with $y = u + v$ such that $n^\alpha (u_n - u_{n-1}) \rightarrow 0$ and $x_n v_n \rightarrow l_2$ ($n \rightarrow \infty$) for some scalars l_1, l_2 . This statement leads to the solvability of the (SSE) defined by $\left(s_{(1/n^\alpha)_n}^0\right)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\beta)_n}^{(c)}$. We obtain the next result which can be obtained by similar arguments as those used above.

Corollary 8. *Let α, β be reals. Then*

(i) *if $\alpha < 1$, then we have*

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c((n^\beta)_n), & \text{if } \beta \leq \alpha - 1, \\ \emptyset, & \text{if } \beta > \alpha - 1. \end{cases}$$

(ii) *If $\alpha = 1$, then we have*

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c((n^\beta)_n), & \text{if } \beta < 0, \\ \emptyset, & \text{if } \beta \geq 0. \end{cases}$$

(iii) *If $\alpha > 1$, then we have*

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c((n^\beta)_n), & \text{if } \beta \leq 0, \\ \emptyset, & \text{if } \beta > 0. \end{cases}$$

Corollary 9. $\mathcal{S}_{\alpha,\beta} \neq \emptyset$ if and only if $\beta \leq \alpha - 1 < 0$, or $\alpha = 1$ and $\beta < 0$, or $\alpha > 1$ and $\beta \leq 0$.

Example 5. As a direct consequence of the preceding, notice that the (SSE) $\left(s_{(n^{-\alpha})_n}^0\right)_\Delta + s_{1/x}^{(c)} = c$ is equivalent to $x_n \rightarrow L$ ($n \rightarrow \infty$) with $L > 0$, for all $\alpha > 1$.

7.4. On the (SSE) $\left(\ell_{(n^{-\alpha})_n}^p\right)_\Delta + s_{1/x}^{(c)} = s_{(n^{-\beta})_n}^{(c)}$

In the next corollary we deal with the statement for reals α and β and $p > 1$: the condition $n^\beta y_n \rightarrow l_1$ holds if and only if there are $u, v \in \omega$ with $y = u + v$ such that $\sum_{k=1}^\infty (k^\alpha |u_k - u_{k-1}|)^p < \infty$ and $x_n v_n \rightarrow l_1$ ($n \rightarrow \infty$) for all $y \in \omega$, and for some scalars l_1, l_2 . This is equivalent to the (SSE)

$$(23) \quad \left(\ell_{(n^{-\alpha})_n}^p\right)_\Delta + s_{1/x}^{(c)} = s_{(n^{-\beta})_n}^{(c)}.$$

We obtain the next result.

Corollary 10. *Let α and β be reals and let S_p^c be the set of all the solutions of the (SSE) determined by (23). Then we have*

(i) if $\alpha q \geq 1$, then we have

$$S_p^c = \begin{cases} cl^c((n^\beta)_n), & \text{if } \beta < 0, \\ \emptyset, & \text{if } \beta \geq 0. \end{cases}$$

(ii) If $\alpha q < 1$, then we have

$$S_p^c = \begin{cases} cl^c((n^\beta)_n), & \text{if } \alpha - \beta \geq \frac{1}{q}, \\ \emptyset, & \text{if } \alpha - \beta < \frac{1}{q}. \end{cases}$$

Proof. The proof comes from the fact that

$$\sigma_n \sim \frac{n^{(\beta-\alpha)q+1}}{1-\alpha q} \quad (n \rightarrow \infty)$$

if $\alpha q \neq 1$; and $\sigma_n \sim n^{\beta q} \ln n$ ($n \rightarrow \infty$) if $\alpha q = 1$. Then it can easily be seen that $\sigma \in \ell_\infty$ if and only if $\alpha - \beta \geq 1/q$ for $\alpha q < 1$, or $\beta < 0$ for $\alpha q \geq 1$. We conclude by Theorem 1. $\blacksquare$

We deal for reals β with the statement: $n^\beta y_n \rightarrow l_1$ if and only if $y = u + v$ with

$$\sum_{k=1}^{\infty} \left(\frac{|u_k - u_{k-1}|}{k} \right)^2 < \infty \quad \text{and} \quad x_n v_n \rightarrow l_2 \quad (n \rightarrow \infty) \quad \text{for all } y$$

and for some scalars l_1, l_2 . This statement is equivalent to the (SSE) defined by $(\ell_{(n)_n}^2)_\Delta + s_{1/x}^{(c)} = s_{(n^{-\beta})_n}^{(c)}$ and this (SSE) has solutions if and only if $\beta \leq -3/2$.

Example 6. Notice that the set of all the solutions of the (SSE) defined by $(\ell_{(1/\sqrt{n})_n}^2)_\Delta + s_x^{(c)} = s_{(\ln n)_n}^{(c)}$ are determined by $\lim_{n \rightarrow \infty} (x_n / \ln n) > 0$.

This result comes from the equivalence $\sum_{k=1}^n (1/\sqrt{k})^2 \sim \ln n$ ($n \rightarrow \infty$).

8. Solvability of the (SSE) of the form $(E_a)_\Delta + s_x^0 = s_b^0$

In this section we solve the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ . For $E = c$, the solvability of the previous (SSE) consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the next statement. For every y the condition $y_n/b_n \rightarrow 0$ ($n \rightarrow \infty$) holds if and only if there are two sequences u, v , with $y = u + v$ such that $(u_n - u_{n-1})/a_n \rightarrow l$ and $v_n/x_n \rightarrow 0$ ($n \rightarrow \infty$) for some scalar l . Here also we may consider the (SSE)

$\left(s_a^{(c)}\right)_\Delta + s_x^0 = s_b^0$ as a perturbed equation of the equation $s_x^0 = s_b^0$, which is equivalent to $K_1 \leq x_n/b_n \leq K_2$ for all n and for some $K_1, K_2 > 0$. We obtain the equivalence of these two equations under some conditions on a and b .

**8.1. Solvability of the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$,
or ℓ_∞ in the general case**

To prove the next result we need a lemma.

Lemma 8. *Let $b \in U^+$ and let T be a triangle. Then we have*

$$s_b^0 \subset Ts_1 \text{ if and only if } s_b \subset Ts_1.$$

Proof. We have $s_b^0 \subset Ts_1$ if and only if

$$(24) \quad T^{-1}D_b \in (c_0, s_1).$$

Since $(c_0, s_1) = (s_1, s_1)$ the condition in (24) is equivalent to $s_b \subset Ts_1$. This concludes the proof. $\blacksquare$

Theorem 2. *The set S_E^0 of all the solutions of the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ is determined by*

$$S_E^0 = \begin{cases} cl^\infty(b), & \text{if } \sigma \in c_0, \\ \emptyset, & \text{if } \sigma \notin c_0. \end{cases}$$

Proof. Let $x \in S_E^0$. Then we have the inclusion $(E_a)_\Delta + s_x^0 \subset s_b^0$. This implies $(E_a)_\Delta \subset s_b^0$ and $D_{1/b}\Sigma D_a \in (E, c_0)$. This implies

$$D_{1/b}\Sigma D_a \in (c, c_0)$$

since $E \supset c$ and

$$(25) \quad \sigma_n \rightarrow 0 \quad (n \rightarrow \infty).$$

Now we have $s_x^0 \subset s_b^0$ and

$$(26) \quad x \in s_b.$$

Then we consider the (SSIE) defined by

$$(27) \quad s_b^0 \subset (E_a)_\Delta + s_x^0.$$

The (SSIE) in (27) with $E \subset s_1$ implies $s_b^0 \subset T s_1$ with $T = \Sigma D_a + D_x$. So the inclusion in (27) implies

$$s_b \subset (s_a)_\Delta + s_x$$

by Lemma 8. Since $(s_a)_\Delta = \Sigma s_a$ there are $h, k \in s_1$ such that

$$\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^n a_k h_k \right) = k_n \text{ for all } n.$$

We have

$$\left| \frac{1}{b_n} \sum_{k=1}^n a_k h_k \right| \leq K \sigma_n \text{ for all } n \text{ and for some } K > 0,$$

and from the condition in (25) we deduce $b/x \in s_1$. Using the condition in (26) we conclude $x \in S_E^0$ implies $x \in cl^\infty(b)$. Conversely, assume $x \in cl^\infty(b)$ and (25) holds. Since $1/b \in c_0$, we have $(E_a)_\Delta \subset s_b^0$ for $E = c_0$, or s_1 , we obtain

$$(E_a)_\Delta + s_x^0 = (E_a)_\Delta + s_b^0 = s_b^0,$$

and $x \in S_E^0$. We conclude $S_E^0 = cl^\infty(b)$. ■

For $a = e$ we easily obtain the next result.

Corollary 11. (i) *The set $S(E_\Delta, c_0)$ of all the solutions of the (SSE) $E_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ is determined by*

$$S(E_\Delta, c_0) = \begin{cases} cl^\infty(b) & \text{if } (n/b_n)_n \in c_0, \\ \emptyset & \text{if } (n/b_n)_n \notin c_0. \end{cases}$$

Example 7. The equation $E_\Delta + s_x^0 = s_{(n^\alpha)_n}^0$ where $E = c$, or ℓ_∞ , has solutions if and only if $\alpha > 1$. So the equation $E_\Delta + s_x^0 = c_0$ has no solution, and the solutions of the equation $E_\Delta + s_x^0 = s_{(n^2)_n}^0$ are determined by $K_1 n^2 \leq x_n \leq K_2 n^2$ for all n and for some $K_1, K_2 > 0$.

8.2. Applications to the solvability of (SSE) of the form

$(E_a)_\Delta + s_x^0 = s_b^0$ for particular sequences a and b

Consider the (SSE) determined by

$$(28) \quad \left(s_{(n^{-\alpha})_n}^{(c)} \right)_\Delta + s_{1/x}^0 = s_{(n^{-\beta})_n}^0$$

$$(29) \quad \left(s_R^{(c)} \right)_\Delta + s_x^0 = s_R^0,$$

$$(30) \quad \left(s_{(n^{-\alpha})_n}^{(c)} \right)_\Delta + s_{1/x}^0 = s_{1/R}^0,$$

$$(31) \quad \left(s_{1/R}^{(c)} \right)_\Delta + s_{1/x}^0 = s_{(n^{-\beta})_n}^0,$$

with reals α and β and $R, \bar{R} > 0$. We obtain the following.

Proposition 1. (i) The (SSE) defined in (28) has solutions if and only if $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.

(ii) The (SSE) defined in (29) has solutions if and only if $R \leq 1 < \bar{R}$, or $1 < R < \bar{R}$.

(iii) The (SSE) defined in (30) has solutions if and only if $R < 1 < \alpha$, or $R < \alpha = 1$, or α and $R < 1$.

(iv) The (SSE) defined in (31) has solutions if and only if $R = 1$ and $\beta < -1$, or $R > 1$ and $\beta < 0$.

Proof. (i) It can easily be seen that $\sigma_n \sim n^{\beta-\alpha+1}/(1-\alpha)$ ($n \rightarrow \infty$) for $\alpha < 1$; $\sigma_n \sim n^\beta \ln n$ ($n \rightarrow \infty$) for $\alpha = 1$; and $\sigma_n \sim Kn^\beta$ ($n \rightarrow \infty$) for $\alpha > 1$. We conclude the (SSE) defined by (28) has solutions if and only if $\sigma_n = o(1)$ ($n \rightarrow \infty$), that is, for $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.

(ii) We have

$$\sigma_n \sim \frac{1}{R-1} \frac{R^{n+1}}{\bar{R}^n} \quad (n \rightarrow \infty) \quad \text{for } R > 1;$$

$$\sigma_n \sim \frac{R}{1-R} \frac{1}{\bar{R}^n} \quad (n \rightarrow \infty) \quad \text{for } R < 1;$$

and

$$\sigma_n \sim \frac{n}{\bar{R}^n} \quad (n \rightarrow \infty) \quad \text{for } R = 1;$$

and we conclude as above.

(iii) This result is a direct consequence of the following. We have $\sigma_n \sim KR^n$ ($n \rightarrow \infty$) for $\alpha > 1$ and for some $K > 0$; $\sigma_n \sim R^n \ln n$ ($n \rightarrow \infty$) for $\alpha = 1$; and $\sigma_n \sim n^{1-\alpha}R^n/(1-\alpha)$ ($n \rightarrow \infty$) for $\alpha < 1$. We conclude by Theorem 2.

(iv) Here we have $\sigma_n \sim n^\beta/(R-1)$ ($n \rightarrow \infty$) for $R > 1$; $\sigma_n \sim n^{\beta+1}$ ($n \rightarrow \infty$) for $R = 1$; and $\sigma_n \sim n^\beta R^{-n}/(1-R)$ ($n \rightarrow \infty$) for $R < 1$ and we conclude by Theorem 2. ■

Example 8. The (SSE) $\left(s_{1/2}^{(c)}\right)_\Delta + s_x^0 = s_{\bar{R}}^0$ has solutions if and only if $\bar{R} > 1$.

Example 9. Let τ, τ' be reals. Then the system of (SSE) defined by

$$\begin{cases} c_\Delta + s_x^0 = s_{(n^\tau)_n}^0, \\ \left(s_{1/2}^{(c)}\right)_\Delta + s_x^0 = s_{(n^{\tau'})_n}^0, \end{cases}$$

where x is the unknown, has solutions if and only if $\tau = \tau' > 1$. Then x is a solution of the system if and only if $x_n \sim Cn^\tau$ ($n \rightarrow \infty$) for some $C > 0$.

This is a direct consequence of Proposition 1 (iv) and of the elementary fact that $s_{(n^\tau)_n} = s_{(n^{\tau'})_n}$ if and only if $\tau = \tau'$.

Example 10. Let S_c^0 be the set of all positive sequences that satisfy the following statement. For every y the condition $y_n/n \rightarrow 0$ ($n \rightarrow \infty$) holds if and only if there are two sequences u, v , with $y = u + v$ such that $\sqrt{n}(u_n - u_{n-1}) \rightarrow L$ and $x_nv_n \rightarrow 0$ ($n \rightarrow \infty$) for some scalar L . By Proposition 1 (i), we have $x \in S_c^0$ if and only if $K_1/n \leq x_n \leq K_2/n$ for all n and for some $K_1, K_2 > 0$.

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