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INEQUALITIES OF JENSEN TYPE FOR φ -CONVEX FUNCTIONS

ABSTRACT. Some inequalities of Jensen type for φ -convex functions defined on real intervals are given.

KEY WORDS: convex functions, integral inequalities, h -convex functions.

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1. Introduction

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([38]). *We say that $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have*

$$(1) \quad f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y).$$

Some further properties of this class of functions can be found in [29], [30], [32], [44], [47] and [48]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([32]). *We say that a function $f : I \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$(2) \quad f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions*, i. e. nonnegative functions satisfying

$$(3) \quad f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P -functions see [32] and [45] while for quasi convex functions, the reader can consult [31].

Definition 3 ([7]). *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense) or Breckner s -convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [27], [28], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h -convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I , respectively.

Definition 4 ([53]). *Let $h : J \rightarrow [0, \infty)$ with h not identical to 0. We say that $f : I \rightarrow [0, \infty)$ is an h -convex function if for all $x, y \in I$ we have*

$$(4) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

Definition 5. *We say that the function $f : I \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1]$, if*

$$(5) \quad f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y),$$

for all $t \in (0, 1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P -functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s -Godunova-Levin functions defined on I , then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(6) \quad (b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}.$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [43]. Since (6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For related results, see [10]-[19], [22]-[26], [33]-[36] and [46].

The following inequality of Hermite-Hadamard type for h -convex function holds [49].

Theorem 1. Assume that the function $f : I \rightarrow [0, \infty)$ is an h -convex function with $h \in L[0, 1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then

$$(7) \quad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq [f(x) + f(y)] \int_0^1 h(t) dt.$$

If we write (7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$(8) \quad f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{2}.$$

If we write (7) for the case of P -type functions $f : I \rightarrow [0, \infty)$, i.e., $h(t) = 1$, $t \in [0, 1]$, then we get the inequality

$$(9) \quad \frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq f(x) + f(y),$$

that has been obtained for functions of real variable in [32].

If f is Breckner s -convex on I , for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (7) we get

$$(10) \quad 2^{s-1} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{s+1},$$

that was obtained for functions of a real variable in [27].

If $f : I \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1)$, then

$$(11) \quad \frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{1-s}.$$

We notice that for $s = 1$ the first inequality in (11) still holds, i.e.

$$(12) \quad \frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty] dt.$$

The case for functions of real variables was obtained for the first time in [32].

2. φ -convex functions

We introduce the following class of h -convex functions.

Definition 6. Let $\varphi : (0, 1) \rightarrow (0, \infty)$ a measurable function. We say that the function $f : I \rightarrow [0, \infty)$ is a φ -convex function on the interval I if for all $x, y \in I$ we have

$$(13) \quad f(tx + (1-t)y) \leq t\varphi(t)f(x) + (1-t)\varphi(1-t)f(y)$$

for all $t \in (0, 1)$.

If we denote $\ell(t) = t$, the identity function, then it is obvious that f is h -convex with $h = \ell\varphi$. Also, all the examples from the introduction can be seen as φ -convex functions with appropriate choices of φ .

If we take $\varphi(t) = \frac{1}{t^{s+1}}$ with $s \in [0, 1]$, then we get the class of s -Godunova-Levin functions. Also, if we put $\varphi(t) = t^{s-1}$ with $s \in (0, 1)$, then we get the concept of Breckner s -convexity. We notice that for all these examples we have

$$\varphi_+(0) := \lim_{t \rightarrow 0+} \varphi(t) = \infty.$$

The case of convex functions, i.e. when $\varphi(t) = 1$ is the only example from above for which $\varphi_+(0)$ is finite, namely $\varphi_+(0) = 1$.

Consider the family of functions, for $p > 1$ and $k > 0$

$$(14) \quad \delta(p, k) : [0, 1] \rightarrow \mathbb{R}_+, \quad \delta(p, k)(t) = k(1-t)^p + 1.$$

We observe that $\delta_+(p, k)(0) = \delta(p, k)(0) = k+1$, $\delta(p, k)$ is strictly decreasing on $[0, 1]$ and $\delta(p, k)(t) \geq \delta(p, k)(1) = 1$.

Definition 7. We say that the function $f : I \rightarrow [0, \infty)$ is a $\delta(p, k)$ -convex function on the interval I if for all $x, y \in I$ we have

$$(15) \quad f(tx + (1-t)y) \leq t[k(1-t)^p + 1]f(x) + (1-t)(kt^p + 1)f(y)$$

for all $t \in (0, 1)$.

It is obvious that any nonnegative convex function is a $\delta^{(p,k)}$ -convex function for any $p > 1$ and $k > 0$.

For $m > 0$ we consider the family of functions

$$\eta(m) : [0, 1] \rightarrow \mathbb{R}_+, \quad \eta(m)(t) := \exp[m(1-t)].$$

We observe that $\eta_+(m)(0) = \eta(m)(0) = \exp(m)$, $\eta(m)$ is strictly decreasing on $[0, 1]$ and $\eta(m)(t) \geq \eta(m)(1) = 1$.

Definition 8. We say that the function $f : I \rightarrow [0, \infty)$ is a $\eta(m)$ -convex function on the interval I if for all $x, y \in I$ we have

$$(16) \quad f(tx + (1-t)y) \leq t \exp[m(1-t)] f(x) + (1-t) \exp(mt) f(y)$$

for all $t \in (0, 1)$.

It is obvious that any nonnegative convex function is a $\eta(m)$ -convex function for any $m > 0$.

There are many other examples one can consider. In fact any continuous function $\varphi : [0, 1] \rightarrow [1, \infty)$ can generate a class of φ -convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1 we can state the following result.

Theorem 2. Assume that the function $f : I \rightarrow [0, \infty)$ is a φ -convex function with $\ell\varphi \in L[0, 1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then

$$(17) \quad \frac{1}{\varphi(\frac{1}{2})} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq [f(x) + f(y)] \int_0^1 t\varphi(t) dt.$$

The proof follows from (7) by taking $h(t) = t\varphi(t)$, $t \in (0, 1)$.

Remark 1. We notice that, since $\int_0^1 t\varphi(t) dt$ can be seen as the expectation of a random variable X with the density function φ , the inequality (17) provides a connection to Probability Theory and motivates the introduction of φ -convex function as a natural concept, having available many examples of density functions φ that arise in applications.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \rightarrow \mathbb{R}$ is said to be *supermultiplicative* if

$$(18) \quad h(ts) \geq h(t)h(s) \quad \text{for any } t, s \in J.$$

If the inequality (18) is reversed, then h is said to be *submultiplicative*. If the equality holds in (18) then h is said to be a multiplicative function on J .

In [53] it has been noted that if $h : [0, \infty) \rightarrow [0, \infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

The case of h -convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable.

Theorem 3. *Let $h : J \rightarrow [0, \infty)$ be a supermultiplicative function on J . If the function $f : I \rightarrow [0, \infty)$ is h -convex on the interval I , then for any $w_i \geq 0$, $x_i \in I$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have*

$$(19) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i).$$

In particular, we have the unweighted inequality

$$(20) \quad f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq h\left(\frac{1}{n}\right) \sum_{i=1}^n f(x_i).$$

Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$(21) \quad \begin{aligned} h(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \quad z \in D(0, 1); \\ h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0, 1). \end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$(22) \quad \begin{aligned} h(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z), \quad z \in \mathbb{C}, \\ h(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right), \quad z \in D(0, 1); \\ h(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} (2n+1) n!} z^{2n+1} = \sin^{-1}(z), \quad z \in D(0, 1); \end{aligned}$$

and

$$(23) \quad h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \quad z \in D(0, 1)$$

$$h(z) = {}_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha, \beta, \gamma > 0, z \in D(0, 1);$$

where Γ is *Gamma function*.

The following result may provide many examples of supemultiplicative functions.

Lemma 1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0, 1] \rightarrow [0, \infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. Then h_r is supemultiplicative on $[0, 1]$.*

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i \in \mathbb{N}}$, $(b_i)_{i \in \mathbb{N}}$ and nonnegative weights $(p_i)_{i \in \mathbb{N}}$:

$$(24) \quad \sum_{i=0}^n p_i \sum_{i=0}^n p_i c_i b_i \geq \sum_{i=0}^n p_i c_i \sum_{i=0}^n p_i b_i,$$

for any $n \in \mathbb{N}$.

Let $t, s \in (0, 1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$(25) \quad \sum_{i=0}^n a_i r^i \sum_{i=0}^n a_i (rts)^i \geq \sum_{i=0}^n a_i (rt)^i \sum_{i=0}^n a_i (rs)^i$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i, \quad \sum_{i=0}^{\infty} a_i (rts)^i, \quad \sum_{i=0}^{\infty} a_i (rt)^i \quad \text{and} \quad \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \rightarrow \infty$ in (25) we get

$$h(r) h(rts) \geq h(rt) h(rs)$$

i.e.

$$h_r(ts) \geq h_r(t) h_r(s).$$

This inequality is also obviously satisfied at the end points of the interval $[0, 1]$ and the proof is completed. ■

Remark 2. Utilising the above theorem, we then conclude that the functions

$$h_r : [0, 1] \rightarrow [0, \infty), \quad h_r(t) := \frac{1-r}{1-rt}, \quad r \in (0, 1)$$

and

$$h_r : [0, 1] \rightarrow [0, \infty), \quad h_r(t) := \exp[-r(1-t)], \quad r > 0$$

are supermultiplicative.

We say that the function $f : I \rightarrow [0, \infty)$ is r -resolvent convex with r fixed in $(0, 1)$, if f is h -convex with $h(t) = \frac{1-r}{1-rt}$, i.e.

$$(26) \quad f(tx + (1-t)y) \leq (1-r) \left[\frac{1}{1-rt} f(x) + \frac{1}{1-r+rt} f(y) \right]$$

for any $x, y \in I$ and $t \in [0, 1]$.

In particular, for $r = \frac{1}{2}$ we have $\frac{1}{2}$ -resolvent convex functions defined by the condition

$$(27) \quad f(tx + (1-t)y) \leq \frac{1}{2-t} f(x) + \frac{1}{1+t} f(y)$$

for any $t \in [0, 1]$ and $x, y \in I$.

Since

$$t < \frac{1}{2-t} < \frac{1}{t} \quad \text{and} \quad 1-t < \frac{1}{1+t} < \frac{1}{1-t} \quad \text{for } t \in (0, 1)$$

it follows that any nonnegative convex function is $\frac{1}{2}$ -resolvent convex which, in its turn, is of Godunova-Levin type.

We say that the function $f : I \rightarrow [0, \infty)$ is r -exponential convex with r fixed in $(0, \infty)$, if f is h -convex with $h(t) = \exp[-r(1-t)]$, i.e.

$$(28) \quad f(tx + (1-t)y) \leq \exp[-r(1-t)] f(x) + \exp(-rt) f(y)$$

for any $t \in [0, 1]$ and $x, y \in C$.

Since

$$t \leq \exp[-r(1-t)] \quad \text{and} \quad 1-t \leq \exp(-rt) \quad \text{for } t \in [0, 1]$$

it follows that any nonnegative convex function is r -exponential convex with $r \in (0, \infty)$.

Corollary 1. Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0, 1] \rightarrow [0, \infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. If the function $f : I \rightarrow [0, \infty)$ is h_r -convex on the interval I , namely

$$(29) \quad f(tx + (1-t)y) \leq \frac{1}{h(r)} [h(rt) f(x) + h(r(1-t)) f(y)]$$

for any $t \in [0, 1]$ and $x, y \in I$, then for any $x_i \in I$, $w_i \geq 0$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$(30) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq \frac{1}{h(r)} \sum_{i=1}^n h\left(r \frac{w_i}{W_n}\right) f(x_i).$$

Remark 3. If the function $f : I \rightarrow [0, \infty)$ is $\frac{1}{2}$ -resolvent convex on I , then for any $x_i \in I$, $w_i \geq 0$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq W_n \sum_{i=1}^n \frac{1}{2W_n - w_i} f(x_i).$$

If the function $f : I \rightarrow [0, \infty)$ is r -exponential convex with r fixed in $(0, \infty)$, then for any $x_i \in I$, $w_i \geq 0$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n \exp\left[-r\left(1 - \frac{w_i}{W_n}\right)\right] f(x_i).$$

We have the following Jensen type inequality for φ -convex functions.

Corollary 2. Let $\varphi : J \rightarrow [0, \infty)$ be a supermultiplicative function on J . If the function $f : I \rightarrow [0, \infty)$ is φ -convex on the interval I , then for any $w_i \geq 0$, $x_i \in I$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$(31) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i \varphi\left(\frac{w_i}{W_n}\right) f(x_i).$$

In particular, we have the unweighted inequality

$$(32) \quad f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \varphi\left(\frac{1}{n}\right) \frac{1}{n} \sum_{i=1}^n f(x_i).$$

The proof follows by Theorem 3 for the supermultiplicative function $h(t) = t\varphi(t)$, $t \in J$.

The inequality (31) will be used further to obtain an integral Jensen type inequality.

3. Some results for differentiable functions

If we assume that the function $f : I \rightarrow [0, \infty)$ is differentiable on the interior of I , denoted by $\overset{\circ}{I}$, then we have the following "gradient inequality" that will play an essential role in the following.

Lemma 2. Let $\varphi : (0, 1) \rightarrow (0, \infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. If the function $f : I \rightarrow [0, \infty)$ is differentiable on $\overset{\circ}{I}$ and φ -convex, then

$$(33) \quad \varphi_+(0) f(x) - [\varphi'_-(1) + 1] f(y) \geq f'(y)(x - y)$$

for any $x, y \in \overset{\circ}{I}$ with $x \neq y$.

Proof. Since f is φ -convex on I , then

$$t\varphi(t)f(x) + (1-t)\varphi(1-t)f(y) \geq f(tx + (1-t)y)$$

for any $t \in (0, 1)$ and for any $x, y \in \overset{\circ}{I}$, which is equivalent to

$$t\varphi(t)f(x) + [(1-t)\varphi(1-t) - 1]f(y) \geq f(tx + (1-t)y) - f(y)$$

and by dividing by $t > 0$ we get

$$(34) \quad \varphi(t)f(x) + \left[\frac{(1-t)\varphi(1-t) - 1}{t} \right] f(y) \geq \frac{f(tx + (1-t)y) - f(y)}{t}$$

for any $t \in (0, 1)$.

Now, since f is differentiable on $y \in \overset{\circ}{I}$, then we have

$$(35) \quad \begin{aligned} \lim_{t \rightarrow 0+} \frac{f(tx + (1-t)y) - f(y)}{t} &= \lim_{t \rightarrow 0+} \frac{f(y + t(x-y)) - f(y)}{t} \\ &= (x-y) \lim_{t \rightarrow 0+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} = (x-y)f'(y) \end{aligned}$$

for any $x \in \overset{\circ}{I}$ with $x \neq y$.

Also since $\varphi_-(1) = 1$ and $\varphi'_-(1)$ exists and is finite, we have

$$(36) \quad \begin{aligned} \lim_{t \rightarrow 0+} \frac{(1-t)\varphi(1-t) - 1}{t} &= \lim_{s \rightarrow 1-} \frac{s\varphi(s) - 1}{1-s} = - \lim_{s \rightarrow 1-} \frac{s\varphi(s) - 1}{s-1} \\ &= - \lim_{s \rightarrow 1-} \frac{s(\varphi(s) - \varphi(1)) + s - 1}{s-1} \\ &= -\varphi'_-(1) - 1. \end{aligned}$$

Taking the limit over $t \rightarrow 0+$ in (34) and utilizing (35) and (36) we get the desired result (33). ■

Remark 4. If we assume that

$$(37) \quad \varphi_+(0) \geq \varphi'_-(1) + 1,$$

then the inequality (33) also holds for $x = y$.

There are numerous examples of such functions, for instance, if, as above we take $\varphi(t) = k(1-t)^p + 1$, $t \in [0, 1]$ ($p > 1, k > 0$) then $\varphi_+(0) = k + 1$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = 0$, which satisfy the condition (37).

If we take $\varphi(t) = \exp[m(1-t)]$ ($m > 0$), then $\varphi_+(0) = \exp m$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = -m$. We have

$$\varphi_+(0) - \varphi_-(1) - \varphi'_-(1) = e^m - 1 + m > 0$$

for $m > 0$.

The following result holds:

Theorem 4. *Let $\varphi : (0, 1) \rightarrow (0, \infty)$ a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that $\varphi'_-(1) > -1$. If the function $f : I \rightarrow [0, \infty)$ is differentiable on $\overset{\circ}{I}$ and φ -convex, then*

$$(38) \quad \frac{\varphi_+(0)}{\varphi'_-(1) + 1} \frac{f(x) + f(y)}{2} \geq \frac{1}{y-x} \int_x^y f(u) du \geq \frac{\varphi'_-(1) + 1}{\varphi_+(0)} f\left(\frac{x+y}{2}\right)$$

for any $x, y \in I$.

Remark 5. It has been shown in [25] that the inequalities (17) and (38) are not comparable, meaning that some time one is better then the other, depending on the φ -convex function involved.

The following discrete Jensen type inequality holds:

Theorem 5. *Let $\varphi : (0, 1) \rightarrow (0, \infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that*

$$(39) \quad \varphi_+(0) \geq \varphi'_-(1) + 1 > 0.$$

If the function $f : I \rightarrow [0, \infty)$ is differentiable on $\overset{\circ}{I}$ and φ -convex, then for any $w_i \geq 0$, $x_i \in \overset{\circ}{I}$, $i \in \{1, \dots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$(40) \quad \frac{\varphi_+(0)}{\varphi'_-(1) + 1} \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i) \geq f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right).$$

If $\frac{1}{W_n} \sum_{i=1}^n w_i x_i \neq x_j$ for any $j \in \{1, \dots, n\}$, then the first condition in (39) can be dropped.

Proof. From (33) we have

$$(41) \quad \varphi_+(0) f(x_j) - [\varphi'_-(1) + 1] f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \\ \geq f'\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \left(x_j - \frac{1}{W_n} \sum_{i=1}^n w_i x_i\right)$$

for any $j \in \{1, \dots, n\}$.

If we multiply (41) by $w_i \geq 0$ and sum over j from 1 to n we get

$$\varphi_+(0) \sum_{j=1}^n w_j f(x_j) - [\varphi'_-(1) + 1] \sum_{j=1}^n w_j f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \\ \geq f'\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \sum_{j=1}^n w_j \left(x_j - \frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) = 0,$$

which proves the desired result (40). ■

4. Integral inequalities

We have the following Jensen inequality for the Riemann integral:

Theorem 6. *Let $u : [a, b] \rightarrow [m, M]$ be a Riemann integrable function. Suppose that $\varphi : J \rightarrow [0, \infty)$ is a supermultiplicative function on J and the function $f : [m, M] \rightarrow [0, \infty)$ is φ -convex and continuous on the interval $[m, M]$. If the right limit $\varphi_+(0)$ exists and is finite, then*

$$(42) \quad f\left(\frac{1}{b-a} \int_a^b u(t) dt\right) \leq \varphi_+(0) \frac{1}{b-a} \int_a^b f(u(t)) dt.$$

Proof. Consider the sequence of divisions

$$d_n : x_i^{(n)} = a + \frac{i}{n} (b-a), \quad i \in \{0, \dots, n\}$$

and the intermediate points

$$\xi_i^{(n)} = a + \frac{i}{n} (b-a), \quad i \in \{0, \dots, n\}.$$

We observe that the norm of the division $\Delta_n := \max_{i \in \{0, \dots, n-1\}} (x_{i+1}^{(n)} - x_i^{(n)}) = \frac{b-a}{n} \rightarrow 0$ as $n \rightarrow \infty$ and since u is Riemann integrable on $[a, b]$, then

$$\int_a^b u(t) dt = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} u(\xi_i^{(n)}) [x_{i+1}^{(n)} - x_i^{(n)}] \\ = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} u\left(a + \frac{i}{n} (b-a)\right).$$

Also, since $f : [m, M] \rightarrow [0, \infty)$ is Riemann integrable, then $f \circ u$ is Riemann integrable and

$$\int_a^b f(u(t)) dt = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f\left[u\left(a + \frac{i}{n}(b-a)\right)\right].$$

Utilising the inequality (31) for $w_i := \frac{b-a}{n}$ and $x_i := u\left(a + \frac{i}{n}(b-a)\right)$ we have

$$\begin{aligned} (43) \quad & f\left(\frac{1}{b-a} \frac{b-a}{n} \sum_{i=0}^{n-1} u\left(a + \frac{i}{n}(b-a)\right)\right) \\ & \leq \frac{1}{b-a} \frac{b-a}{n} \sum_{i=0}^{n-1} \varphi\left(\frac{1}{n}\right) f\left(u\left(a + \frac{i}{n}(b-a)\right)\right) \\ & = \frac{1}{b-a} \varphi\left(\frac{1}{n}\right) \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(u\left(a + \frac{i}{n}(b-a)\right)\right) \end{aligned}$$

for any $n \geq 1$.

Since f is continuous, then

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{b-a} \frac{b-a}{n} \sum_{i=0}^{n-1} u\left(a + \frac{i}{n}(b-a)\right)\right) = f\left(\frac{1}{b-a} \int_a^b u(t) dt\right).$$

Also

$$\lim_{n \rightarrow \infty} \varphi\left(\frac{1}{n}\right) = \varphi_+(0) < \infty.$$

Therefore, taking the limit over $n \rightarrow \infty$ in the inequality (43) we deduce the desired result (42). ■

We have the following Hermite-Hadamard type inequality:

Corollary 3. *Suppose that $\varphi : J \rightarrow [0, \infty)$ is a supermultiplicative function on J and the function $f : I \rightarrow [0, \infty)$ is φ -convex and continuous on the interval I . If the right limit $\varphi_+(0)$ exists and is finite with $\varphi_+(0) > 0$, then for any $x, y \in I$ with $x \neq y$ we have*

$$(44) \quad \frac{1}{\varphi_+(0)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u(t)) dt.$$

Remark 6. If the function $f : [m, M] \rightarrow [0, \infty)$ is a $\delta(p, k)$ -convex and continuous function on the interval $[m, M]$ ($p > 1$ and $k > 0$, see

Definition 7) then for any $u : [a, b] \rightarrow [m, M]$ a Riemann integrable function on $[a, b]$ we have

$$(45) \quad \frac{1}{k+1} f \left(\frac{1}{b-a} \int_a^b u(t) dt \right) \leq \frac{1}{b-a} \int_a^b f(u(t)) dt.$$

If the function $f : [m, M] \rightarrow [0, \infty)$ is a $\eta(s)$ -convex and continuous function on the interval $[m, M]$ ($s > 0$, see Definition 8) then for any $u : [a, b] \rightarrow [m, M]$ a Riemann integrable function on $[a, b]$ we have

$$(46) \quad \frac{1}{e^s} f \left(\frac{1}{b-a} \int_a^b u(t) dt \right) \leq \frac{1}{b-a} \int_a^b f(u(t)) dt.$$

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω , a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ -measurable function $w : \Omega \rightarrow \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} w(x) |f(x)| d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w d\mu$ instead of $\int_{\Omega} w(x) d\mu(x)$.

Theorem 7. *Let $\varphi : (0, 1) \rightarrow (0, \infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that*

$$(47) \quad \varphi_+(0) \geq \varphi'_-(1) + 1 > 0.$$

If the function $f : I \rightarrow [0, \infty)$ is differentiable on $\overset{\circ}{I}$ and φ -convex, then for any $u : \Omega \rightarrow [m, M] \subset \overset{\circ}{I}$ so that $f \circ u, u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ -a.e. (almost everywhere) on Ω with $\int_{\Omega} w d\mu = 1$ we have

$$(48) \quad \frac{\varphi_+(0)}{\varphi'_-(1) + 1} \int_{\Omega} w(f \circ u) d\mu \geq f \left(\int_{\Omega} w u d\mu \right).$$

If $\int_{\Omega} w u d\mu \neq u(x)$ for μ -a.e. $x \in \Omega$, then we can drop the first condition in (47).

Proof. From (33) and since $\int_{\Omega} w u d\mu \in [m, M] \subset \overset{\circ}{I}$ we have

$$(49) \quad \begin{aligned} & \varphi_+(0) f(u(x)) - [\varphi'_-(1) + 1] f \left(\int_{\Omega} w u d\mu \right) \\ & \geq f' \left(\int_{\Omega} w u d\mu \right) \left(u(x) - \int_{\Omega} w u d\mu \right), \text{ for any } x \in \Omega. \end{aligned}$$

If we multiply (49) by $w \geq 0$ μ -a.e. on Ω and integrate over the positive measure μ we get

$$\begin{aligned} \varphi_+(0) \int_{\Omega} w(x) f(u(x)) d\mu(x) - [\varphi'_-(1) + 1] f\left(\int_{\Omega} w u d\mu\right) \int_{\Omega} w(x) d\mu(x) \\ \geq f'\left(\int_{\Omega} w u d\mu\right) \int_{\Omega} w(x) \left(u(x) - \int_{\Omega} w u d\mu\right) d\mu(x) = 0, \end{aligned}$$

which produces the desired result (48). ■

Remark 7. If the function $f : [m, M] \rightarrow [0, \infty)$ is a $\delta(p, k)$ -convex and continuous function on the interval $[m, M]$, then for any $u : \Omega \rightarrow [m, M] \subset \overset{\circ}{I}$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ -a.e. on Ω with $\int_{\Omega} w d\mu = 1$ we have

$$(50) \quad \int_{\Omega} w(f \circ u) d\mu \geq \frac{1}{k+1} f\left(\int_{\Omega} w u d\mu\right).$$

If the function $f : [m, M] \rightarrow [0, \infty)$ is a $\eta(s)$ -convex and continuous function on the interval $[m, M]$ then for any $u : \Omega \rightarrow [m, M] \subset \overset{\circ}{I}$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ -a.e. on Ω with $\int_{\Omega} w d\mu = 1$ we have

$$(51) \quad \int_{\Omega} w(f \circ u) d\mu \geq \frac{1}{e^s} f\left(\int_{\Omega} w u d\mu\right).$$

These results generalize the inequalities (45) and (46).

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