A NEW ALGORITHM TO CONSTRUCT S-BOXES WITH HIGH DIFFUSION

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ABSTRACT

In this paper is proposed a new algorithm to construct S-Boxes over $GF(28)$ with Branch Number value at least 3. This is an important property that guarantees a high diffusion in the S-Box [12]. Also are introduced some definition and properties that show the behavior of S-Boxes under the composition with affine functions. Finally is presented a comparison between this algorithm and the method proposed by Tavares [12].

1 INTRODUCTION

In his paper "Communication Theory of Secrecy Systems", Claude Shannon defines confusion and diffusion as two essential properties for block cipher design. S-Boxes to be used in this cryptographic algorithms require the compliance of criteria such as Nonlinearity, Branch Number, Algebraic Degree and Differential Characteristic, which bring security against differential and linear cryptanalysis [9][10].

In 1994 Tavares [12] proposes a method to and S-Boxes with a desired value of Branch Number. The principal problem of this proposal is that the obtained S-Boxes can have a big number of fixed points and his Nonlinearity values were small to be used in Block Ciphers.

In this paper we propose a new algorithm to construct S-Boxes with Branch Number at least 3. In this method the obtained S-Boxes doesn't have fixed points and his Nonlinearity values are acceptable. Also we present the definition of Parameter Set and introduce some properties that allow show when the parameter set of a S-Box is invariant under the composition with affine transformations. This have an special importance because we can obtain dynamic S-Boxes for Block Ciphers constructing one with good Parameter Set and making the composition with affine functions.

2 THEORETICAL ASPECTS

In this section are presented some definitions related with S-boxes design that are very important for a better understanding of the paper.

Definition 1. A function $\Pi : GF(2)^n \rightarrow GF(2)^m$ is called an $n \times m$ S-Box, in which any input of n bits is transformed into output of m bits

There are many criteria dedicated to measure the cryptographic security of S-Boxes. However, does not exists an universal measure that allow affirm which of this are the most important to take into account in S-Box design. We will work with the four criteria presented bellow [11]:

Definition 2 (Nonlinearity). Let $f : (GF(2))^n \rightarrow (GF(2))^n$ the nonlinearity of this function denoted by $nl(f)$ is the magnitude:

$$nl(f) = \min_{g \in A_n} wt(f + g)$$

when A_n is the set of affine functions and wt is the Hamming weight.

Definition 3 (Algebraic Degree). Let $f(x_1; x_2, \dots, x_n)$ a boolean function of n variables, then f could be represented as (ANF):

$$f(x) = a_0 \oplus \bigoplus_{i=1}^n a_i x_i \oplus \bigoplus_{1 \leq i \neq j \leq n} a_{ij} x_i x_j \oplus \dots \oplus a_{12\dots n} x_1 x_2 \dots x_n$$

when the coefficients $a_0, a_i, a_{ij}, \dots, a_{12\dots n} \in \{0,1\}$. The number of variables in the product of higher order with non-zero coefficients is known as algebraic degree of f .

The algebraic degree of a vectorial boolean function is the minimum algebraic degree of all *non-zero* linear combinations of his component functions.

$$deg(S) = \min_f deg\{f | f = \bigoplus_{i=1}^m j_i f_i(x)\}$$

Definition 4 (Differential Characteristic). Let $GF(2)^n$ y $GF(2)^m$ two finite vectorial spaces. An application $\Pi : GF(2)^n \rightarrow GF(2)^m$ is called differential δ - uniform (or differential characteristic δ) if $\forall \alpha \in GF(2)^n, \alpha \neq 0$ and $\beta \in GF(2)^m, |\{z \in GF(2)^n | \Pi(z + \alpha) + \Pi(z) = \beta\}| \leq \delta$.

Definition 5 (Branch Number). The branch number (diffusion coefficient) of $f : GF(2)^n \rightarrow GF(2)^n$ is defined as the value obtained from the expression:

This four criteria: branch number, algebraic degree, nonlinearity and differential characteristic are very important because they provide the necessary elements against the linear and differential cryptanalysis.

Linear Cryptanalysis The main objective of linear cryptanalysis is to find a linear probabilistic relation for a part of the cipher equation that connects the input and the output data. Also this relation must be complied with a probability significantly different of 1/2.

If this probability is denoted by p_L is necessary minimize the magnitude $|p_L - 1/2|$ to obtain a good security against this kind of attack. If we assume that the X was selected randomly, then; the better linear approximation of an S-Box in the form $a_1 X_n \oplus \dots \oplus a_n X_n = b_1 Y_n \oplus \dots \oplus b_n Y_n$ be obtained with probability p_e . Also

$$|p_e - 1/2| = \frac{2^{n-1} - NL(\Pi)}{2^n} \quad (1)$$

For example if we have a SP network with 10 rounds and an 8×8 S-Box with $NL(\Pi) = 112$ like in AES, will be necessary at least 2^{62} known plain-text to obtain a bit of the key.

Differential Cryptanalysis In many of his papers Biham and Shamir show the witness of some ciphers against the differential cryptanalysis for this reason is fundamental guaranties the protection against this kind of attack.

In [12] is defined the probability of the difference par $(\Delta X, \Delta Y)$ in an S-Box to the probability that ΔY occurs when one of the input values is selected randomly and the other is deduced from the relation ΔX . The probability of the most frequently par was denoted as p_δ .

Many authors affirm that the minimization of p_δ brings resistance against differential cryptanalysis. O'Connor [13] shows that for a values of n , The probability of the difference par in an S-Box is expected less than the value $n/2^{n-1}$. For example in an 8×8 S-Box $p_\delta \leq 2^{-4}$.

Another frequently cause of high probability characteristic is the poor diffusion in the S-Box bits [13][12].

3 S-BOX CHARACTERIZATION

In this section we introduce some definitions that allow us characterize S-Boxes from the values obtained in the four criteria shown before.

Definition 6. Let $\Pi : GF(2^n) \rightarrow GF(2^n)$ we called Parameter Set of Π to the vector

$$\Pi^* = (p_e, bn, p_\delta, deg)$$

when the parameters p_e, bn, p_δ, deg denote the static probability, branch number, characteristic probability and the algebraic degree of Π respectively.

The definition shown before brings the necessary tools to introduce the next definition from whom will be characterize the S-Boxes.

Definition 7. Let $\Pi_1 : GF(2^n) \rightarrow GF(2^n)$ and $\Pi_2 : GF(2^n) \rightarrow GF(2^n)$. We say that Π_1 and Π_2 are parametric equivalents if:

$$\Pi_1^* = \Pi_2^*$$

Another important approach would be posses dynamic S-Boxes in the cipher. However in practice is very difficult find S-Boxes with an acceptable parameter set. With this porpoise is presented the next definition.

Definition 8. Let $\Pi_1 : GF(2^n) \rightarrow GF(2^n)$ and $\Pi_2 : GF(2^n) \rightarrow GF(2^n)$. We say that Π_1 y Π_2 are left affine equivalents if exists an affine function $A : GF(2^n) \rightarrow GF(2^n)$ so that:

$$\Pi_1 = A \circ \Pi_2.$$

Affine functions are very easy to construct, for this reason we could maybe think about obtain S-Boxes with an acceptable parameter set from the composition of one with an acceptable parameter set selected a priory and an affine function. Take this into account is important to know the be haviour of this composition. For this reason we consider the next example.

if we have that $A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ then to $j_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $j_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ we obtain that:

$$A^{-1}j_1 = A^{-1}j_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The above example serves as a basis to demonstrate the following properties, which are an essential contribution to the development of this study.

Proposition 1. Let $\Pi_1 : GF(2^n) \rightarrow GF(2^n)$ and $\Pi_2 : GF(2^n) \rightarrow GF(2^n)$ S-Boxes left affine equivalent. Then we have that:

$$p_\delta(\Pi_1) \geq p_\delta(\Pi_2)$$

Proof. Given that Π_1 and Π_2 are S-Boxes left affine equivalent then exists $A(x) = \hat{A}(x) \oplus C$ so that $\Pi_1 = A \circ \Pi_2$.

$$\begin{aligned} p_\delta(\Pi_1) &= \max P(\Pi_1(x) \oplus \Pi_1(x \oplus i) = j) \\ &= XOR(\Pi_1)/2^n \\ &= \max |x : \hat{A}\Pi_2(x) \oplus C \oplus \hat{A}\Pi_2(x \oplus i) \oplus C = j|/2^n \\ &= \max |x : \Pi_2(x) \oplus \Pi_2(x \oplus i) = \hat{A}^{-1}j|/2^n \end{aligned}$$

Now taking $j' = \hat{A}^{-1}j$ we obtain that for each (i, j) exists (i, j') so that:

$$\begin{aligned} |x : \Pi_2(x) \oplus \Pi_2(x \oplus i) = \hat{A}^{-1}j| &= |x : \hat{A}\Pi_2(x) \oplus C \oplus \hat{A}\Pi_2(x \oplus i) \oplus C = j'| \\ &= |x : \Pi_2(x) \oplus \Pi_2(x \oplus i) = j'| \end{aligned}$$

However could happend that for two differents par (i, j) exist two equal par (i, j') as we can see in the example before. For this reason we have that:

$$\begin{aligned} \max XOR_{\Pi_1}(i, j) &\geq \max XOR_{\Pi_2}(i, j') \\ &\geq p_\delta(\Pi_2) \end{aligned}$$

For static probability a similar result could be obtain:

Proposition 2. Let $\Pi_1 : GF(2^n) \rightarrow GF(2^n)$ and $\Pi_2 : GF(2^n) \rightarrow GF(2^n)$ S-Boxes left affine equivalents. Then we have that:

$$p_e(\Pi_1) \geq p_e(\Pi_2).$$

Proof. Given that Π_1 and Π_2 are left affine equivalents then exist $A(x) = \hat{A}(x) \oplus C$ so that $\Pi_1 = A \circ \Pi_2$.

$$\begin{aligned} p_e(\Pi_1) &= 1 - \frac{NL(\Pi_1)}{2^n} \\ &= 1 - \frac{NL(\hat{A}\Pi_2 \oplus C)}{2^n} \end{aligned}$$

In the other side we have that:

$$\begin{aligned}
 &= NL(\hat{A}\Pi_2 \oplus C) \\
 &= \min NL(z) \\
 &= \min |x : a(\hat{A}\Pi_2 \oplus C) = \omega x \oplus c| \\
 &= \min |x : a\hat{A}\Pi_2 = \omega x \oplus (c \oplus aC)|
 \end{aligned}$$

Now taking $a' = a\hat{A}$ y $c' = c \oplus aC$ we obtain that for any $z = (a, \omega, c)$ exist $z' = (a', \omega, c')$ so that:

$$|x : a(\hat{A}\Pi_2 \oplus C) = \omega x \oplus c| = |x : a'(\Pi_1 \oplus C) = \omega x \oplus c'|$$

However could happend that for two different tern (a, ω, c) exist two equal tern (a', ω, c') . Then $NL(\Pi_1) \leq NL(\Pi_2)$ whereby

$$p_e(\Pi_2) \leq p_e(\Pi_1)$$

To see how is the behaviour of branch number of an S-BOX under left affine equivalents transformations we consider the next example:

Example 2. Let $\Pi_1 : GF(2^8) \rightarrow GF(2^8)$ and $\Pi_2 : GF(2^8) \rightarrow GF(2^8)$ left affine equivalents S-Boxes so that: $\Pi_1 = A \circ \Pi$,

$$A(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot x$$

And

$\Pi = \{209, 88, 197, 173, 93, 12, 116, 247, 210, 249, 125, 231, 169, 32, 2, 109, 65, 8, 79, 158, 60, 204, 105, 224, 233, 196, 96, 166, 175, 149, 213, 16, 18, 31, 97, 177, 48, 174, 45, 154, 243, 195, 1, 76, 92, 218, 200, 167, 135, 153, 68, 110, 121, 70, 141, 251, 203, 144, 220, 201, 14, 55, 108, 245, 13, 75, 21, 72, 238, 24, 132, 142, 178, 22, 57, 216, 129, 61, 191, 241, 17, 206, 36, 127, 190, 181, 112, 9, 41, 148, 255, 103, 170, 102, 4, 194, 217, 122, 56, 239, 117, 192, 185, 147, 33, 137, 90, 104, 43, 165, 160, 53, 146, 140, 171, 67, 54, 119, 214, 208, 62, 80, 23, 29, 69, 44, 159, 232, 221, 50, 234, 133, 237, 120, 11, 227, 244, 128, 207, 236, 163, 143, 63, 115, 19, 7, 20, 176, 38, 235, 151, 91, 49, 83, 183, 52, 64, 187, 161, 212, 35, 10, 86, 162, 100, 189, 40, 89, 71, 188, 46, 113, 253, 85, 37, 0, 58, 131, 219, 254, 25, 99, 156, 3, 47, 51, 136, 215, 240, 77, 155, 168, 30, 59, 123, 134, 34, 225, 226, 179, 198, 164, 87, 182, 39, 107, 211, 124, 193, 229, 66, 94, 172, 73, 28, 98, 152, 246, 106, 252, 130, 114, 145, 27, 95, 111, 5, 230, 184, 42, 199, 74, 84, 157, 139, 250, 81, 223, 248, 118, 138, 126, 180, 82, 242, 228, 78, 205, 186, 6, 101, 15, 26, 150, 202, 222, \}$

However is easy to see that: $(bn(\Pi_1) = 2) \neq (bn(\Pi_2) = 3)$ Attending to the results parametric equivalents transformations are not possible to classify by the left composition with affine transformations.

4 PRESENTATION OF THE NEW ALGORITHM

As we show before S-Boxes with good diffusion in the output bits posses a better resistance against differential cryptanalysis. The branch number is an S-Box design criterion that results as a diffusion measure. Unfortunately experimental results show that S-Boxes with non-trivial diffusion orders cant be determinate by randomly search ($bn > 2$).

Table 1. Probability of randomly selection of S-Boxes with $bn \geq 3$

n	Valor estimado
3	1.2×10^{-3}
4	4.9×10^{-5}
5	7.7×10^{-7}
6	14.5×10^{-9}
7	9.3×10^{-12}
8	6.7×10^{-15}

A new algorithm for the construction of S-Boxes with $bn = 3$ is presented below.

Algorithm 1: Algorithm to construct S-Boxes with $bn = 3$.

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Data:  $\Gamma = \{0, 1, 2, \dots, 2^n - 1\}$ ,  $\Delta_0 = \Gamma$ 
Result: S-caja con  $bn = 3$ 
1 for  $i = 0, 1, \dots, 2^n - 1$  do
2   while  $s[i] \neq -1$  do
3     if  $i=0$  then
4        $s[i] = rand(\Delta_0)$ 
5        $\Delta_i = \Delta_i - \{s[i]\}$ 
6     else
7        $C = \{i : wt(i, j) = 1, \forall j < i\}$ 
8       if  $|C|=0$  then
9          $s[i] = rand(\Delta_i) - i$ 
10         $\Delta_i = \Delta_i - \{s[i]\}$ 
11      else
12         $s[i] = v : wt(v, s[m]) \neq 1 \quad \forall m < i \text{ o } s[i] = -1$ 

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For a better understanding of the algorithm we consider the next sets:

$$L_i = \{(a, b) : wt(a \oplus b) = i\}$$

$B(i)$ is the set formed by the images of the elements less equal than i .

Then the construction is performed as follows:

A random value different to 0 is assigned to the S-box for input 0.

$$f(0) = rand(255) \neq 0$$

Then if $(i, k) \in L_2$, $\forall k < i$ is assigned to $f(i)$ a random value different to i not belonging to the set $f(k) : k < i$.

$$f(i) = rand(255) \notin B(i - 1) \neq i$$

In the other way if $\exists K < i$ so that $(i, k) \in L_1$.

$$f(i) = x_i \neq i : (x_i, f(k)) \in L_2$$

4.1 Comparative results

In [12] Tavares propose an algorithm for the construction of S-Boxes with $bn \geq 3$, however this algorithm present some inconvenient. The 8×8 constructed S-Boxes could posses a high quantity of fixed points. Also the nonlinearity values for this constructions are always less equal than 96 as we can see in the table 3.

Table 2. Nonlinearity of 8×8 S-Boxes in the new construction.

Parameters	bn	Min NL	Max NL	$\%NL = 96$	$\%NL = 98$	$\%NL = 100$
Tavares Algorithm	3	86	96	26	0	0
New Algorithm	3	92	100	36	14	0.5

We generate a set of 50 S-Boxes in which we find his nonlinearity values and the results are showed in the table 3.

Is easy to see that if we compare the result of table 3 the output S-boxes of the new algorithm posses nonlinearity values higher than the S-boxes obtained by the Tavares algorithm.

Table 3. Comparision between the new algorithm and the Tavares proposal.

Parameters	Tavares algorithm	New Algorithm
Max nonlinearity	96	100
Fixed Points	yes	no
Kind of algorithm	search	constructive

5 CONCLUSIONS

In this paper is presented an algorithm to generate S-boxes with high diffusion. Also was shown that these outputs could have acceptables values of nonlinearity and do not have fixed point wich allows that these S-boxes could be used as part of the cipher in a secure way.

As a theoretical result was proved that the Parameter Set of an S-box is not invariant under left affine transformations. Take this into account is not possible obtain S-boxes with a good Parameter Set by the left composition with an affine function and a good S-box.

Finally was shown that the outputs of the new algorithm have better results than the outputs of the Tavares algorithm.

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