EXACT BOUND STATE SOLUTION OF Q-DEFORMED WOODS-SAXON PLUS MODIFIED COULOMB POTENTIAL USING CONVENTIONAL NIKIFOROV-UVAROV METHOD

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ABSTRACT

In this work, we obtained an exact solution to Schrodinger equation using q-deformed Woods-Saxon plus modified Coulomb potential Using conventional Nikiforov-Uvarov method. We also obtained the energy eigen value and its associated total wave function . This potential with some suitable conditions reduces to two well known potentials namely: the Yukawa and coulomb potential. Finally, we obtained the numerical results for energy eigen value with different values of q as dimensionless parameter. The result shows that the values of the energies for different quantum number(n) is negative(bound state condition) and increases with an increase in the value of the dimensionless parameter(arbitrary constant). The graph in figure (1) shows the different energy levels for a particular quantum number.

KEYWORDS

Schrodinger, Nikiforov-Uvarov method, Woods-Saxon potential plus modified exponential potential.

1. INTRODUCTION

Schrodinger equation forms the non-relativistic part of the quantum mechanics and obtaining both exact and arbitrary solution with some chosen potentials has been of great interest because of its enormous applications [1-2] Obtaining the total wave function and its corresponding eigen value is essential and very important as its provides the necessary information for the quantum mechanical systems[3-5]. Many authors have provided both exact and approximate solutions to Schrodinger equation using different solvable potentials such as Pseudoharmonic, Rosen-Morse, coulomb , Yukawa, Kratzer ,Hellmann potentials [6-9]. Some of these potentials are solvable in Schrodinger equation using some method. These methods are: variational principles[10-11], Factorization method[12], Nikiforov-Uvarov method[13-16].

2. q-DEFORMED WOODS-SAXON PLUS MODIFIED COULOMB POTENTIAL

Woods-Saxon potential for more than a decade has been of practical interest especially as it is widely used in nuclear nuclear physics to describe nuclear shell model [17-20]. Nuclear shell model describes the interaction of nucleon with heavy nucleus [21-23]. Understanding the behavior of electrons in the valence shell has a major role in investigating metallic system [24]. This potential is express as a function of the distance from the centre of the nucleus. Different authors have worked with Woods-Saxon potential using different techniques especially in relativistic quantum mechanics. C. Roja and V.M. Villalba [25] developed an approach to obtain the bound state solution of one dimensional Dirac equation for Woods-Saxon potential. A. Ikot and I. Akpan [26] solve the radial Schrodinger equation for more general Woods-Saxon potential (MGWSP) using Nikiforov-Uvarov method . Also J. Sadeghi and M. Pahlavani [27] formulated an hierarchy of Hamiltonian for Spherical Woods-Saxon potential

The q-deformed Woods-Saxon plus modified Coulomb potential is given by

$$V(r) = \frac{-v_0}{\left(1 + qe^{\frac{r-R_0}{a}}\right)} - \frac{Ae^{\alpha r}}{r} \quad (1)$$

Where q is the dimensionless parameter that takes different values . A is a constant value.

v_0 is the potential well depth having the dimension of energy.

R_0 is the nuclear radius express as

$$R_0 = r_0 B^{\frac{1}{3}} \quad (2)$$

B in equation (2) is the mass number while a in equation (1) is a length representing the surface thickness of the nucleus. If with the condition that $r - R_0 \equiv r$ and $a = \frac{1}{2\alpha}$. Then equation (1)

reduces to

$$V(r) = \frac{-v_0}{(1 + qe^{2\alpha r})} - \frac{Ae^{\alpha r}}{r} \quad (3)$$

The second term in equation (3) is the modified exponential term. This term takes care of the coulomb force that exist between the valence electron and the protons in the nucleus of an atom, and hence, give a complete description of an atom.

3. CONVENTIONAL FORM OF NIKIFOROV-UVAROV METHOD.

The Nikiforov-Uvarov method popularly known as (NU) involves reducing second order linear differential equation into hyper-geometric type [28, 29] .This method provides exact solutions in terms of special orthogonal functions as well as corresponding energy eigen value. This method can be used to provide solutions to both relativistic and non-relativistic equations in quantum mechanics with some suitable potentials. This equation can be express as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi(s) = 0 \quad (4)$$

To find the solution to equation (4), the total wave function can be express as

$$\Psi(s) = \phi(s) \chi(s) \quad (5)$$

Substituting, equation (5) into equation (4) reduces equation (4) into hyper-geometric type.

$$\sigma(s) \chi''(s) + \tau(s) \chi'(s) + \lambda \chi(s) = 0 \quad (6)$$

Where the wave function $\psi(s)$ is defined as the logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (7)$$

Where $\pi(s)$ is at most first-degree polynomials.

Equation (6) can be express in Rodrigues relation as

$$Y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma''(s) \rho(s)] \quad (8)$$

B_n Is the normalization constant and the weight function $\rho(s)$ satisfies the condition.

$$\frac{d}{ds} (\sigma(s) \rho(s)) = \tau(s) \rho(s) \quad (9)$$

Such that

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (10)$$

In order to accomplish the conditions imposed on the weight function $\rho(s)$, it is necessary that the classical orthogonal polynomials $\tau(s)$ be equal to zero, so that its derivative be less than zero. This implies that

$$\frac{d\tau(s)}{ds} < 0 \quad (11)$$

Therefore, the function $\pi(s)$ and the parameter λ required for the NU-method are defined as follows:

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} \quad (12)$$

$$\lambda = k + \pi'(s) \quad (13)$$

The k-values in equation (12) are possible to evaluate if the expression under the square root must be square of polynomials. This is possible, if and only if its discriminant is zero. With this, a new eigen-value equation becomes

$$\lambda_n = \frac{nd\tau}{ds} - \frac{n(n-1)}{2} \frac{d^2\sigma}{ds^2}, n = 0, 1, 2 \dots (14)$$

where $\tau(s)$ is as defined in equation (10) and on comparing equation (13) and equation (14), we obtain the energy Eigen values.

4. SOLUTION OF SCHRODINGER EQUATION WITH THE PROPOSED POTENTIAL

The Schrodinger equation is given by

$$\frac{d^2\Psi(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - V(r) - \frac{\hbar^2 l(l+1)}{2\mu r^2} \right] \Psi(r) = 0 \quad (15)$$

Substituting equation (3) into equation (15) for $l = 0$ gives

$$\frac{d^2\Psi(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E + \frac{v_0}{(1+qe^{2\alpha r})} + \frac{Ae^{\alpha r}}{r} \right] \Psi(r) = 0 \quad (16).$$

Using the transformation.

$$s = -e^{2\alpha r} \Rightarrow s^2 = e^{4\alpha r} \text{ and } \frac{1}{r} = \frac{2\alpha e^{\alpha r}}{(1+qe^{2\alpha r})} \text{ as the approximation to the centrifugal term, then}$$

equation (16) reduces to equation (17) .

$$s^2 \frac{d^2\Psi(s)}{ds^2} + s \frac{d\Psi(s)}{ds} + \frac{\mu}{2\hbar^2\alpha^2} \left[E + \frac{v_0}{(1-qs)} - \frac{2A\alpha s}{(1-qs)} \right] \Psi(s) = 0 \quad (17) \quad .$$

Further simplification reduces equation (17) to

$$\frac{d^2\Psi(s)}{ds^2} + \frac{1}{s} \frac{d\Psi(s)}{ds} + \frac{\mu}{2\hbar^2 s^2 \alpha^2} \left[E + \frac{v_0}{(1-qs)} - \frac{2A\alpha s}{(1-qs)} \right] \Psi(s) = 0 \quad (18) \quad .$$

Equation (18) can further be reduced to

$$\frac{d^2\Psi(s)}{ds^2} + \frac{(1-qs)}{s(1-qs)} \frac{d\Psi(s)}{ds} + \frac{1}{s^2(1-qs)^2} \left[(\gamma q - \varepsilon q^2)s^2 + (2\varepsilon q - \delta q - \gamma)s + (\delta - \varepsilon) \right] \Psi(s) = 0 \quad (19) \quad .$$

Where $\varepsilon = \frac{-\mu E}{2\hbar^2 \alpha^2}$, $\delta = \frac{\mu v_0}{2\hbar^2 \alpha^2}$, and $\gamma = \frac{\mu A}{\hbar^2 \alpha}$ (19a). Then comparing equation (19) with (4), we obtain the followings:

$$\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1 - qs), \quad \tilde{\sigma}(s) = (\gamma q - \varepsilon q^2)s^2 + (2\varepsilon q - \delta q - \gamma)s + (\delta - \varepsilon).$$

Then using equation (12), The polynomial equation of $\pi(s)$ become

$$\pi(s) = \frac{-qs}{2} \pm \frac{1}{2} \sqrt{(q^2 + 4\varepsilon q^2 - 4kq - 4\gamma q)s^2 + 4(k + \gamma + \delta q - 2\varepsilon q)s + 4(\varepsilon - \delta)} \quad (20).$$

To find the value of k we consider the discriminant, that is, only the term under the square root sign such that $b^2 - 4ac = 0$. Hence,

$$k_1 = (\delta q - \gamma) + q\sqrt{\varepsilon - \delta} \quad \text{and} \quad k_2 = (\delta q - \gamma) - q\sqrt{\varepsilon - \delta} \quad (21).$$

Substituting the value of k_1 for k in equation (20), then.

$$\pi(s) = \frac{-qs}{2} \pm \frac{1}{2} \left[(1 - 2\sqrt{\varepsilon - \delta})qs + 2\sqrt{\varepsilon - \delta} \right] \quad (22).$$

Also, substituting the value of k_2 for k in equation (20) then

$$\pi(s) = \frac{-qs}{2} \pm \frac{1}{2} \left[(1 + 2\sqrt{\varepsilon - \delta})qs - 2\sqrt{\varepsilon - \delta} \right] \quad (23).$$

The four values of $\pi(s)$ is then given by

$$\pi(s) = \frac{-qs}{2} \pm \frac{1}{2} \left[\begin{array}{l} (1 - 2\sqrt{\varepsilon - \delta})qs + 2\sqrt{\varepsilon - \delta}, \quad k_1 = (\delta q - \gamma) + q\sqrt{\varepsilon - \delta} \\ (1 + 2\sqrt{\varepsilon - \delta})qs - 2\sqrt{\varepsilon - \delta}, \quad k_2 = (\delta q - \gamma) - q\sqrt{\varepsilon - \delta} \end{array} \right] \quad (24)$$

$\pi(s)$ has four solutions and one of the solutions satisfied bound state condition which is

$$\pi(s) = \frac{-qs}{2} - \frac{1}{2} \left[(1 + 2\sqrt{\varepsilon - \delta})qs - 2\sqrt{\varepsilon - \delta} \right] \quad (25)$$

For Bound state condition to be satisfied then equation (11) must be satisfied. But using equation (10) then

$$\tau(s) = 1 - 2qs - \left[(1 + 2\sqrt{\varepsilon - \delta})qs - 2\sqrt{\varepsilon - \delta} \right] \quad (26).$$

Using equation (11)

$$\tau'(s) = -q \left[\left(3 + 2\sqrt{\varepsilon - \delta} \right) \right] < 0 \quad (26).$$

This then satisfies the bound state condition. Using equation (13)

$$\lambda = k + \pi'(s) = \delta q - \gamma - q - 2q\sqrt{\varepsilon - \delta} \quad (27) \quad .$$

Using equation (14)

$$\lambda_n = 3nq + 2nq\sqrt{\varepsilon - \delta} + qn(n-1) \quad (28)$$

5. CALCULATION OF THE ENERGY EIGEN VALUE

The energy eigen-value is obtained by equating equation (27) to equation (28) Thus,

$$\varepsilon = \left[\frac{\delta}{2(n+1)} - \frac{(n+1)}{2} - \frac{\gamma}{2q(n+1)} \right]^2 + \delta \quad (29).$$

Substituting (19a) into equation (29) gives the energy to be

$$E_n = \frac{-\hbar^2}{2a^2\mu} \left[\left(\frac{a^2\mu v_0}{\hbar^2(n+1)} - \frac{a\mu A}{q\hbar^2(n+1)} - \frac{(n+1)}{2} \right)^2 + \frac{2a\mu v_0}{\hbar^2} \right] \quad (30).$$

6. CALCULATION OF THE WAVE FUNCTION

Using equation (7)

$$\frac{\phi^1(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \Rightarrow \frac{\phi^1(s)}{\phi(s)} = \frac{-qs}{2(s-qs^2)} - \frac{1}{2(s-qs^2)} \left[\left(1 + 2\sqrt{\varepsilon - \delta} \right) qs - 2\sqrt{\varepsilon - \delta} \right] \quad (31).$$

By integrating equation (31) and simplifying reduces it to

$$\phi(s) = \left[s^{\sqrt{\varepsilon - \delta}} (1-qs)^{(1+2\sqrt{\varepsilon - \delta})} (1-qs)^{-\frac{1}{2}(1+\sqrt{\varepsilon - \delta})} \right]^{\frac{1}{2}} \Rightarrow \phi(s) = s^{\frac{1}{2}\sqrt{\varepsilon - \delta}} (1-qs)^{\frac{1}{2}\sqrt{\varepsilon - \delta}} \quad (32).$$

Equation (32) gives the first part of the wave function. To determine the second part of the wave function, we first of all calculate the weight function using equation (9). Thus:

$$\begin{aligned} \frac{d}{ds}(\sigma(s)\rho(s)) &= \tau(s)\rho(s) \Rightarrow \int \frac{\rho'(s)}{\rho(s)} ds = \int \frac{\tau(s) - \sigma'(s)}{\sigma(s)} ds \\ \Rightarrow \rho(s) &= s^{-2\sqrt{\varepsilon-\delta}} (1-qs)^{(1+4\sqrt{\varepsilon-\delta})} \end{aligned} \quad (33)$$

Rewriting equation (33) using Rodrique relation (8) gives

$$\chi_n(s) = B_n(s) s^{2\sqrt{\varepsilon-\delta}} (1-qs)^{-(1+4\sqrt{\varepsilon-\delta})} \frac{d^n}{ds^n} \left[s^{n-2\sqrt{\varepsilon-\delta}} (1-qs)^{n+(1+4\sqrt{\varepsilon-\delta})} \right] \quad (34)$$

Associated Laguerre polynomial is given as.

$$Y_n(s) = B_n(s) s^{2\nu} (1-s)^{-\mu} \frac{d^n}{ds^n} \left[s^{n-2\nu} (1-s)^{n+\mu} \right] = B_n P_n^{[\mu+2\nu, \mu]}(1-s) \quad (35)$$

Comparing equations (34) and (35) then

$$\chi_n(s) = B_n(s) P_n^{[(1+8\sqrt{\varepsilon-\delta}), (1+4\sqrt{\varepsilon-\delta})]}(1-qs) \quad (36) .$$

Using equation (5), we obtain the total wave function as given in equation (37).

$$\Psi_n(s) = B_n(s) s^{\frac{1}{2}\sqrt{\varepsilon-\delta}} (1-qs)^{-\frac{1}{2}\sqrt{\varepsilon-\delta}} P_n^{[(1+8\sqrt{\varepsilon-\delta}), (1+4\sqrt{\varepsilon-\delta})]}(1-qs) \quad (37)$$

7. RESULT AND DISCUSSION

Considering the proposed potential in equation (3), if $\nu_0 = \alpha = 0$, then the potential reduced to Coulomb potential $V(r) = \frac{-A}{r}$.

If the constant A is negative, α in the exponential function is also negative while $\nu_0 = 0$ then, the proposed potential reduces to Yukawa potential.

$$V(r) = \frac{Ae^{-\alpha r}}{r} .$$

8. NUMERICAL RESULT

n	q	E_n	n	q	E_n	n	q	E_n	n	q	E_n	n	q	E_n
1	1	-103.55556	2	1	-45.17014	3	1	-24.753473	4	1	-15.323473	5	1	-10.222222
1	2	-107.17014	2	2	-46.75347	3	2	-25.625868	4	2	-15.866806	5	2	-10.586805
1	3	-108.38889	2	3	-47.28742	3	3	-25.920140	4	3	-16.050138	5	3	-10.709876
1	4	-109.00087	2	4	-47.55556	3	4	-26.067926	4	4	-16.142221	5	4	-10.771702
1	5	-109.36889	2	5	-47.71681	3	5	-26.156807	4	5	-16.197607	5	5	-10.808888
1	6	-109.61458	2	6	-47.82446	3	6	-26.216146	4	6	-16.234583	5	6	-10.833719
1	7	-109.79025	2	7	-47.90143	3	7	-26.258574	4	7	-16.261024	5	7	-10.851475
1	8	-109.92210	2	8	-47.95920	3	8	-26.290419	4	8	-16.280860	5	8	-10.864800

Table 1: Computed values of energy for the diffuse parameter $a=6.0$ and $v_0=5\text{ev}$

Fig 1

9. CONCLUSION

In this work, we have used the proposed potential to determine the energy and the wave function of the Schrodinger wave equation using conventional Nikiforov-Uvarov method. The numerical values of this energy increases significantly with an increase in the dimensionless parameter and this is illustrated graphically to show the different energy levels of electron in an atom. The numerical result also shows that the energy level increases with an increase in quantum state. However, the negative energies in the numerical result conform to bound state condition for a non-relativistic wave equation.

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