AXISYMMETRIC BESSEL LIGHT BEAMS

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Abstract :

Bessel light beams satisfy a wave equation but, in addition, they are solutions of Maxwell's equations. The light power they convey is proportional to the square modulus of the electric field $\mathbf{E}$. In cylindrical coordinates, Bessel light beams depend on the polarization of $\mathbf{E}$ and, assuming an axisymmetric beam, we give the expressions of $\mathbf{E}$ when its polarization is, azimuthal, radial and linear or circular. The energy flux carried by these beams is obtained from the time averaged Poynting vector.

Keywords:

Light beams, Bessel, Electric field, Polarization.

I. INTRODUCTION

Introduced by Durnin [1,2] as nondiffractive, Bessel beams have been the object of numerous works [3-6] and of some experiments with quasi Bessel beams (quasi since Bessel beams require an infinite amount of power). Only a few works were devoted to Bessel *light* beams [8,9,10] which, in addition to satisfy a wave equation, are solutions of Maxwell's equations.

The light power conveyed by a Bessel light beam depends on the square modulus $|\mathbf{E}|^2$ of the electric field and, in a homogeneous, isotropic medium with permittivity ϵ and permeability μ , $\mathbf{E}$ satisfies the Helmholtz and divergence equations

$$(\Delta + k^2)\mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0, \quad k^2 = \omega^2 \epsilon \mu / c^2 \quad (1)$$

In cylindrical coordinates r, ϕ, z , the Helmholtz equation is different for each coordinate of $\mathbf{E}$ and with $\nabla = 1/r \partial_r (r \partial_r) + 1/r^2 \partial_\phi^2 + \partial_z^2$, we get [11] for the components E_r, E_ϕ, E_z

$$\begin{aligned} \Delta E_r &= \nabla^2 E_r - 1/r^2 E_r - 2/r^2 \partial_\phi E_\phi \\ \Delta E_\phi &= \nabla^2 E_\phi - 1/r^2 E_\phi + 2/r^2 \partial_\phi E_r \\ \Delta E_z &= \nabla^2 E_z \end{aligned} \quad (2)$$

while the divergence equation becomes

$$(\partial_r + 1/r) E_r + 1/r \partial_\phi E_\phi + \partial_z E_z = 0 \quad (2a)$$

So, according to (2), Bessel light beams depend on the polarization of the electric field and, we shall consider successively azimuthal, radial, and linear or circular polarizations. In addition, Bessel light beams are supposed axisymmetric so that the electric field is only function of r and z . Finally we look for the solutions of the Helmholtz equation in the form

$$\mathbf{E}(r, z) = E_0 \exp(ik_z z) \mathbf{E}(r) \quad (3)$$

with the constant amplitude E_0 .

The energy flux carried by axisymmetric Bessel light beams is obtained from the time averaged Poynting vector.

2. ELECTRIC FIELD POLARIZATION

The polarization of the electric field $\mathbf{E}$ in Bessel light beams is carefully discussed in [8,12]. We follow closely [12] and we impose $\partial_\phi \mathbf{E} = 0$ to satisfy the axisymmetric condition.

2.1 The azimuthal polarization is characterized by $E_r = E_z = 0$ so that according to (2) and since $\partial_\phi E_\phi = 0$, the Helmholtz equation satisfied by E_ϕ is

$$(\partial_r^2 + 1/r \partial_r - 1/r^2 + \partial_z^2 + k^2) E_\phi(r, z) = 0 \quad (4)$$

with the solutions in which $k_r^2 + k_z^2 = k^2$

$$E_\phi(r, z) = E_0 J_1(k_r r) \exp(ik_z z) \quad (5)$$

J_1 is the Bessel function of the first kind of order one and the divergence equation (2a) is trivially satisfied.

2.2 For the radial polarization, only the component E_ϕ is null. The component E_r , according to (2) and taking into account $\partial_\phi E_r = 0$, satisfies the Helmholtz equation (4) so that

$$E_r(r, z) = E_0 J_1(k_r r) \exp(ik_z z) \quad (6)$$

Now, the component E_z still from (2) and from $\partial_\phi E_z = 0$, is solution of the Helmholtz equation

$$(\partial_r^2 + 1/r \partial_r + \partial_z^2 + k^2) E_z(r, z) = 0 \quad (7)$$

with the solutions

$$E_z(r, z) = -iE_0 k_r/k_z J_0(k_r r) \exp(ik_z z) \quad (8)$$

in which J_0 is the Bessel function of the first kind of order zero.

And using the relation $(\partial_r + 1/r) J_1(k_r r) = k_r J_0(k_r r)$, it is checked at once that the divergence equation (2) is satisfied.

2.3 For linear and circular polarizations, the electric field, in cartesian coordinates is obtained from Eq.(12) of [12] in the form

$$\mathbf{E}(r, z) = E_0 \exp(ik_z z) [(\alpha u_x + \beta u_y) J_0(k_r r) + ik_r/2k_z \{(\alpha + i\beta) \exp(-i\phi) J_{-1}(k_r r) + (\alpha - i\beta) \exp(i\phi) J_1(k_r r)\} u_z] \quad (9)$$

u_x, u_y, u_z are the unit vectors in cartesian coordinates Then, since $J_{-1} = -J_1$, we get from (9)

$$\begin{aligned} E_x(r, z) &= \alpha E_0 \exp(ik_z z) J_0(k_r r) \\ E_y(r, z) &= \beta E_0 \exp(ik_z z) J_0(k_r r) \\ E_z(r, z) &= -ik_r/k_z (\alpha \cos\phi + \beta \sin\phi) E_0 \exp(ik_z z) J_1(k_r r) \end{aligned} \quad (10)$$

For a linear polarization the parameters α, β are both real while for right and left circular polarizations, they satisfy $\beta = i\alpha$ and $\beta = -i\alpha$ respectively.

In these last expressions $r = (x^2 + y^2)^{1/2}$, and a simple calculation proves that the divergence equation

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 0 \quad (11)$$

is fulfilled.

3. BESSEL LIGHT BEAMS

These results are summarized in the following table leaving aside $E_0 \exp(ik_z z)$

Table 1

Polarization	:	electric field $\mathbf{E}(\mathbf{r})$
Azimuthal	:	$E_r(\mathbf{r}) = E_z(\mathbf{r}) = 0$, $E_\phi(\mathbf{r}) \approx J_1(k_r r)$
Radial	:	$E_\phi(\mathbf{r}) \approx 0$, $E_r(\mathbf{r}) \approx J_1(k_r r)$, $E_z(\mathbf{r}) \approx J_0(k_r r)$
Linear	:	$E_x(\mathbf{r}) \approx \alpha J_0(k_r r)$, $E_y(\mathbf{r}) \approx \beta J_0(k_r r)$, $E_z(\mathbf{r}) = -i\alpha k_r / k_z (\alpha \cos\phi + \beta \sin\phi) J_1(k_r r)$.

This table shows how the structure of Bessel light beams depends on the electric field polarization and, in any case, this structure reduces to that of conventional Bessel beams.

The importance of Bessel light beams lies in the energy flux they carry which may be obtained from the time averaged Poynting vector [12,13]

$$\mathbf{S} = 1/4i\omega\mu [\mathbf{E} \wedge \text{curl} \mathbf{E}^* + \text{cc}] \quad (12)$$

the asterisk denotes the complex conjugation and $\{\text{cc}\} = \mathbf{E}^* \wedge \text{curl} \mathbf{E}$.

For an axisymmetric beam $\partial_\phi \mathbf{E} = 0$ and the components of $\text{curl} \mathbf{E}^*$ are

$$\begin{aligned} (\text{curl} \mathbf{E}^*)_r &= -\partial_z E_\phi^* \\ (\text{curl} \mathbf{E}^*)_\phi &= \partial_z E_r^* - \partial_r E_z^* \\ (\text{curl} \mathbf{E}^*)_z &= (\partial_r + 1/r) E_\phi^* \end{aligned} \quad (13)$$

so that

$$\begin{aligned} (\mathbf{E} \wedge \text{curl} \mathbf{E}^*)_r &= E_\phi (\text{curl} \mathbf{E}^*)_z - E_z (\text{curl} \mathbf{E}^*)_\phi \\ &= E_\phi (\partial_r + 1/r) E_z^* - E_z (\partial_z E_r^* - \partial_r E_z^*) \\ (\mathbf{E} \wedge \text{curl} \mathbf{E}^*)_\phi &= E_z (\text{curl} \mathbf{E}^*)_r - E_r (\text{curl} \mathbf{E}^*)_z \\ &= -E_z \partial_z E_\phi^* - E_r \partial_r + 1/r E_\phi^* \\ (\mathbf{E} \wedge \text{curl} \mathbf{E}^*)_z &= E_r (\text{curl} \mathbf{E}^*)_\phi - E_\phi (\text{curl} \mathbf{E}^*)_r \\ &= E_r (\partial_z E_r^* - \partial_r E_z^*) - E_\phi \partial_z E_\phi^* \end{aligned} \quad (14)$$

Then, for azimuthal polarization $E_r = E_z = 0$ and taking into account (5) and (14) we get from (12) for the components of the Poynting vector

$$S_r = S_\phi = 0 \quad S_z = [E_0]^2 k_z / 2\omega\mu |J_1(k_r r)|^2 \quad (15)$$

A result also valid for the radial polarization since $E_\phi = 0$ and since according to (8)

$$E_z(\partial_z E_r^* - \partial_r E_z^*) + \{cc\} = 0 \quad (16)$$

So, from a Poynting point of view, taking (15) into account, an axisymmetric Bessel light beam could be characterized by the Bessel function J_1 and not by J_0 as it is usually assumed. We leave aside the most intricate Poynting vector for linear and circular polarizations also discussed in [12].

4. CONCLUSION

Works on conventional Bessel beams with J_0 are flourishing [1-6] while those on Bessel light beams are few. Nevertheless, starting with Stratton [14] (see [15]) some authors have considered non-diffractive higher order Bessel beams with J_m m integer. But although these fields satisfy the Helmholtz equation, they do not necessarily satisfy the Maxwell's equations. In [8], some of the results given here, are obtained for a vector potential, written as a superposition of the solutions to the vector Helmholtz equation, nevertheless this work [8] is not referenced in [15].

Remark : Bessel light beams depend on a conicity angle and when this angle is small ($\leq$ qq degrees) their z -component becomes negligible and they can be described as a field with the r -component proportional to J_0 .

The results obtained here may be generalized in two directions. In [16], a scalar, axisymmetric, pulsed Bessel beam has the representation

$$\psi(r, z, \omega) = J_0(r/r_0) \exp[i\beta(z)z] f(\omega - \omega_0) \quad (17)$$

r_0 is constant, $f(\omega - \omega_0)$ is the spectral distribution with $f(\omega_0)$ at $z = 0$ and

$$\beta(z) = \beta_0 + \sum_{m=1}^{\infty} \beta_m / m! (\omega - \omega_0)^m \quad (17a)$$

Then, assuming

$$f(\omega - \omega_0) = 1/\sqrt{2\pi} T_0 \exp[-T_0^3 (\omega - \omega_0)^3 / 3] \quad (18)$$

the inverse Fourier transform of (17) supplies the Airy Bessel pulsed beam $\Psi(r, z, t)$.

This technique could be applied to a Bessel light beam with azimuthal polarization just by changing J_0 into J_1 (see Table 1 of Sec.3) and giving an Airy-Bessel pulsed light beam.

Now, using Hertzian vector potentials of electric and magnetic types $\mathbf{\Pi}_e$, $\mathbf{\Pi}_m$ solutions of the Helmholtz equation $(\Delta + k^2) \mathbf{\Pi} = 0$, we get [16] for the electric and magnetic fields

$$\begin{aligned} \mathbf{E}_e &= \nabla \wedge \nabla \wedge \mathbf{\Pi}_e = \nabla \nabla \cdot \mathbf{\Pi}_e + k^2 \mathbf{\Pi}_e & , & & \mathbf{E}_m &= -i\mu_0 \nabla \wedge \mathbf{\Pi}_m & \text{a)} \\ \mathbf{E}_m &= \nabla \wedge \nabla \wedge \mathbf{\Pi}_m = \nabla \nabla \cdot \mathbf{\Pi}_m + k^2 \mathbf{\Pi}_m & , & & \mathbf{H}_m &= i\mu_0 \nabla \wedge \mathbf{\Pi}_m & \text{b)} \end{aligned} \quad (19)$$

This technique is used in [17] to get TE, TM Bessel beams.

Let us now use cylindrical coordinates, then with a judicious choice of $\mathbf{\Pi}$, we get further expressions of Bessel light beams. Suppose first $\Pi_r = \Pi_\phi = 0$, then we get [18] from (19a,b) the following expressions of the electric field

$$E_r = -i\omega\mu/r \partial_\phi \Pi_z, \quad , \quad E_r = \partial_r \partial_z \Pi_z$$

$$\begin{aligned} E_\phi &= i\omega\mu \partial_r \Pi_z & , & & E_\phi &= 1/r \partial_\phi \partial_z \Pi_z \\ E_z &= 0 & (20b) & , & E_z &= (k^2 + \partial_z^2) \Pi_z \end{aligned} \quad (20a)$$

But, Π_z is solution of the wave equation (7) so that (A_z is a constant amplitude)

$$\Pi_z = A_z \exp(ik_z z) J_0(k_r r) \quad , \quad k_r^2 + k_z^2 = k^2 \quad (21)$$

Substituting (21) into (20a,b) and using the relations

$$\partial_r J_0(k_r r) = -k_r J_1(k_r r) \quad , \quad (\partial_r + 1/r) J_1(k_r r) = k_r J_0(k_r r) \quad (22)$$

give

$$\begin{aligned} E_r &= -ik_r k_z A_z \exp(ik_z z) J_1(k_r r) & , & & E_r &= 0 \\ E_\phi &= 0 & , & & E_\phi &= i\omega\mu k_r A_z \exp(ik_z z) J_1(k_r r) \\ E_z &= k_r^2 A_z \exp(ik_z z) J_0(k_r r) & (23a) & , & E_z &= 0 \end{aligned} \quad (23b)$$

Similarly for $\Pi_\phi = \Pi_z = 0$, we get from (19a,b)

$$\begin{aligned} E_r &= 0 & , & & E_r &= [k^2 + \partial_r(\partial_r + 1/r)] \Pi_r \\ E_\phi &= -i\omega\mu \partial_z \Pi_r & , & & E_\phi &= 1/r \partial_\phi \partial_r (\Pi_r) \\ E_z &= i\omega\mu/r \partial_\phi \Pi_r & (24b) & , & E_z &= 1/r \partial_z \partial_r (\Pi_r) \end{aligned} \quad (24a)$$

with Π_r solution of the wave equation (4) so that

$$\Pi_r = A_r \exp(ik_z z) J_1(k_r r) \quad (25)$$

Substituting (25) into (24a,b) and using (22) give

$$\begin{aligned} E_r &= k_z^2 A_r \exp(ik_z z) J_1(k_r r) & , & & E_r &= 0 \\ E_\phi &= 0 & , & & E_\phi &= i\omega\mu k_z A_r \exp(ik_z z) J_1(k_r r) \\ E_z &= ik_z k_r A_r \exp(ik_z z) J_0(k_r r) & (26a) & , & E_z &= 0 \end{aligned} \quad (26b)$$

Finally for $\Pi_z = \Pi_r = 0$, it comes [18]

$$\begin{aligned} E_r &= i\omega\mu \partial_z \Pi_\phi & , & & E_r &= 1/r \partial_r \partial_\phi \Pi_\phi \\ E_\phi &= 0 & , & & E_\phi &= (k^2 + 1/r^2 \partial_\phi^2) \Pi_\phi \\ E_z &= i\omega\mu/r \partial_r (r \Pi_\phi) & (27a) & , & E_z &= 1/r \partial_z \partial_\phi \Pi_\phi \end{aligned} \quad (27b)$$

In which Π_ϕ is solution of the wave equation (4) so that

$$\Pi_\phi = A_\phi \exp(ik_z z) J_1(k_r r) \quad (28)$$

Substituting (28) into (27a,b) and still using (22) give

$$\begin{aligned} E_r &= 0 & , & & E_r &= \omega\mu k_z A_\phi \exp(ik_z z) J_1(k_r r) \\ E_\phi &= k^2 A_\phi \exp(ik_z z) J_1(k_r r) & , & & E_\phi &= 0 \\ E_z &= 0 & (29a) & , & E_z &= i\omega\mu k_r A_\phi \exp(ik_z z) J_0(k_r r) \end{aligned} \quad (29b)$$

Comparing these results with those of the Table 1 in Sec.3 shows that (23b), (26b), (29a) correspond to Bessel light beams with azimuthal polarization while for (23a), (26a), (29b) the polarization is radial. We may now combine these fields to get Bessel light beams with a more

elaborated structure. For instance, substituting (26a) and (23a) on one hand and (23b) and (26b) on the other hand gives

$$\begin{aligned} E_r &= (-ik_r k_z A_z + k_z^2 A_r) \exp(ik_z z) J_1(k_r r) \\ E_\phi &= 0 \\ E_z &= (k_r^2 A_z + ik_z k_r A_r) \exp(ik_z z) J_0(k_r r) \end{aligned} \quad (30a)$$

and

$$\begin{aligned} E_r &= 0 \\ E_\phi &= (i\omega\mu k_r A_z + i\omega\mu k_z A_r) \exp(ik_z z) J_1(k_r r) \\ E_z &= 0 \end{aligned} \quad (30b)$$

The polarization radial in (30a) is azimuthal in (30b). Similarly with (26a), (29a), (26b), (29b),

$$\begin{aligned} E_r &= k_z^2 A_r \exp(ik_z z) J_1(k_r r) \\ E_\phi &= k_r^2 A_\phi \exp(ik_z z) J_1(k_r r) \\ E_z &= ik_z k_r A_z \exp(ik_z z) J_0(k_r r) \end{aligned} \quad (31a)$$

and

$$\begin{aligned} E_r &= \omega\mu k_z A_\phi \exp(ik_z z) J_1(k_r r) \\ E_\phi &= i\omega\mu k_z A_r \exp(ik_z z) J_1(k_r r) \\ E_z &= i\omega\mu k_r A_\phi \exp(ik_z z) J_0(k_r r) \end{aligned} \quad (31b)$$

They represent Bessel light beams with polarizations different from those of Table 1.

Let us now come back to the conventional Bessel beams made of nondiffractive waves defined in terms of the Bessel function J_0 . Concentrate around the propagation axis, their transverse shape is patterned by J_0 . Thus, these beams can be described as a bullseye surrounded by an infinite (finite for a quasi Bessel beam) number of concentrating rings.

So, when people, working with lasers, generate a beam, nondiffractive on some propagation distance, with an annular ring structure [7], their claim to have produced a Bessel beam could be an abuse of language since it is not proved that their annular rings have the representation J_0 .

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