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OPTIMAL CONTROL OF SOLUTIONS TO THE MULTIPOINT INITIAL-FINAL PROBLEM FOR NONSTATIONARY RELATIVELY BOUNDED EQUATIONS OF SOBOLEV TYPE

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We study the problem of optimal control of solutions to an operator-differential equation, which is not solved with respect to the time derivative, together with a multipoint initial-final condition. In this case, one of the operators in the equation is multiplied by a scalar function of time. By the properties of the operators involved, the stationary equation has analytical resolving group. We construct a solution to the multipoint initial-final problem for the nonstationary equation. We show that a unique optimal control of solutions to this problem exists.

Apart from the introduction and bibliography, the article consists of three sections. The first section provides the essentials of the theory of relatively p-bounded operators. In the second section we construct a strong solution to the multipoint initial-final problem for nonstationary Sobolev-type equations. The third section contains our proof that there exists a unique optimal control of solutions to the multipoint initial-final problem.

Keywords: optimal control; multipoint initial-final problem; Sobolev-type equations; relatively bounded operator.

Introduction

Suppose that $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{U}$ are Hilbert spaces, and then take bounded linear operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $B \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, assuming that the kernel of L is non-trivial. Take also a closed linear operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ whose domain is dense in $\mathfrak{X}$.

Consider the Sobolev-type equation [1–4]

$$L\dot{x}(t) = a(t)Mx(t) + f(t) + Bu(t) \tag{1}$$

with a control vector function $u : [0, T] \to \mathfrak{U}$, a vector function $f : [0, T] \to \mathfrak{Y}$ of exterior force, and a scalar function $a : [0, T] \to \mathbb{R}_+$, to be specified later, characterizing the change in time of the parameters of (1). The operators L and M generate the analytic resolving group for the homogeneous stationary equation (1), which means that $a(t) \equiv 1$.

We consider an optimal control problem for (1). Namely, we aim to find a pair $(\hat{x}, \hat{u}) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ with

$$J(\hat{x}, \hat{u}) = \inf_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u). \tag{2}$$

Here $\mathfrak{U}_{ad}$ is a closed convex set of admissible controls in the Hilbert space $\mathfrak{U}$ of controls, all pairs (x, u) satisfy the multipoint initial-final problem [5] for (1), and $J(x, u)$ is a certain penalty functional in special form.

Previously the authors studied the optimal control problem for solutions to non-stationary Sobolev-type equations (1) with the Showalter–Sidorov condition [6, 7]. In this paper we study the optimal control of solutions to the multipoint initial–final problem [5], which is a generalized Showalter–Sidorov problem [8] for (1).

1. Relatively Spectrally Bounded Operators

Recall the standard notation of the theory of relatively p-bounded operators [3].

Starting with two Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$, take a bounded linear operator $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ with non-trivial kernel and a closed linear operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ whose domain is dense in $\mathfrak{X}$. Consider the stationary equation

$$L\dot{x}(t) = Mx(t) + f(t), \tag{3}$$

called a *Sobolev-type equation* [3].

Definition 1. The sets $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y};\mathfrak{X})\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ are called the *L-resolvent set* and the *L-spectrum* of M respectively.

Definition 2. The operator-valued functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are respectively called the *resolvent*, *right resolvent*, and *left resolvent* of M *with respect to* L (or briefly the *L-resolvent*, *right L-resolvent*, and *left L-resolvent* of M).

Lemma 1. *Given $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, the L-resolvent, right and left L-resolvents of M are analytic on $\rho^L(M)$.*

Definition 3. An operator M is called *spectrally bounded with respect to* an operator L (or briefly (L,σ)-*bounded*) whenever $\exists r_0 > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r_0) \;\Rightarrow\; (\mu \in \rho^L(M))$.

Put $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > r_0\}$. The Riesz-type integrals

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)\,d\mu, \qquad Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)\,d\mu$$

exist by Lemma 1 for every (L,σ)-bounded operator M. The operators $P \in \mathcal{L}(\mathfrak{X})$ and $Q \in \mathcal{L}(\mathfrak{Y})$ are projections [3]. Put $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$; $\mathfrak{X}^1 = \mathrm{im} P$, and $\mathfrak{Y}^1 = \mathrm{im} Q$. Denote the restriction of L (M) to $\mathfrak{X}^k$ by L_k (M_k) for $k = 0, 1$.

Theorem 1. *The following claims hold for every (L,σ)-bounded operator M:*

(i) *the operators $L_k, M_k : \mathfrak{X}^k \to \mathfrak{Y}^k$ for $k = 0, 1$;*

(ii) *the operators $M_0 \in \mathcal{L}(\mathfrak{X}^0;\mathfrak{Y}^0)$ and $M_1 \in \mathcal{C}l(\mathfrak{X}^1;\mathfrak{Y}^1)$;*

(iii) *there exists operators $L_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1;\mathfrak{X}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0;\mathfrak{X}^0)$;*

(iv) *there exist analytic resolving operator groups $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}\}$ for the homogeneous equation* (3) *and $\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$ for the equation $R^L_\beta(M)\dot{y}(t) = M(\beta L - M)^{-1}y(t)$, where $\beta \in \rho^L(M)$, which are of the form*

$$X^t = e^{tL_1^{-1}M_1}P = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)e^{\mu t}d\mu \qquad Y^t = e^{tM_1L_1^{-1}}Q = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)e^{\mu t}d\mu.$$

Theorem 1 implies the existence of the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{X}^1)$.

Definition 4. An (L, σ)-bounded operator M is called

(i) $(L, 0)$-*bounded* whenever the point ∞ is a *removable singularity* of the L-resolvent of M, that is, $H \equiv \mathbb{O}$;

(ii) (L, p)-*bounded* whenever the point ∞ is an *order* $p \in \mathbb{N}$ *pole* of the L-resolvent of M, that is, $H^p \neq \mathbb{O}$ and $H^{p+1} \equiv \mathbb{O}$;

(iii) (L, ∞)-*bounded* whenever the point ∞ is an *essential singularity* of the L-resolvent of M, that is, $H^q \neq \mathbb{O}$ for all $q \in \mathbb{N}$.

2. Strong Solutions of the Multipoint Problem

Take two Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$. For two operators $L \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ and $M \in \mathcal{C}l(\mathfrak{X}; \mathfrak{Y})$, where M is (L, p)-bounded for $p \in \{0\} \cup \mathbb{N}$, introduce the condition

$$\left.\begin{gathered} \sigma^L(M) = \bigcup_{j=0}^{n} \sigma_j^L(M),\ n \in \mathbb{N}, \text{ and } \sigma_j^L(M) \neq \emptyset, \text{ there is a closed loop } \gamma_j \subset \mathbb{C} \\ \text{and } \gamma_j = \partial D_j, \text{ where } D_j \supset \sigma_j^L(M), \text{ such that } \overline{D_j} \cap \sigma_0^L(M) = \emptyset \\ \text{and } \overline{D_k} \cap \overline{D_l} = \emptyset \text{ for all } j, k, l = \overline{1, n}, k \neq l. \end{gathered}\right\} \tag{4}$$

Define the operators $P_j \in \mathcal{L}(\mathfrak{X})$ and $Q_j \in \mathcal{L}(\mathfrak{Y})$ for $j = \overline{j, n}$ as

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} R_\mu^L(M) d\mu, \quad Q_j = \frac{1}{2\pi i} \int_{\gamma_j} L_\mu^L(M) d\mu, \quad j = \overline{1, n}$$

thanks to the relative spectral theorem [9], and moreover, the results of [9], and the operators $\quad P_0 = P - \sum\limits_{j=1}^{n} P_j, \quad Q_0 = Q - \sum\limits_{j=1}^{n} Q_j.$

Consider the multipoint initial-final problem

$$P_j(x(\tau_j) - x_j) = 0, \quad (\tau_j < \tau_{j+1}) \quad j = \overline{0, n} \tag{5}$$

for (3). Applying to (3) the projections $\mathbb{I} - Q$ and Q_j for $j = \overline{0, n}$ yields the equivalent system

$$H\dot{x}^0 = x^0 + M_0^{-1} f^0, \tag{6}$$

$$\dot{x}_j^1 = S_{1j} x_j^1 + L_{1j}^{-1} f_j^1, \tag{7}$$

where $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{X}^0)$ is a degree $p \in \{0\} \cup \mathbb{N}$ nilpotent operator, the operator $S_{1j} = L_{1j}^{-1} M_{1j} \in \mathcal{C}l(\mathfrak{X}_j^1)$ has the range $\sigma(S_j) = \sigma_j^L(M)$, while $f^0 = (\mathbb{I} - Q)f$, $f_j^1 = Q_j f$, $x^0 = (\mathbb{I} - P)x$, and $x_j^1 = P_j x$ for $j = \overline{0, n}$.

Put $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ and construct the space

$$H^{p+1}(\mathfrak{Y}) = \{\xi \in L_2(0, \tau; \mathfrak{Y}) : \xi^{(p+1)} \in L_2(0, T; \mathfrak{Y}),\ p \in \mathbb{N}_0\}$$

which is a Hilbert space with the inner product $\quad [\xi, \eta] = \sum\limits_{q=0}^{p+1} \int\limits_0^\tau \left\langle \xi^{(q)}, \eta^{(q)} \right\rangle_{\mathfrak{Y}} dt.$

Definition 5. A vector-valued function $x \in H^1(\mathfrak{X})$ is called a *strong solution* to the multipoint initial-final problem (3), (5) whenever it satisfies (3) and the terms of $P_j(x(\tau_j) - x_j) = 0$ for $j = \overline{0, n}$ almost everywhere.

Lemma 2. *If an operator M is (L,p)-bounded, with $p \in \mathbb{N}_0$, then for every vector function $f^0 \in H^{p+1}(\mathfrak{Y}^0)$ there exists a unique solution $x^0 \in H^1(\mathfrak{X}^0)$ to* (6):

$$x^0(t) = -\sum_{q=0}^{p} H^q M_0^{-1} \frac{d^q}{dt^q} f^0(t).$$

Lemma 3. *Under the assumptions of Lemma* 2*, if condition* (4) *is fulfilled then for every vector $x_j \in \mathfrak{X}$ and every vector function $f_j^1 \in H(\mathfrak{Y}_j^1)$ there exists a unique solution $x_j^1 \in H^1(\mathfrak{X}_j^1)$ to the problem $P_j(x(\tau_j) - x_j) = 0$ for* (7)*:* $x_j^1(t) = X_j^{t-\tau_j} x_{\tau_j} - \int\limits_t^{\tau_j} X_j^{t-s} L_{1j}^{-1} f_j^1(s)ds.$

Theorem 2. *Given vectors $x_j \in \mathfrak{X}$ for $j = \overline{0,n}$ and a vector function $f : [0,\tau] \to \mathfrak{Y}$ satisfying the assumptions of Lemmas* 2 *and* 3*, there exists a unique solution $x \in H^1(\mathfrak{X})$:*

$$x(t) = -\sum_{q=0}^{p} H^q M_0^{-1} \frac{d^q}{dt^q} f^0(t) + \sum_{j=0}^{n} \left(X_j^{t-\tau_j} x_{\tau_j} - \int\limits_t^{\tau_j} X_j^{t-s} L_{1j}^{-1} f_j^1(s)ds \right).$$

3. Optimal Control of the Multipoint Problem

For a Hilbert space $\mathfrak{X}$ consider the equation

$$L\dot{x}(t) = a(t)Mx(t) + f(t) + Bu(t) \tag{8}$$

with operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, and $B \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, a scalar function $a : [0,\tau) \to \mathbb{R}_+$, as well as vector functions $u : [0,\tau) \to \mathfrak{U}$ and $f : [0,\tau) \to \mathfrak{Y}$ to be specified later.

Take a Hilbert space $\mathfrak{Z}$ and an operator $C \in \mathcal{L}(\mathfrak{X};\mathfrak{Z})$. Consider the *penalty functional*

$$J(u) = \sum_{q=0}^{1} \int\limits_0^{\tau} \|z^{(q)} - z_d^{(q)}\|_{\mathfrak{Z}}^2 dt + \sum_{q=0}^{k} \int\limits_0^{\tau} \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt, \qquad z = Cx, \tag{9}$$

where $0 \le k \le p+1$. The operators $N_q \in \mathcal{L}(\mathfrak{U})$ for $q = 0, 1, \ldots, p+1$ are self-adjoint and positive definite, while $z_d = z_d(t,s)$ is an observation from some *space of observations* $\mathfrak{Z}$. Note that if $x \in H^1(\mathfrak{X})$ then $z \in H^1(\mathfrak{Z})$. By analogy with $H^{p+1}(\mathfrak{Y})$, define the space $H^{p+1}(\mathfrak{U})$, which is a Hilbert space because so is $\mathfrak{U}$. We distinguish a convex and closed subset $H_{ad}^{p+1}(\mathfrak{U})$ of the space $H^{p+1}(\mathfrak{U})$, called the *set of admissible controls.*

Definition 6. A vector function $v \in H_{ad}^{p+1}(\mathfrak{U})$ is called an *optimal control* of solutions to problem (5), (8) whenever

$$J(v) = \min_{(x(u),u) \in \mathfrak{X} \times H_{ad}^{p+1}(\mathfrak{U})} J(u), \tag{10}$$

where the pairs $(x(u), u) \in \mathfrak{X} \times H_{ad}^{p+1}(\mathfrak{U})$ satisfy (5), (8).

By Theorem 2, a unique solution $x \in H^1(\mathfrak{X})$ to problem (5), (8) exists for all vectors $x_j \in \mathfrak{X}$ for $j = \overline{0,n}$, vector functions $f \in H^{p+1}(\mathfrak{Y})$, $u \in H^{p+1}(\mathfrak{U})$ and a function $a \in C^{p+1}([0,T);\mathbb{R}_+)$ separated from zero:

$$x(t) = -\sum_{q=0}^{p} H^q M_0^{-1}(I-Q) \left(\frac{1}{a(t)} \frac{d}{dt} \right)^k \frac{f(t) + Bu(t)}{a(t)} +$$

$$+\sum_{j=0}^{n}\left(X_j^{A(t)-A(\tau_j)}x_{\tau_j}-\int_t^{\tau_j}X_j^{A(t)-A(s)}L_{1j}^{-1}Q_j(f(s)+Bu(s))ds\right) \tag{11}$$

by analogy with [6]. Here $A(t)=\int_0^t a(\varsigma)d\varsigma$. We now fix $x_j\in\mathfrak{X}$ for $j=\overline{0,n}$ and $f\in H^{p+1}(\mathfrak{Y})$ and consider (11) as a mapping $D: u\to x(u)$.

Lemma 4. *Given Hilbert spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{U}$, take an (L,p)-bounded operator M, with $p\in\mathbb{N}_0$, a function $a\in C^{p+1}(\overline{\mathbb{R}}_+;\mathbb{R}_+)$ separated from zero, and fix vectors $x_j\in\mathfrak{X}$ for $j=\overline{0,n}$ and $f\in H^{p+1}(\mathfrak{Y})$. Then the mapping $D: H^{p+1}(\mathfrak{U})\to H^1(\mathfrak{X})$ defined by* (11) *is continuous.*

Proof. Since $B\in\mathcal{L}(H^{p+1}(\mathfrak{U});H^{p+1}(\mathfrak{Y}))$ and (11) is the solution to (8), this lemma holds by the properties of the operator group X^t and the continuity of $a(t)$ for $t\in\overline{\mathbb{R}}_+$, by analogy with the proof of Theorem 2.

□

Theorem 3. *Take an (L,p)-bounded operator M with ($p\in\mathbb{N}_0$) and a function $a\in C^{p+1}([0,\tau);\mathbb{R}_+)$ separated from zero. Then for all vectors $x_j\in\mathfrak{X}$ for $j=\overline{0,n}$, $f\in H^{p+1}(\mathfrak{Y})$, and $z_d\in\mathfrak{Z}$, there exists a unique solution $v\in H^{p+1}_{ad}(\mathfrak{U})$ to the optimal control problem* (5), (8)–(10).

Proof. Using the mapping D of Lemma 4, we see that the functional (9) becomes

$$J(u)=\|Cx(t;u)-z_d\|^2_{H^1(\mathfrak{Z})}+[\eta,u],$$

where $\eta^{(k)}(t)=N_k u^{(k)}$ for $k=0,\ldots,p+1$. Therefore,

$$J(u)=\pi(u,u)-2\theta(u)+\|z_d-Cx(t;0)\|^2_{H^1(\mathfrak{Z})},$$

where $\pi(u,u)=\|C(x(t;u)-x(t;0))\|^2_{H^1(\mathfrak{Z})}+[\eta,u]$ is a coercive continuous bilinear form on $H^{p+1}(\mathfrak{U})$, and

$$\theta(u)=\langle z_d-Cx(t;0),C(x(t;u)-x(t;0))\rangle_{H^1(\mathfrak{Z})}$$

is a continuous linear form on $H^{p+1}(\mathfrak{U})$. Thus, the theorem is valid by analogy with [6].

□

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ РЕШЕНИЯМИ МНОГОТОЧЕЧНОЙ НАЧАЛЬНО-КОНЕЧНОЙ ЗАДАЧИ ДЛЯ НЕСТАЦИОНАРНЫХ ОТНОСИТЕЛЬНО ОГРАНИЧЕННЫХ УРАВНЕНИЙ СОБОЛЕВСКОГО ТИПА

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В статье рассматривается оптимальное управление решениями начально-конечной задачи для операторно-дифференциального уравнения, неразрешенного относительно производной. При этом в уравнении один из операторов умножен на скалярную функцию переменной t, и свойства операторов таковы, что стационарное уравнение обладает аналитической разрешающей группой. В статье строится сильное решение начально-конечной задачи для нестационарного уравнения соболевского типа в случае относительной ограниченности. Используя построенное решение, доказывается существование единственного оптимального управления решениями указанной задачи. Статья кроме введения и списка литературы содержит три части. В первой из них приводятся необходимые сведения теории относительно p-ограниченных операторов, во второй — строится сильное решение многоточечной начально-конечной задачи для нестационароного уравнения соболевского типа. Наконец, в третьей части доказывается существование и единственность оптимального управления решениями начально-конечной задачи для нестационарного уравнения соболевского типа.

Ключевые слова: оптимальное управление; многоточечная начально-конечная задача; уравнения соболевского типа; относительно ограниченный оператор.

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