Fourth-Order Accurate Alternating Group Schemes for the Diffusion Problem

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Abstract

Based on the alternating group strategies, a new high accurate Alternating Group 8-point scheme (AG-8p) for the diffusion problem is presented. The scheme is proved to be unconditionally stable. In numerical experiments, we compare the AG-8p scheme with the AGE and the ASC-N schemes. Finally, numerical results show that the AG-8p scheme has high accuracy.

Keywords: *Diffusion problem, Alternating group schemes, Fourth order accuracy, Parallel computing*

1. Introduction

In many fields of natural science, many phenomena are described by diffusion equation, such as chemical diffusion, biochemistry and certain biological process etc. So the solution of the diffusion equation is one of the most classic problem in the numerical approximation of partial differential equations. Among modern numerical methods, the finite difference method is the most classic one. With the parallel computer's birth and development, some disadvantages appear in different means. For example, explicit scheme is easy to implement and suit for parallel computing, but for stability constraints, it requires small time steps. Implicit scheme is absolutely stable, but for computing slowly unsteady problems, it needs to march with large time steps.

For numerically solving heat equations on parallel computers, two kinds of finite difference methods are constructed. One is the alternating method (see [1-10]). The method uses alternately the explicit scheme and implicit scheme in the time and space direction, which can implement the parallel computation and is unconditionally stable; the other is the domain decomposition method (see [11-12]). In the further research of the alternating method, D. J. Evans introduces the Alternating Group Explicit (AGE) method in [1-2] for solving the parabolic equation. Based on the works of Evans, B. L. Zhang develops the Alternating Segment Explicit-Implicit (ASEI) method and the Alternating Segment Crank-Nicolson (ASCN) method ([3-4]). After that the study of the AGE scheme has been introduced into solving the hyperbolic equation and the dispersive equation respectively in [4] and [9]. All these methods are all unconditionally stable, and in space direction, their truncation errors are all nearly second order. Recently, S. Zhu develops a higher order truncation error approximately, which results from the use of asymmetric three-level approximations ([13]).

In this paper we design a new alternating group scheme with the truncation error of higher order by using asymmetric two-level approximations for the following diffusion problem with periodic solution. The scheme has fourth order accuracy, being unconditionally stable and naturally parallel.

2. The Finite Difference Approximation

We consider the following diffusion problem,

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad 0 < t < T, \quad (1)$$

with the initial condition $u(x,0) = f(x)$, $0 < x < l$ and the boundary condition $u(x,t) = u(x+l,t)$, $0 < t < T$, where l is the constant period.

Let h and τ be the spatial step size and the time step size respectively. We use $u(x,t)$ to represent the exact solution of Eq. (1) and U_i^n to denote the numerical approximation of $u(x_i, t_n) = u_i^n$, where $x_i = (i-1)h$, $i = 1, 2, \dots, J+1$, $t_n = n\tau$, $n = 0, 1, \dots, [T/\tau]$. In particular, $h = \frac{l}{J}$ for some positive integer $J = 8m$.

In order to increase the order of error in spatial, we introduce the following fourth-order difference scheme

$$\begin{aligned} \left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+\frac{1}{2}} &= \frac{2}{3} \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \right] - \\ &\quad \frac{1}{6} \left[\frac{u_{i+2}^{n+1} - 2u_{i+1}^{n+1} + u_i^{n+1}}{4h^2} + \frac{u_{i+2}^n - 2u_{i+1}^n + u_i^n}{4h^2} \right] + O(\tau^2 + h^4) \end{aligned} \quad (2)$$

On the basis of the above formula, and combining with $\left(\frac{\partial u}{\partial t}\right)_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\tau} + O(\tau^2)$, we can get the following eight basic schemes approximating Eq. (1):

$$\begin{aligned} A_1 : & (1+15r)U_i^{n+1} - 16rU_{i+1}^{n+1} + rU_{i+2}^{n+1} \\ &= -2rU_{i-2}^n + 32rU_{i-1}^n + (1-45r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \end{aligned} \quad (3)$$

$$\begin{aligned} A_2 : & -16rU_{i-1}^{n+1} + (1+31r)U_i^{n+1} - 16rU_{i+1}^{n+1} + rU_{i+2}^{n+1} \\ &= -2rU_{i-2}^n + 16rU_{i-1}^n + (1-29r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \end{aligned} \quad (4)$$

$$\begin{aligned} A_3 : & rU_{i-2}^{n+1} - 16rU_{i-1}^{n+1} + (1+29r)U_i^{n+1} - 16rU_{i+1}^{n+1} + 2rU_{i+2}^{n+1} \\ &= -rU_{i-2}^n + 16rU_{i-1}^n + (1-31r)U_i^n + 16rU_{i+1}^n \end{aligned} \quad (5)$$

$$\begin{aligned} A_4 : & rU_{i-2}^{n+1} - 16rU_{i-1}^{n+1} + (1+45r)U_i^{n+1} - 32rU_{i+1}^{n+1} + 2rU_{i+2}^{n+1} \\ &= -rU_{i-2}^n + 16rU_{i-1}^n + (1-15r)U_i^n \end{aligned} \quad (6)$$

$$\begin{aligned} A_5 : & 2rU_{i-2}^{n+1} - 32rU_{i-1}^{n+1} + (1+45r)U_i^{n+1} - 16rU_{i+1}^{n+1} + rU_{i+2}^{n+1} \\ &= (1-15r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \end{aligned} \quad (7)$$

$$\begin{aligned} A_6 : & 2rU_{i-2}^{n+1} - 16rU_{i-1}^{n+1} + (1+29r)U_i^{n+1} - 16rU_{i+1}^{n+1} + rU_{i+2}^{n+1} \\ &= 16rU_{i-1}^n + (1-31r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \end{aligned} \quad (8)$$

$$\begin{aligned} A_7 : & rU_{i-2}^{n+1} - 16rU_{i-1}^{n+1} + (1+31r)U_i^{n+1} - 16rU_{i+1}^{n+1} \\ &= -rU_{i-2}^n + 16rU_{i-1}^n + (1-29r)U_i^n + 16rU_{i+1}^n - 2rU_{i+2}^n \end{aligned} \quad (9)$$

$$\begin{aligned} A_8 : & rU_{i-2}^{n+1} - 16rU_{i-1}^{n+1} + (1+15r)U_i^{n+1} \\ &= -rU_{i-2}^n + 16rU_{i-1}^n + (1-45r)U_i^n + 32rU_{i+1}^n - 2rU_{i+2}^n \end{aligned} \quad (10)$$

where $r = \frac{\tau}{24h^2}$.

3. The New Alternating Group 8-point (AG-8p) Scheme

We assume that $J = 8m$, we provide the implementation of the AG 8-p algorithm for solving the diffusion problem as follows:

I) At the odd time levels, grid nodes can be divided into m groups and use the schemes $A_1 \sim A_8$ for each group from the first node to the last node respectively.

II) At the even time levels, the nodes include a left 4-point group, $m-1$ groups of eight points and a right 4-point group; and we use the schemes $A_5 \sim A_8$ for the left group and the schemes $A_1 \sim A_4$ for the right group. The other $m-1$ groups can be solved by the schemes $A_1 \sim A_8$. The solving scheme is constructed. Such arrangement results in the following linear system.

When n is even number, for solving the $n+1$ time level for (x_j, t_{n+1}) , $j = i+1, \dots, i+8$; $i = 8k$, $k = 0, 1, \dots, m-1$, we use the schemes $A_1 \sim A_8$, and we rewrite the formulae in the following matrix form:

$$(I + rP_k)\bar{U}^{n+1} = B,$$

where

$$\bar{U}^{n+1} = (U_{i+1}^{n+1}, U_{i+2}^{n+1}, U_{i+3}^{n+1}, U_{i+4}^{n+1}, U_{i+5}^{n+1}, U_{i+6}^{n+1}, U_{i+7}^{n+1}, U_{i+8}^{n+1})^T,$$

P_k is a non-negative definite symmetric matrix as follows:

$$P_k = \begin{bmatrix} 15 & -16 & 1 & & & & & \\ -16 & 31 & -16 & 1 & & & & \\ 1 & -16 & 29 & -16 & \vdots & 2 & & \\ & 1 & -16 & 45 & \vdots & -32 & 2 & \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ & & 2 & -32 & \vdots & 45 & -16 & 1 \\ & & & 2 & \vdots & -16 & 29 & -16 & 1 \\ & & & & \vdots & 1 & -16 & 31 & -16 \\ & & & & \vdots & & 1 & -16 & 15 \end{bmatrix} = \begin{bmatrix} E_k & F_k \\ F_k^T & H_k \end{bmatrix},$$

$$B = (I - rQ_k)\bar{U}^n + b,$$

$$Q_k = \begin{bmatrix} H_k & 0 \\ 0 & E_k \end{bmatrix},$$

$$b = (-2rU_{i-1}^n + 32rU_i^n, -2rU_i^n, 0, 0, 0, 0, -2rU_{i+9}^n, 32rU_{i+9}^n - 2rU_{i+10}^n)^T.$$

Because of $J = 8m$, m is a positive integer. So we divide the mesh points $x_1, x_2, \dots, x_J$ which are located in one period into m groups. Later we propose the AG-8p method by using the formulae $A_1 \sim A_8$ for the eight points respectively in every group. Let $U^n = (U_1^n, U_2^n, \dots, U_J^n)^T$, considering the periodic property of boundary condition, we can obtain:

$$(I + rG_1)U^{n+1} = (I - rG_2)U^n,$$

where

$$G_1 = \begin{bmatrix} P_k & & & \\ & P_k & & \\ & & \ddots & \\ & & & P_k \end{bmatrix}_{J \times J}, G_2 = \begin{bmatrix} E_k & & & F_k^T \\ & P_k & & \\ & & \ddots & \\ & & & P_k \\ F_k & & & H_k \end{bmatrix}_{J \times J},$$

But for the $n+2$ time level, we can obtain the matrix equation as follows by the same processing method:

$$(I + rG_2)U^{n+2} = (I - rG_1)U^{n+1},$$

So we get the complete scheme as follows, namely the Alternating Group 8-points scheme(AG-8p).

$$\begin{aligned} B_1: (I + rG_1)U^{n+1} &= (I - rG_2)U^n, \\ B_2: (I + rG_2)U^{n+2} &= (I - rG_1)U^{n+1}, \\ n &= 0, 2, 4, \dots \end{aligned}$$

4. Stability and Truncation Error Analyses

With the lemma of Kellogg (see [14]), we can give the stability analysis of the presented algorithm, that is:

Now eliminating U^{n+1} from B_1 and B_2 gives:

$$U^{2n} = TU^{2n-2} = \dots = T^n U^0,$$

where

$$T = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2),$$

Since G_1, G_2 are all non-negative definite, then the following inequalities are derived for any positive integer n and positive r

$$\begin{aligned} \|T^n\| &\leq \|(I + rG_2)^{-1}\|_2 \|(I - rG_1)(I + rG_1)^{-1}\|_2 \|(I - rG_2)(I + rG_2)^{-1}\|_2 \\ &\dots \|(I - rG_1)(I + rG_1)^{-1}\|_2 \|(I - rG_2)\|_2 \\ &\leq \|(I - rG_2)\|_2 \leq 1 + 4r_0, \quad (r_0 = \frac{\tau}{h^2}) \end{aligned}$$

So we prove the AG-8p scheme is unconditional stable. Then we have the next theorem.

Theorem Assume that n is an even number, r is any positive real number, the AG-8p method $B_1 \sim B_2$ is unconditionally stable.

We perform the truncation error analysis for the AG-8p scheme. The actual computation involves four pairs of schemes which are alternately used between two time levels. These pairs are: A_1 and A_5 , A_2 and A_6 , A_3 and A_7 , A_4 and A_8 . In fact, we also may regard the AG-8p schemes $B_1 \sim B_2$ as the schemes of three layers, *i.e.*, formulae A_1 and A_5 have been combined into the formulae C_1 and C_5 as follows, similarly, formulae A_2 and A_6 into C_2 and C_6 , formulae A_3 and A_7 into C_3 and C_7 , formulae A_4 and A_8 into C_4 and C_8 .

$$\begin{aligned}
 C_1 : & (1+15r)U_i^{n+2} - 16rU_{i+1}^{n+2} + rU_{i+2}^{n+2} \\
 & = -4rU_{i-2}^{n+1} + 64rU_{i-1}^{n+1} - 90rU_i^{n+1} + 32rU_{i+1}^{n+1} \\
 & \quad - 2rU_{i+2}^{n+1} + (1-15r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \\
 C_2 : & -16rU_{i-1}^{n+2} + (1+31r)U_i^{n+2} - 16rU_{i+1}^{n+2} + rU_{i+2}^{n+2} \\
 & \quad = -4rU_{i-2}^{n+1} + 32rU_{i-1}^{n+1} - 58rU_i^{n+1} + 32rU_{i+1}^{n+1} - 2rU_{i+2}^{n+1} \\
 & \quad \quad + 16rU_{i-1}^n + (1-31r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \\
 C_3 : & rU_{i-2}^{n+2} - 16rU_{i-1}^{n+2} + (1+29r)U_i^{n+2} - 16rU_{i+1}^{n+2} + 2rU_{i+2}^{n+2} \\
 & \quad = -2rU_{i-2}^{n+1} + 32rU_{i-1}^{n+1} - 62rU_i^{n+1} + 32rU_{i+1}^{n+1} - rU_{i-2}^n \\
 & \quad \quad + 16rU_{i-1}^n + (1-29r)U_i^n + 16rU_{i+1}^n - 2rU_{i+2}^n \\
 C_4 : & rU_{i-2}^{n+2} - 16rU_{i-1}^{n+2} + (1+45r)U_i^{n+2} - 32rU_{i+1}^{n+2} + 2rU_{i+2}^{n+2} \\
 & \quad = -2rU_{i-2}^{n+1} + 32rU_{i-1}^{n+1} - 30rU_i^{n+1} - rU_{i-2}^n \\
 & \quad \quad + 16rU_{i-1}^n + (1-45r)U_i^n + 32rU_{i+1}^n - 2rU_{i+2}^n \\
 C_5 : & 2rU_{i-2}^{n+2} - 32rU_{i-1}^{n+2} + (1+45r)U_i^{n+2} - 16rU_{i+1}^{n+2} + rU_{i+2}^{n+2} \\
 & \quad = -30rU_i^{n+1} + 32rU_{i+1}^{n+1} - 2rU_{i+2}^{n+1} - 2rU_{i-2}^n \\
 & \quad \quad + 32rU_{i-1}^n + (1-45r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \\
 C_6 : & 2rU_{i-2}^{n+2} - 16rU_{i-1}^{n+2} + (1+29r)U_i^{n+2} - 16rU_{i+1}^{n+2} + rU_{i+2}^{n+2} \\
 & \quad = 32rU_{i-1}^{n+1} - 62rU_i^{n+1} + 32rU_{i+1}^{n+1} - 2rU_{i+2}^{n+1} - 2rU_{i-2}^n \\
 & \quad \quad + 16rU_{i-1}^n + (1-29r)U_i^n + 16rU_{i+1}^n - rU_{i+2}^n \\
 C_7 : & rU_{i-2}^{n+2} - 16rU_{i-1}^{n+2} + (1+31r)U_i^{n+2} - 16rU_{i+1}^{n+2} \\
 & \quad = -2rU_{i-2}^{n+1} + 32rU_{i-1}^{n+1} - 58rU_i^{n+1} + 32rU_{i+1}^{n+1} - 4rU_{i+2}^{n+1} \\
 & \quad \quad - rU_{i-2}^n + 16rU_{i-1}^n + (1-31r)U_i^n + 16rU_{i+1}^n \\
 C_8 : & rU_{i-2}^{n+2} - 16rU_{i-1}^{n+2} + (1+15r)U_i^{n+2} \\
 & \quad = -2rU_{i-2}^{n+1} + 32rU_{i-1}^{n+1} - 90rU_i^{n+1} + 64rU_{i+1}^{n+1} \\
 & \quad \quad - 4rU_{i+2}^{n+1} - rU_{i-2}^n + 16rU_{i-1}^n + (1-15r)U_i^n
 \end{aligned}$$

Thus, it is sufficient to analyze the truncation errors of the formulae $C_1 \sim C_8$. We use $L_h^{(1)} - L_h^{(8)}$ to represent the discrete operators of L basing on the schemes $C_1 \sim C_8$ respectively, for example, $L_h^{(1)}$ can be defined as the following form:

$$L_h^{(1)} u_i^{n+1} = \frac{1}{2\tau} \{ (1+15r)u_i^{n+2} - 16ru_{i+1}^{n+2} + ru_{i+2}^{n+2} + 4ru_{i-2}^{n+1} - 64ru_{i-1}^{n+1} \\ + 90ru_i^{n+1} - 32ru_{i+1}^{n+1} + 2ru_{i+2}^{n+1} - (1-15r)u_i^n - 16ru_{i+1}^n + ru_{i+2}^n \}$$

Notice to that the formulae $C_1 \sim C_8$ are all symmetrical schemes with their axis of symmetry $t = t_{n+1}$, then from the Taylor series expansions at the point (x_i, t_{n+1}) , we can obtain their truncation errors as below concise forms.

$$L_h^{(k)} u_i^{n+1} - [Lu]_i^{n+1} = -7rh\tau \left(\frac{\partial^3 u}{\partial x \partial t^2} \right)_i^{n+1} - \frac{7}{12} rh\tau^3 \left(\frac{\partial^5 u}{\partial x \partial t^4} \right)_i^{n+1} + O(\tau^2 + h^4 + h^\alpha \tau^\beta)$$

$$k = 1, 4$$

$$L_h^{(k)} u_i^{n+1} - [Lu]_i^{n+1} = 7rh\tau \left(\frac{\partial^3 u}{\partial x \partial t^2} \right)_i^{n+1} + \frac{7}{12} rh\tau^3 \left(\frac{\partial^5 u}{\partial x \partial t^4} \right)_i^{n+1} + O(\tau^2 + h^4 + h^\alpha \tau^\beta)$$

$$k = 5, 8$$

$$L_h^{(k)} u_i^{n+1} - [Lu]_i^{n+1} = rh\tau \left(\frac{\partial^3 u}{\partial x \partial t^2} \right)_i^{n+1} + \frac{1}{12} rh\tau^3 \left(\frac{\partial^5 u}{\partial x \partial t^4} \right)_i^{n+1} + O(\tau^2 + h^4 + h^\alpha \tau^\beta)$$

$$k = 2, 3$$

$$L_h^{(k)} u_i^{n+1} - [Lu]_i^{n+1} = -rh\tau \left(\frac{\partial^3 u}{\partial x \partial t^2} \right)_i^{n+1} - \frac{1}{12} rh\tau^3 \left(\frac{\partial^5 u}{\partial x \partial t^4} \right)_i^{n+1} + O(\tau^2 + h^4 + h^\alpha \tau^\beta)$$

$$k = 6, 7$$

where $\alpha + \beta = 4$.

In addition, through further analysis we see also that formulae C_1 and C_5 are symmetric about the axis that is the vertical line of $x = \text{constant}$, and the signs of the first two terms of formulae C_1 and C_5 of their truncation errors are opposite, and the errors can be eliminated partly. Similarly,

formulae C_2 and C_6 , C_3 and C_7 , C_4 and C_8 have the same property as C_1 and C_5 . Thus the accuracy of the schemes has been increased.

5. Numerical Experiments

We perform the numerical experiments using the following diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad 0 < t < T, \\ u(0, t) = u(2, t), \quad 0 < t < T, \\ u(x, 0) = \cos(\pi x), \quad 0 < x < 2$$

The exact solution is: $u(x, t) = e^{-a\pi^2 t} \cos(\pi x)$, $0 < x < 2$.

Let $u_i^n = u(x_i, t_n)$ and U_i^n be the exact solution and the approximate solution of the problem respectively, and we calculate the errors in L_∞ -norm and L_2 -norm as:

$$E_{\infty, \Delta x}^n = \max_i (e_{\Delta x, \Delta t}^n(i)) = \max_i |u_i^n - U_i^n|, \quad E_{2, \Delta x}^n = \|e_{\Delta x, \Delta t}^n\|_2 = \sqrt{\sum_i |u_i^n - U_i^n|^2 \Delta x},$$

where Δx is the step size in space and Δt is the step size in time. The ratios of convergence in space are calculated by

$$\text{Ratio of convergence} \approx \frac{\log(E_{l, \Delta x_1} / E_{l, \Delta x_2})}{\log(\Delta x_1 / \Delta x_2)}, \quad l = 2, \infty,$$

when very small time step size Δt is taken in computation.

We use a very small time step size $\Delta t = 0.1 \times 10^{-5}$ for the three methods in computation of the ratios of convergence in space, the spatial step sizes are taken as $\Delta x = 1/8, 1/16, 1/32, 1/56$ and $1/64$. Table 1 and 2 show results of three methods at the time step number $n = 10^5$. From the tables, it is clear that the AG-8p scheme has a ratio that is nearly fourth order in space, but the AGE [1] and AGE 4-p [8] have a ratio of nearly second order. In terms of errors at different times and mesh parameters, we have compared these numerical methods. Table 3 and 4 give the absolute errors (*ae*) and the relative errors (*re*) of numerical solutions. It is obvious that the AG-8p scheme is more accurate than both the AGE method and the AGE 4-p method. From Table 4 and Fig 1, we also see the AG-8p method always has better accuracy than the other two methods for large step size in time ($\lambda = 4, 2, 5$).

Table 1. The L_{∞} – error and Ratio in Space for the Diffusion Problem

Δx	AG-8p	AGE	AGE-4p
	L_{∞} – error Ratio	L_{∞} – error Ratio	L_{∞} – error Ratio
1/8	$9.5883e-5$ --	$4.7328e-3$ --	$4.7328e-3$ --
1/16	$6.0540e-6$ 3.9853	$1.1822e-3$ 2.0013	$1.1822e-3$ 2.0013
1/32	$3.7872e-7$ 4.0025	$2.9547e-4$ 2.0002	$2.9547e-4$ 2.0002
1/56	$3.8501e-8$ 4.1964	$9.6465e-5$ 2.0004	$9.6474e-5$ 2.0001
1/64	$2.1151e-8$ 4.4858	$7.3850e-5$ 2.0007	$7.3861e-5$ 2.0002

Table 2. The L_2 – error and Ratio in Space for the Diffusion Problem

Δx	AG-8p	AGE	AGE-4p
	L_2 – error Ratio	L_2 – error Ratio	L_2 – error Ratio
1/8	$9.0624e-5$ --	$4.4732e-3$ --	$4.4732e-3$ --
1/16	$5.8692e-6$ 3.9487	$1.1461e-3$ 1.9646	$1.1461e-3$ 1.9646
1/32	$3.7281e-7$ 3.9845	$2.9086e-4$ 1.9822	$2.9086e-4$ 1.9822
1/56	$3.8149e-8$ 4.1872	$9.5603e-5$ 1.9907	$9.5611e-5$ 1.9904

1/64	$2.0979e-8$ 4.4783	$7.3272e-5$ 1.9922	$7.3283e-5$ 1.9918
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Table 3. The Comparison of the Numerical Errors for
 $J = 200, \lambda = \tau/h^2 = 0.1, n = 1000$

$x(i)$	exact solution ($e-1$)	AG-8p		AGE-4p[8]		AGE[1]	
		ae. ($e-7$)	re. ($e-7$)	ae. ($e-6$)	re. ($e-6$)	ae. ($e-6$)	re. ($e-6$)
0.2	7.3298	0.37199	0.50750	5.1535	7.0309	5.7515	7.8467
0.6	-2.7997	1.8675	6.6701	2.1336	7.6205	2.2381	7.9940
1.0	-9.0602	1.5261	1.6844	6.4721	7.1435	7.1348	7.8748
1.4	-2.7997	0.92425	3.3012	1.8664	6.6665	2.1714	7.7557
1.8	7.3298	2.0974	2.8614	5.3186	7.2561	5.7928	7.9030

Table 4 The Comparison of the Numerical Errors for
 $J = 400, \lambda = \tau/h^2 = 4, n = 1200$

$x(i)$	exact solution ($e-1$)	AG-8p		AGE-4p[8]		AGE[1]	
		ae. ($e-4$)	re. ($e-4$)	ae. ($e-4$)	re. ($e-3$)	ae. ($e-3$)	re. ($e-3$)
0.2	2.4751	2.0169	8.1487	2.8484	1.1508	1.1594	4.6840
0.6	-0.94542	0.75677	8.0046	1.0601	1.1213	0.43165	4.5657
1.0	-3.0594	2.4846	8.1212	3.5036	1.1452	1.4261	4.6614
1.4	2.4751	0.77881	8.2377	1.1052	1.1690	0.44975	4.7572
1.8	-0.94542	2.0033	8.0937	2.8205	1.1395	1.1482	4.6388

Figure 1. The Comparison of the Absolute Errors at Same Time (Different Numbers of Time Step) and for Different Mesh Parameters $\lambda (= \tau/h^2)$

6. Conclusions

In this paper, we first construct a group of high-order unsymmetrical schemes $A_1 \sim A_8$ based on the idea of the alternating group method. Next we propose a new alternating group method, whose ratio of convergence is of order 4 in space. The results of our numerical experiments show that the present AG-8p method is efficient and accurate. The method of the AG-8p also can be applied to other partial differential equations.

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