Analysis of 90/150 CA Concatenated using Inverse Symmetric Transition Rule

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Abstract

In symmetric cryptosystems, key sequence generation is a very important problem. Various key sequence generation methods have been proposed by many researchers. A shrinking generator consisting of two LFSRs with maximum lengths generates a key sequence in nonlinear manners. Since the characteristic polynomial of the shrunk sequence is of the form $\{f(x)\}^k$ ($k \in \mathbb{N}$), it can be modeled as a linear 90/150 Cellular Automata (CA). In this paper, we analyze a linear 90/150 CA using inverse symmetric transition rules of various types. By the proposed method we generate $(2n+i)$ -cell ($i=0,1,2, \dots, n$) 90/150 CA based on n -cell 90/150 CA. Especially we synthesize 90/150 CA corresponding to $\{f(x)\}^k$ using inverse symmetric transition rules. Also, we show that the probabilities of finding $(2n+i)$ -cell CA $\mathbb{C}$ with maximum lengths by the proposed method are greater than the probabilities of finding the $\mathbb{C}$ by an exhaustive search.

Keywords: 90/150 Cellular Automata; Characteristic Polynomial; Inverse Symmetric Transition Rule; Shrinking Generator; Key Sequence.

1. Introduction

Wolfram in early 1980's suggested a simplified two-state three-neighborhood 1-D CA with cells arranged linearly in one dimension [1]. CA has a simple, regular, modular and cascable structure with logical neighborhood interconnection. The simple structure of CA with logical interconnections is ideally suited for hardware implementation. For these reasons CA have been used for various applications such as pattern classification ([2], [3]), cryptography ([4] ~ [8]), and pseudorandom-number generation, etc. ([9] ~ [12]). Cho et al. ([11], [13]) analyzed characteristic polynomials of group CA and non-group CA. Moreover they proposed efficient methods of synthesis of 90/150 maximum length CA [13]. Choi et al. [14] proposed the 90/150 CA synthesis method whose characteristic polynomial has maximum weight. Cho et al. [11] and Sabater et al. [15] proposed classes of cryptographic interleaved sequences generated by linear 90/150 CA obtained by concatenating the basic automaton. The interleaved sequence generator in [11] and [15] has a limitation that characteristic polynomial of the generator is $\{f(x)\}^{2^a}$ ($a \in \mathbb{N}$). To

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overcome this restriction, we propose the method to model the shrinking generator as 90/150 CA whose characteristic polynomial is of the form $\{f(x)\}^k$ ($k \in \mathbb{N}$) in this paper. And we analyze 90/150 CA using inverse symmetric transition rule of various types. Also we give a method of generating $(2n+i)$ -cell ($i=0,1,2,\dots,n$) 90/150 CA based on n -cell 90/150 CA. Especially we give $(2n+i)$ -cell 90/150 CA $\mathbb{C}$ whose characteristic polynomials are irreducible or primitive. The probabilities of finding $(2n+i)$ -cell CA $\mathbb{C}$ are greater than the probabilities of finding the $\mathbb{C}$ by an exhaustive search.

2. Preliminaries

A CA consists of a number of interconnected cells arranged spatially in a regular manner [1]. In most simple case, a CA cell can exhibit two different states (0 or 1) and the next state of each cell depends on the present states of its three neighborhoods including itself. The state s_i^{t+1} of the i th cell at time $(t+1)$ is denoted as

$$s_i^{t+1} = f_i(s_{i-1}^t, s_i^t, s_{i+1}^t),$$

where s_i^t denotes the state of the i th cell at time t and f_i is the next state function called the rule of the CA. If the next state generating logic employs only XOR logic, then it is called a *linear rule*. And a CA with all the cells having linear rules is called a *linear CA* [16]. Since a linear CA employs XOR logic only as the next state function, it can be represented as a matrix referred to as the *state transition matrix* over the finite field $GF(2)$. An n -cell CA is characterized by an $n \times n$ state transition matrix. The state transition matrix T is constructed as

$$T = \begin{pmatrix} d_1 & a_{1,2} & 0 & \cdots & 0 & 0 \\ a_{2,1} & d_2 & a_{2,3} & \cdots & 0 & 0 \\ 0 & a_{3,2} & d_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & d_n \end{pmatrix}$$

where $a_{i,j} = 1$ if the next state of the i th cell depends on the present state of the j th cell, and $a_{i,j} = 0$ otherwise. And $a_{i,i} = d_i$, $i=1,2,\dots,n$. In this paper, a CA is a null-boundary 90/150 CA fully specified by which cells use 90 and 150. The transition rules 90 and 150 are defined as follows:

$$\text{Rule 90 : } s_i^{t+1} = s_{i-1}^t \oplus s_{i+1}^t$$

$$\text{Rule 150 : } s_i^{t+1} = s_{i-1}^t \oplus s_i^t \oplus s_{i+1}^t$$

According to rule 90, the value of a particular cell i is the sum modulo 2 of the values of its two neighboring cells(left and right) on the previous time step t . And in rule 150, the value of a particular cell i is the sum modulo 2 of the values of its two neighboring cells and the value of cell i on the previous time step t . A natural form for the specification of 90/150 CA is an n -tuple $\langle d_1, d_2, \dots, d_n \rangle$, called the rule vector, where $d_i = 0$ (resp. 1) if cell i uses rule 90 (resp. 150).

A polynomial is said to be a *CA-polynomial* if it is the characteristic polynomial of some 90/150 CA. All irreducible polynomials are CA-polynomials [17]. In [17], Cattell et al. proposed a method for the synthesis of one-dimensional 90/150 Linear Hybrid Group CA (LHGCA) for an irreducible polynomial. Cho et al. [13] proposed

a new efficient method for the synthesis of one-dimensional 90/150 LHGCA for any CA-polynomial as well as irreducible polynomial by using Lanczos tridiagonalization algorithm. This algorithm is efficient and suitable for all practical applications. Cho et al. [11] and Sabater et al. [15] proposed a method of constructing a linear 90/150 CA with characteristic polynomial $\{f(x)\}^2$ by concatenating the basic automaton whose characteristic polynomial is $f(x)$, which is irreducible.

3. Recurrence relation of 90/150 CA using inverse symmetric transition rule

In this section, we provide theoretical information about concatenation of 90/150 CA using inverse symmetric transition rules of various types. The characteristic polynomial Δ_n of an n -cell 90/150 CA $\mathbb{C}_n$ is defined by $\Delta_n = |T_n \oplus xI_n|$ where x is an indeterminate, I_n is the $n \times n$ identity matrix and T_n is the state transition matrix of $\mathbb{C}_n$. For any n -cell 90/150 CA whose state transition matrix is T_n , the minimal polynomial for T_n is the same as the characteristic polynomial for T_n [12]. Let Δ_n be the characteristic polynomial of T_n . Then the following recurrence relation holds:

$$\Delta_n = (x + d_n)\Delta_{n-1} + \Delta_{n-2} \quad (1)$$

where $\Delta_1 = x + d_1$, $\Delta_0 = 1$ [17]. Eq. (1) provides an efficient algorithm to compute Δ_n of a given 90/150 CA from its rule vector. We denote the characteristic polynomial of *sub-CA* consisting of cells i through j by $\Delta_{i,j}$, where $i \leq j$. We simply denote $\Delta_{1,j}$ by Δ_j . Δ_j is said to be a *CA-subpolynomial*. Let $T_n = \langle a_1, a_2, \dots, a_n \rangle$ (resp. $T_n^* = \langle a_n, \dots, a_2, a_1 \rangle$) be the state transition matrix of an n -cell 90/150 CA and Δ_n (resp. Δ_n^*) be the characteristic polynomial of T_n (resp. T_n^*). Then $\Delta_n^* = \Delta_n$. T_n^* is called the state transition matrix corresponding to T_n .

Definition 3.1. [18] Let T_n be the state transition matrix of an n -cell 90/150 CA and let T_n^* be the state transition matrix corresponding to T_n . Let S_{2n} be the following state transition matrix of a $2n$ -cell 90/150 CA $\mathbb{C}_{2n}$. Then the state transition matrix $S_{2n} = \langle T_n, T_n^* \rangle = \langle a_1, a_2, \dots, a_n, a_n, \dots, a_2, a_1 \rangle$ is called the *symmetric transition rule* of $\mathbb{C}_{2n}$.

Let $T_n = \langle a_1, a_2, \dots, a_n \rangle$ be the state transition matrix of an n -cell 90/150 CA and $T_n^- = \langle a_1, \dots, a_{n-1}, \overline{a_n} \rangle$. And let V_{2n} be the characteristic polynomial of $M_{2n} = \langle T_n^-, T_n^* \rangle$. Then the following holds:

$$V_{2n} = (\Delta_n + \Delta_{n-1})^2 = (\Delta_n^-)^2$$

where Δ_n^- is the characteristic polynomial of T_n^- . Let D_1 be the characteristic polynomial of the 1-cell 90/150 CA with the rule vector $\langle d \rangle$. Let $M_{2n+1} = \langle T_n, d, T_n^* \rangle$ be the state transition matrix of a $(2n+1)$ -cell 90/150 CA $\mathbb{C}_{2n+1}$. And let V_{2n+1} be the characteristic polynomial of M_{2n+1} . Then $V_{2n+1} = D_1 \Delta_n^2$ [18].

Definition 3.2. [19] Let T_n be the state transition matrix of an n -cell 90/150 CA. Then $\overline{T_n^*} = \langle \overline{a_n}, \dots, \overline{a_2}, \overline{a_1} \rangle$ is called the *inverse symmetric transition rule* of T_n .

If $\Delta_n = C_n(x)$ is the characteristic polynomial of T_n and $\overline{\Delta_n^*}$ is the characteristic polynomial of $\overline{T_n^*}$, then $\overline{\Delta_n^*} = C_n(x+1)$. Thus $\overline{\Delta_n^*}$ is irreducible whenever Δ_n is irreducible. For example, let $T_6 = \langle 0, 0, 0, 1, 1, 0 \rangle$. Then $\overline{T_6^*} = \langle 1, 0, 0, 1, 1, 1 \rangle$. Since $\Delta_6 = C_6(x) = x^6 + x + 1$ is irreducible, $\overline{\Delta_6^*} = (x+1)^6 + (x+1) + 1 = x^6 + x^4 + x^2 + x + 1$ is also irreducible.

For $\overline{T_n^*}$, the following recurrence relation holds:

$$\overline{\Delta_n^*} = (x + \overline{a_n})\overline{\Delta_{n-1}^*} + \overline{\Delta_{n-2}^*} \quad (2)$$

where $\overline{\Delta_1^*} = x + \overline{a_1}$, $\overline{\Delta_0^*} = 1$. Now we investigate the properties about concatenation of the 90/150 CA using the inverse symmetric transition rule.

Theorem 3.3. Let $T_n = \langle a_1, a_2, \dots, a_n \rangle$ be the state transition matrix of an n -cell 90/150 CA. Then the characteristic polynomial $V_{2n}(x)$ of $R_{2n} = \langle T_n, \overline{T_n^*} \rangle$ is

$$V_{2n}(x) = \Delta_n \overline{\Delta_n^*} + \Delta_{n-1} \overline{\Delta_{n-1}^*}$$

where $\overline{\Delta_{n-1}^*}$ is the characteristic polynomial of $\langle \overline{a_{n-1}}, \dots, \overline{a_2}, \overline{a_1} \rangle$.

Proof. Since the $(n+1)$ th row of $R_{2n} + x I_{2n}$ is of the form

$$\langle 0, \dots, 0, 1, \underbrace{x + \overline{a_n}}_{(n+1)\text{th}}, 1, 0, \dots, 0 \rangle$$

, by cofactor expansion along the $(n+1)$ th row of $R_{2n} + x I_{2n}$ we obtain the following:

$$\begin{aligned} V_{2n}(x) &= (x + \overline{a_n}) \Delta_n \overline{\Delta_{n-1}^*} + 1 \cdot |A| + 1 \cdot |B| \\ &= (x + \overline{a_n}) \Delta_n \overline{\Delta_{n-1}^*} + \Delta_{n-1} \overline{\Delta_{n-1}^*} + \Delta_n \overline{\Delta_{n-2}^*} \\ &= \left\{ (x + \overline{a_n}) \overline{\Delta_{n-1}^*} + \overline{\Delta_{n-2}^*} \right\} \Delta_n + \Delta_{n-1} \overline{\Delta_{n-1}^*} \\ &= \Delta_n \overline{\Delta_n^*} + \Delta_{n-1} \overline{\Delta_{n-1}^*}, \end{aligned} \quad (3)$$

where A is the submatrix obtained by removing the $(n+1)$ th row and the n th column of $R_{2n} + x I_{2n}$ and B is the submatrix obtained by removing the n th row and the $(n+1)$ th column of $R_{2n} + x I_{2n}$.

Since $\Delta_n(x+1) = \overline{\Delta_n^*}(x)$ and $\Delta_{n-1}(x+1) = \overline{\Delta_{n-1}^*}(x)$, $V_{2n}(x+1) = V_{2n}(x)$ in Theorem 3.3. Let $F_n(x) = x^{2^n} + x + 1$. Then the degree of every irreducible factor of $F_n(x)$ over $GF(2)$ divides $2n$ [16]. Since $F_n(x+1) = F_n(x)$, every irreducible factor of $F_n(x)$ is the one of $V_{2n}(x)$.

For example, for $x^2 + x + 1 = (x^8 + x^6 + x^5 + x^4 + x^3 + x + 1)(x^8 + x^6 + x^5 + x^3 + 1)$, 90/150 CA corresponding to $x^8 + x^6 + x^5 + x^4 + x^3 + x + 1$ is <00110011> and 90/150 CA corresponding to $x^8 + x^6 + x^5 + x^3 + 1$ (primitive) is <00001111>.

To model shrinking generator as 90/150 CA, we need to synthesize the 90/150 CA corresponding to $\{f(x)\}^k$ (k : odd). It is enough to get only the first consecutive $nk-1$ bits $a_1, a_2, \dots, a_{nk-1}$ for finding $2nk$ -cell 90/150 CA $\langle a_1, a_2, \dots, a_{nk-1}, a_{nk}, \overline{a_{nk}}, \overline{a_{nk-1}}, \dots, \overline{a_2}, \overline{a_1} \rangle$ corresponding to $\{f(x)\}^k$ and $n = \deg f(x)$. The linear complexity of the shrunken sequence generated by $\{f(x)\}^k$ is $2^{L_1} \cdot L_2$ where $2^{L_1-1} < k \leq 2^{L_1}$ and $L_2 = 2^{2n} - 1$. Table 1 shows 90/150 CA corresponding to $(x^8 + x^6 + x^5 + x^3 + 1)^k$ ($k = 3, 5, 7, 9, 11, 19$).

Table 1. 90/150 CA corresponding to $(x^8 + x^6 + x^5 + x^3 + 1)^k$

k	90/150 CA (given in hexadecimal notation)
3	37D413
5	3C1F2B07C3
7	0D18C2B2BCE74F
9	0DDF790D4D4F61044F
11	0E48CE7903333F618CED8F
$\vdots$	$\vdots$
19	37D4318BBEDB52860A71AF9EB524822E73D413

Now we extend the transition rules by adding the 1-cell CA $\langle d \rangle$ or the 2-cell CA $\langle d_1, d_2 \rangle$.

Theorem 3.4. Let $T_n = \langle a_1, a_2, \dots, a_n \rangle$ be the state transition matrix of an n -cell 90/150 CA. Then the following hold:

- (1) The characteristic polynomial V_{2n+1} of $M_{2n+1} = \langle T_n, d, \overline{T_n^*} \rangle$ is

$$V_{2n+1} = \overline{\Delta_n^*}^{\overline{\Delta_{n+1}^*}^{l_d}} + \Delta_{n-1} \overline{\Delta_n^*}$$

where $\overline{\Delta_{n+1}^*}^{l_d}$ is the characteristic polynomial of rule vector $\langle d, \overline{a_n}, \dots, \overline{a_2}, \overline{a_1} \rangle$.

- (2) The characteristic polynomial V_{2n+2} of $M_{2n+2} = \langle T_n, d_1, d_2, \overline{T_n^*} \rangle$ is

$$V_{2n+2} = D_2 \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} + \Delta_{n-1} \overline{\Delta_{n+1}^*}^{l_{d_2}}$$

where D_2 is the characteristic polynomial of the 2-cell 90/150 CA with rule vector $\langle d_1, d_2 \rangle$ and $\Delta_{n+1}^{r_{d_1}}$ is the characteristic polynomial of rule vector $\langle a_1, a_2, \dots, a_n, d_1 \rangle$, and $\overline{\Delta_{n+1}^*}^{l_{d_2}}$ is the characteristic polynomial of rule vector $\langle d_2, \overline{a_n}, \dots, \overline{a_2}, \overline{a_1} \rangle$.

Proof. We only prove for the case (2). Since the $(n+2)$ th row of $M_{2n+2} + x I_{2n+2}$ is of the form

$$\langle 0, \dots, 0, 1, \overbrace{x + d_2}^{(n+2)\text{th}}, 1, 0, \dots, 0 \rangle$$

by cofactor expansion along the $(n+2)$ th row of $M_{2n+2} + x I_{2n+2}$ we obtain the following:

$$\begin{aligned}
 V_{2n+2} &= \Delta_{n+1}^{r_{d_1}} (x + d_2) \overline{\Delta_n^*} + 1 \cdot |A| + 1 \cdot |B| \\
 &= \Delta_{n+1}^{r_{d_1}} (x + d_2) \overline{\Delta_n^*} + \Delta_n \overline{\Delta_n^*} + \Delta_{n+1}^{r_{d_1}} \overline{\Delta_{n-1}^*} \\
 &= (x + d_2) \{ (x + d_1) \Delta_n + \Delta_{n-1} \} \overline{\Delta_n^*} + \Delta_n \overline{\Delta_n^*} + \{ (x + d_1) \Delta_n + \Delta_{n-1} \} \overline{\Delta_{n-1}^*} \\
 &= (x + d_1)(x + d_2) \Delta_n \overline{\Delta_n^*} + (x + d_2) \Delta_{n-1} \overline{\Delta_n^*} + \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} \\
 &\quad + \Delta_{n-1} \overline{\Delta_{n-1}^*} \\
 &= D_2 \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} + (x + d_2) \Delta_{n-1} \overline{\Delta_n^*} + \Delta_{n-1} \overline{\Delta_{n-1}^*} \\
 &= D_2 \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} + \Delta_{n-1} \{ (x + d_2) \overline{\Delta_n^*} + \overline{\Delta_{n-1}^*} \} \\
 &= D_2 \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} + \Delta_{n-1} \overline{\Delta_{n+1}^{l_{d_2}}} \tag{4}
 \end{aligned}$$

where A is the submatrix obtained by removing the $(n+2)$ th row and the $(n+1)$ th column of $M_{2n+2} + x I_{2n+2}$ and B is the submatrix obtained by removing the $(n+2)$ th row and the $(n+3)$ th column of $M_{2n+2} + x I_{2n+2}$, and $\Delta_{n+1}^{r_{d_1}}$ denotes the characteristic polynomial of rule vector $\langle a_1, a_2, \dots, a_n, d_1 \rangle$, and $\overline{\Delta_{n+1}^{l_{d_2}}}$ denotes the characteristic polynomial of rule vector $\langle d_2, a_n, \dots, a_2, a_1 \rangle$.

In Theorem 3.4(2), if $d_2 = \overline{d_1}$, then

$$V_{2n+2} = \Delta_{n+1}^{r_{d_1}} \overline{\Delta_{n+1}^{r_{d_1}}} + \Delta_n \overline{\Delta_n^*}$$

Otherwise,

$$\begin{aligned}
 V_{2n+2} &= (x + d_1)^2 \Delta_n \overline{\Delta_n^*} + \Delta_n \overline{\Delta_n^*} + (x + d_1) \Delta_n \overline{\Delta_{n-1}^*} + \Delta_{n-1} \{ (x + d_1) \overline{\Delta_n^*} + \overline{\Delta_{n-1}^*} \} \\
 &= (x + \overline{d_1})^2 \Delta_n \overline{\Delta_n^*} + (x + d_1) (\Delta_n \overline{\Delta_{n-1}^*} + \overline{\Delta_n^*} \Delta_{n-1}) + \Delta_{n-1} \overline{\Delta_{n-1}^*}
 \end{aligned}$$

Table 2 shows the characteristic polynomial V_{21} for the case $d = 0$.

Table 2. Characteristic Polynomial V_{21} for the Case $d = 0$

$T_{10} = \langle a_1, \dots, a_{10} \rangle$	Characteristic polynomial V_{21} of $\langle T_{10}, 0, \overline{T_{10}^*} \rangle$
$\langle 0, 0, 0, 0, 0, 0, 0, 0, 1 \rangle$	21, 19, 18, 16, 15, 13, 12, 7, 6, 4, 0 (irreducible)
$\langle 0, 0, 0, 0, 0, 0, 0, 1, 1, 1 \rangle$	21, 19, 18, 16, 13, 9, 7, 5, 2, 1, 0 (primitive)
$\langle 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 \rangle$	21, 19, 18, 17, 16, 14, 13, 3, 0 (primitive)
$\langle 0, 0, 0, 0, 0, 1, 1, 0, 0, 1 \rangle$	21, 19, 18, 16, 15, 12, 10, 9, 6, 4, 3, 1, 0 (primitive)
$\langle 0, 0, 0, 0, 1, 0, 0, 0, 0, 1 \rangle$	21, 19, 18, 16, 15, 13, 12, 9, 5, 2, 0 (primitive)
$\langle 0, 0, 0, 0, 1, 0, 0, 1, 0, 0 \rangle$	21, 19, 18, 17, 16, 14, 11, 10, 8, 3, 2, 1, 0 (primitive)
$\langle 0, 0, 0, 0, 1, 0, 0, 1, 0, 1 \rangle$	21, 19, 18, 16, 15, 13, 12, 9, 2, 1, 0 (primitive)
$\langle 0, 0, 0, 0, 1, 0, 1, 0, 1, 1 \rangle$	21, 19, 18, 16, 11, 10, 9, 8, 7, 1, 0 (primitive)

$\langle 0,0,0,0,1,1,0,1,0,0 \rangle$	21,19,18,17,16,14,11,10,8,6,4,1,0 (primitive)
$\langle 0,0,0,0,1,1,1,0,1,1 \rangle$	21,19,18,16,11,10,9,8,7,6,4,3,2,1,0 (primitive)

Table 3 shows the characteristic polynomial V_{21} for the case $d = 1$.

Table 3. Characteristic Polynomial V_{21} for the Case $d = 1$

$T_{10} = \langle a_1, \dots, a_{10} \rangle$	Characteristic polynomial V_{21} of $\langle T_{10}, 1, \overline{T_{10}^*} \rangle$
$\langle 0,0,0,0,0,0,0,0,0,1 \rangle$	21,20,19,15,14,12,11,10,8,6,4,3,2,1,0 (primitive)
$\langle 0,0,0,0,0,0,0,1,1,1 \rangle$	21,20,19,13,12,7,6,1,0 (primitive)
$\langle 0,0,0,0,0,0,1,0,0,0 \rangle$	21,20,19,17,16,14,13,10,9,6,4,1,0 (primitive)
$\langle 0,0,0,0,0,1,1,0,0,1 \rangle$	21,20,19,15,14,13,11,7,4,3,0 (irreducible)
$\langle 0,0,0,0,1,0,0,0,0,1 \rangle$	21,20,19,15,14,12,11,10,9,7,6,4,0 (primitive)
$\langle 0,0,0,0,1,0,0,1,0,0 \rangle$	21,20,19,17,16,14,12,11,10,9,6,5,3,2,0 (primitive)
$\langle 0,0,0,0,1,0,0,1,0,1 \rangle$	21,20,19,15,14,12,11,10,9,7,6,5,0 (primitive)
$\langle 0,0,0,0,1,0,1,0,1,1 \rangle$	21,20,19,11,7,6,3,2,0 (primitive)
$\langle 0,0,0,0,1,1,0,1,0,0 \rangle$	21,20,19,17,16,14,12,11,10,9,5,1,0 (primitive)
$\langle 0,0,0,0,1,1,1,0,1,1 \rangle$	21,20,19,11,7,1,0 (primitive)

Table 4 shows the characteristic polynomial V_{22} for the case $(d_1, d_2) = (0, 0)$.

Table 4. Characteristic Polynomial V_{22} for the Case $(d_1, d_2) = (0, 0)$

$T_{10} = \langle a_1, \dots, a_{10} \rangle$	Characteristic polynomial V_{22} of $\langle T_{10}, 0, 0, \overline{T_{10}^*} \rangle$
$\langle 0,0,0,0,0,0,0,0,0,0 \rangle$	22,19,18,15,14,12,7,6,3,2,0 (primitive)
$\langle 0,0,0,0,0,0,0,0,1,1 \rangle$	22,19,16,14,13,9,6,2,0 (primitive)
$\langle 0,0,0,1,0,1,1,0,1,1 \rangle$	22,19,16,13,8,5,4,1,0 (primitive)
$\langle 0,0,0,1,1,0,0,1,0,0 \rangle$	22,19,18,15,14,11,8,5,4,1,0 (irreducible)
$\langle 0,0,0,1,1,0,1,0,0,0 \rangle$	22,19,18,15,14,12,11,10,9,8,7,6,5,4,3,1,0 (primitive)
$\langle 0,0,0,1,1,1,0,0,0,1 \rangle$	22,19,14,12,11,9,8,7,6,5,4,3,0 (primitive)
$\langle 0,0,0,1,1,1,0,0,1,0 \rangle$	22,19,18,16,15,14,13,11,7,6,0 (irreducible)
$\langle 0,0,0,1,1,1,0,1,1,0 \rangle$	22,19,18,16,15,13,12,9,7,6,4,3,0 (primitive)
$\langle 0,0,1,0,0,0,0,0,0,1 \rangle$	22,19,14,12,11,9,7,4,0 (primitive)
$\langle 0,0,1,0,0,0,0,0,1,0 \rangle$	22,19,18,16,15,14,13,11,10,3,0 (irreducible)

Table 5 shows the characteristic polynomial V_{22} for the case $(d_1, d_2) = (1, 1)$.

Table 5. Characteristic Polynomial V_{22} for the Case $(d_1, d_2) = (1, 1)$

$T_{10} = \langle a_1, \dots, a_{10} \rangle$	Characteristic polynomial V_{22} of $\langle T_{10}, 1, 1, \overline{T_{10}^*} \rangle$
$\langle 0,0,0,0,0,0,0,0,0,0 \rangle$	22,20,19,18,17,16,15,13,12,11,9,8,6,2,0 (primitive)
$\langle 0,0,0,0,0,0,0,0,1,1 \rangle$	22,20,19,17,16,14,13,10,8,6,5,3,2,1,0 (primitive)
$\langle 0,0,0,1,0,1,1,0,1,1 \rangle$	22,20,19,17,16,13,12,9,6,3,0 (primitive)
$\langle 0,0,0,1,1,0,0,1,0,0 \rangle$	22,20,19,18,17,16,15,13,10,7,6,4,3,1,0 (irreducible)
$\langle 0,0,0,1,1,0,1,0,0,0 \rangle$	22,20,19,18,17,16,15,13,12,9,8,6,5,3,0 (primitive)
$\langle 0,0,0,1,1,1,0,0,0,1 \rangle$	22,20,19,17,14,11,8,7,4,2,0 (primitive)
$\langle 0,0,0,1,1,1,0,0,1,0 \rangle$	22,20,19,18,17,15,12,10,9,6,5,1,0 (primitive)

$\langle 0,0,0,1,1,1,0,1,1,0 \rangle$	22,20,19,18,17,15,14,12,11,10,9,5,4,2,0 (irreducible)
$\langle 0,0,1,0,0,0,0,0,0,1 \rangle$	22,20,19,17,14,11,7,6,5,4,3,2,0 (irreducible)
$\langle 0,0,1,0,0,0,0,0,1,0 \rangle$	22,20,19,18,17,15,12,9,8,7,6,1,0 (primitive)

In Theorem 3.4, $\langle d_1, d_2 \rangle$ can be extended to $D_k = \langle d_1, d_2, \dots, d_k \rangle$ ($k \leq n$).

Remark. If the characteristic polynomial $C_{2n+k}(x)$ of a $(2n+k)$ -cell 90/150 CA $\langle T_n, D_k, \overline{T_n^*} \rangle = \langle a_1, \dots, a_n, d_1, \dots, d_k, \overline{a_n}, \dots, \overline{a_1} \rangle$ is irreducible, then the characteristic polynomial of $\langle T_n, \overline{D_k^*}, \overline{T_n^*} \rangle$ is irreducible.

We can obtain $(2n+i)$ -cell ($i=0,1,2,\dots,n$) 90/150 CA using T_n and D_k ($k \leq n$) by Theorem 3.3, Theorem 3.4 and Remark.

By Remark, we obtain Tables 6 and 7. Table 6 provides a list of n -cell ($8 \leq n \leq 12$) 90/150 CA $\mathbb{C}$ and irreducible polynomials corresponding to $\mathbb{C}$, which are obtained by the proposed method using 4-cell 90/150 CA and D_k ($k=1,2,3,4$). Also Table 7 shows the comparison of frequency rates of irreducible polynomials by exhaustive method and the proposed method. It is shown that the probability of finding the irreducible polynomial by the proposed method is greater than the probability of finding the irreducible polynomials by the exhaustive method.

Table 6. 90/150 CA $\mathbb{C}$ and Irreducible Polynomials corresponding to $\mathbb{C}$

Degree	T_n	D_k	Characteristic Polynomial
8	$\langle 0, 0, 0, 0 \rangle$	-	8,6,5,3,0
	$\langle 1, 1, 0, 0 \rangle$	-	8,6,5,4,3,1,0
	$\langle 1, 1, 1, 1 \rangle$	-	8,6,5,3,0
9	$\langle 0, 0, 0, 1 \rangle$	$\langle 0 \rangle$	9,6,5,3,2,1,0
	$\langle 0, 1, 1, 1 \rangle$	$\langle 0 \rangle$	9,6,5,3,0
	$\langle 1, 0, 0, 0 \rangle$	$\langle 0 \rangle$	9,6,5,3,0
	$\langle 1, 1, 1, 0 \rangle$	$\langle 0 \rangle$	9,6,5,3,2,1,0
	$\langle 0, 0, 0, 1 \rangle$	$\langle 1 \rangle$	9,8,6,5,3,2,0
	$\langle 0, 1, 1, 1 \rangle$	$\langle 1 \rangle$	9,8,6,5,3,1,0
	$\langle 1, 0, 0, 0 \rangle$	$\langle 1 \rangle$	9,8,6,5,3,1,0
	$\langle 1, 1, 1, 0 \rangle$	$\langle 1 \rangle$	9,8,6,5,3,2,0
10	$\langle 0, 0, 0, 0 \rangle$	$\langle 0, 0 \rangle$	10,8,7,6,5,4,3,1,0
	$\langle 1, 1, 1, 1 \rangle$	$\langle 0, 0 \rangle$	10,8,7,6,5,4,3,1,0
	$\langle 0, 1, 0, 1 \rangle$	$\langle 0, 1 \rangle$	10,9,8,6,5,1,0
	$\langle 1, 0, 1, 1 \rangle$	$\langle 0, 1 \rangle$	10,9,8,3,2,1,0
	$\langle 1, 1, 0, 0 \rangle$	$\langle 0, 1 \rangle$	10,9,8,4,3,2,0
	$\langle 0, 0, 1, 1 \rangle$	$\langle 1, 0 \rangle$	10,9,8,4,3,2,0
	$\langle 0, 1, 0, 0 \rangle$	$\langle 1, 0 \rangle$	10,9,8,3,2,1,0
	$\langle 1, 0, 1, 0 \rangle$	$\langle 1, 0 \rangle$	10,9,8,6,5,1,0
	$\langle 0, 0, 0, 0 \rangle$	$\langle 1, 1 \rangle$	10,7,0
	$\langle 1, 1, 1, 1 \rangle$	$\langle 1, 1 \rangle$	10,7,0
11	$\langle 0, 1, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	11,8,5,4,2,1,0
	$\langle 1, 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	11,8,5,4,2,1,0
	$\langle 0, 0, 1, 0 \rangle$	$\langle 0, 0, 1 \rangle$	11,10,8,7,5,4,3,1,0
	$\langle 1, 1, 0, 1 \rangle$	$\langle 0, 0, 1 \rangle$	11,10,8,6,5,1,0
	$\langle 0, 0, 0, 0 \rangle$	$\langle 0, 1, 0 \rangle$	11,10,8,6,4,2,0
	$\langle 1, 1, 1, 1 \rangle$	$\langle 0, 1, 0 \rangle$	11,10,8,6,4,2,0
	$\langle 0, 0, 1, 0 \rangle$	$\langle 0, 1, 1 \rangle$	11,9,8,7,6,4,3,1,0
	$\langle 1, 1, 0, 1 \rangle$	$\langle 0, 1, 1 \rangle$	11,9,8,6,5,3,2,1,0
	$\langle 0, 0, 1, 0 \rangle$	$\langle 1, 0, 0 \rangle$	11,10,8,6,5,1,0
	$\langle 1, 1, 0, 1 \rangle$	$\langle 1, 0, 0 \rangle$	11,10,8,7,5,4,3,1,0
	$\langle 0, 0, 0, 0 \rangle$	$\langle 1, 0, 1 \rangle$	11,9,8,6,3,1,0
	$\langle 1, 1, 1, 1 \rangle$	$\langle 1, 0, 1 \rangle$	11,9,8,6,3,1,0

	< 0, 0, 1, 0 > < 1, 1, 0, 1 > < 0, 1, 1, 1 > < 1, 0, 0, 0 >	< 1, 1, 0 > < 1, 1, 0 > < 1, 1, 1 > < 1, 1, 1 >	11,9,8,6,5,3,2,1,0 11,9,8,7,6,4,3,1,0 11,10,9,5,3,1,0 11,10,9,5,3,1,0
12	< 0, 1, 0, 0 > < 0, 1, 0, 1 > < 0, 1, 1, 1 > < 1, 0, 0, 0 > < 1, 0, 1, 0 > < 1, 0, 1, 1 > < 0, 0, 0, 0 > < 0, 1, 0, 1 > < 1, 0, 1, 0 > < 1, 0, 1, 1 > < 0, 0, 0, 1 > < 0, 0, 1, 0 > < 0, 1, 0, 0 > < 0, 1, 0, 1 > < 0, 1, 1, 1 > < 1, 0, 1, 1 > < 0, 0, 0, 0 > < 1, 1, 1, 1 > < 0, 1, 0, 0 > < 1, 0, 0, 0 > < 1, 0, 1, 0 > < 0, 1, 0, 1 > < 1, 0, 1, 0 > < 1, 0, 1, 1 > < 1, 1, 0, 1 > < 1, 1, 1, 0 > < 0, 1, 0, 0 > < 0, 1, 0, 1 > < 1, 1, 1, 1 > < 0, 1, 0, 0 > < 0, 1, 0, 1 > < 1, 1, 1, 1 > < 1, 0, 0, 0 > < 1, 0, 1, 0 > < 1, 0, 1, 1 >	< 0, 0, 0, 0 > < 0, 0, 0, 0 > < 0, 0, 0, 0 > < 0, 0, 0, 0 > < 0, 0, 0, 0 > < 0, 0, 0, 0 > < 0, 0, 0, 1 > < 0, 0, 1, 0 > < 0, 0, 1, 0 > < 0, 0, 1, 0 > < 0, 0, 1, 1 > < 0, 0, 1, 1 > < 0, 1, 0, 0 > < 0, 1, 0, 0 > < 0, 1, 0, 1 > < 0, 1, 0, 1 > < 0, 1, 0, 1 > < 0, 1, 1, 1 > < 1, 0, 0, 0 > < 1, 0, 1, 0 > < 1, 0, 1, 0 > < 1, 0, 1, 1 > < 1, 0, 1, 1 > < 1, 0, 1, 1 > < 1, 1, 0, 0 > < 1, 1, 0, 1 > < 1, 1, 1, 0 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 > < 1, 1, 1, 1 >	12,10,9,7,6,5,4,1,0 12,10,9,8,7,3,0 12,10,9,8,7,6,0 12,10,9,8,7,6,0 12,10,9,8,7,3,0 12,10,9,7,6,5,4,1,0 12,11,10,2,0 12,11,10,6,3,2,0 12,11,10,5,2,1,0 12,11,10,8,7,4,0 12,9,8,5,4,1,0 12,9,8,5,0 12,11,10,8,7,4,0 12,11,10,5,2,1,0 12,11,10,6,3,2,0 12,9,5,4,2,1,0 12,9,5,2,0 12,9,8,6,3,2,0 12,11,9,8,4,3,2,1,0 12,11,10,2,0 12,9,8,6,3,2,0 12,9,5,2,0 12,9,5,4,2,1,0 12,11,9,8,6,2,0 12,11,9,8,5,3,2,1,0 12,11,9,7,6,5,4,2,0 12,9,8,5,0 12,9,8,5,4,1,0 12,11,9,7,6,5,4,2,0 12,11,9,8,5,3,2,1,0 12,11,9,8,6,2,0 12,11,9,8,4,3,2,1,0 12,10,9,8,7,4,3,2,0 12,10,9,7,6,5,2,1,0 12,10,9,7,5,4,3,2,0 12,10,9,7,5,4,3,2,0 12,10,9,7,6,5,2,1,0 12,10,9,8,7,4,3,2,0

Table 7. Comparison of Frequency Rates of Irreducible Polynomials by Exhaustive Method and the proposed Method

Degree	Frequency Rate of Irreducible Polynomials by Exhaustive Method	Frequency Rate of Irreducible Polynomials by the Proposed Method
8	23.44%	37.50%
9	21.88%	50.00%
10	19.34%	31.25%
11	18.16%	25.00%
12	16.36%	29.69%

4. Conclusion

In this paper, we analysed the characteristic polynomials of 90/150 CA using inverse symmetric transition rule of various types. By the proposed method, we synthesized $(2n+i)$ -cell $(i=0,1,2,\dots,n)$ 90/150 CA using T_n and D_k $(k \leq n)$.

Especially, we modelled the shrinking generator as 90/150 CA whose characteristic polynomial is $\{f(x)\}^k$ ($k \leq n$) using inverse symmetric transition rules. Also, we showed that the probabilities of finding $(2n+i)$ -cell CA $\mathbb{C}$ with the maximum length by the proposed method are greater than the probabilities of finding $\mathbb{C}$ by an exhaustive search.

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